AFCRL-66-153

AD 634 070

NUMERICAL INVESTIGATION OF ELECTROMAGNETIC SCATTERING AND DIFFRACTION BY CONVEX OBJECTS

Nelson A. Logan

Lockheed Missiles & Space Company
A Group Division of Lockheed Aircraft Corporation
Sunnyvale, California

Contract No. AF19(628)-4393

Project No. 4600

Task No. 460004

FINAL REPORT

Period Covered: 16 Nov 1964 through 15 Nov 1965

Date of Report
December, 1965

Prepared
for

AIR FORCE CAMBRIDGE RESEARCH LABORATORIES
OFFICE OF AEROSPACE RESEARCH
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LOCKHEED MISSILES & SPACE COMPANY

ABSTRACT

The solutions $D(x,y,y_0,q)$ of the partial differential equation $D_{yy} + iD_x + yD = 0$ are employed as a mathematical model for the fields reflected from and diffracted around an opaque convex surface. This solution involves the Airy function $w_1(t)$ which is a solution of $w_1''(t) - tw_1(t) = 0$. FORTRAN computer programs and tables are given for the appropriate Airy functions. A considerable portion of the report is devoted to discussions of the representations for the roots of the equation $w_1'(t_s) - qw_1(t_s) = 0$ which play a fundamental role in the theory. As a function of q, these roots satisfy the Riccati equation $(t - q^2)(dt/dq) = 1$. The Riccati equation is used to develop several representations for these roots. FORTRAN programs are presented for the evaluation of these constants. Tables of the roots are given for certain cases which have been shown to be of practical significance. Asymptotic expansions are developed for the height gain function $F(y)$ which is a solution of $F''(y) + (k^2 f(y) - \lambda_s)F(y) = 0$. The theory of representing functions by a series of Chebyshev polynomials is reviewed and a discussion is given of the significance of these expansions for future work in diffraction theory.

ACKNOWLEDGMENTS

Many of the results contained in this report had their beginnings in the period of several years which preceded the one-year period of Contract AF 19(628)-4393. The author wishes to express his gratitude to the three department managers under whom he served during these years when the research was supported by the Lockheed Independent Research Program. Without the encouragement and understanding of Messrs. E. A. Blasi, A. S. Dunbar and N. J. Gamara this work would never have been carried out to the degree of completeness at which is now stands. During the past year (while the work has been supported by the contract with the Air Force Cambridge Research Laboratories), the author has had the privilege of working under the management of Mr. A. S. Dunbar.

Although the work of the last year has been carried out solely by the author, many of the results reported herein are the fruits of the labors of Mr. R. S. Mason and Dr. K. S. Yee.

Some of the equations in this report were typed by Miss Ann Youngs and the author gratefully acknowledges his endebtedness to her for her diligence in transcriting handwritten equations into typescript. The author is also endebted to Mrs. Bette DeGand for the typing of several of the tables and to Mrs. Marion Hegeman for assistance in administrative details.

RELATED CONTRACTS AND PUBLICATIONS

1. Logan, N.A., "Survey of Some Early Studies of the Scattering of Plane Waves by a Sphere," Proc. of IEEE, Vol. 3, Aug 65, pp. 773-785. Contract No. AF 19(628)-4393

PREFACE

Although this report is officially entitled a "final" report upon Contract AF 19(628)-4393, the report is in fact an "interim" report upon a subject which has attracted the author's attention for over ten years. Prior to the one-year period during which the research has been carried out under the present contract, the author undertook work along these lines as part of the Independent Research Program of the Lockheed Missiles and Space Company. The attainments made to date fall far short of the author's goals. However, the author hopes to continue to strive towards filling some of the gaps that exist between the results in <u>both</u> the asymptotic theory of diffraction by convex surfaces and the numerical analysis which is required in order to translate the esoteric mathematical results into curves and tables that will meet the needs of the practical engineer. The author readily admits that the opaque convex surface is an over-idealized model for any except the simplest of propagation and diffraction problems. However, the actual problems that confront most engineers in this field are so complex that there is a great demand for information concerning the phenomena associated with a smooth convex surface. The present research efforts have been directed towards extending the algorithms and the computer programs that will permit us to obtain numerical results when the scatterer is an opaque surface which is characterized by an impedance boundary condition.

The problem of the diffraction of electromagnetic waves by convex surfaces has been the subject of hundreds of papers since the appearance over 100 years ago of a classic paper by A. Clebsch which can be considered to be the "first" paper upon this subject. The

author recently had the honor of having a survey of some of the early history of this problem to appear as the lead article in a "Special Issue on Radar Reflectivity" which was published as the August, 1965, issue of the Proceedings of the Institute of Electrical and Electronics Engineers (Proc. IEEE, Vol. 53, No. 8, Aug., 1965). The author would like to encourage his readers to read this historical survey. If the reader is not already familiar with this special issue, the author is certain that the very fact that the reader has been interested enough in this report to have read this far is a very sure indication that the "Special Issue on Radar Reflectivity" will provide the reader with a "gold mine" of information.

The theory of diffraction by convex surfaces cannot be readily separated from a large body of work in applied mathematics. Therefore, the engineer who finds himself facing a practical problem involving diffraction by a convex surface finds that he is confronted with an almost chaotic mass of literature in which there are many gaps in the practical information to say nothing of the fact that there is a lack of a coherent and effective general theory. It is with regrets that the present author finds himself in the position of adding to the confusion and the frustration of the "newcomers" by releasing the present material in a form such that a considerable background is required in order to pick out results which may be helpful in a particular practical problem.

The author would like to illustrate his sympathy for the reader who is a "newcomer" to the diffraction field by relating an experience which occurred during the course of this contract. The author is frequently asked to recommend books, reports, and papers which can provide an engineer with a means of learning what results are currently available for him to use in connection with problems associated with diffraction by a convex surface.

For several years, the author has been telling his colleagues that the book that they were seeking was about to be published as Vol. 1, Diffraction by Convex Surfaces in the International Series of Monographs on Electromagnetic Waves which was begun several years ago by Pergamon Press. The author of this book was to have been Academician V. A. Fock of Leningrad University. The present author* had been considerably influenced by a series of papers by Fock which he had discovered in the mid-1950's. With the cooperation of P. Blacksmith, M. D. Friedman, and others, the author had made available to a limited circle of readers a collection** of 13 papers by Fock, together with a summary of Fock's contributions by Academician V. I. Smirnov and an appendix which contained a pertinent paper by Academician M. A. Leontovich. It was the understanding of the present author that when Fock visited the United States in the late 1950's that he had refused permission for the collection of papers to be issued in book-form on the grounds that the papers were incoherent and that he intended to prepare a monograph in which a unified presentation of his research would be presented. However, even behind the "Iron Curtain" the scientists apparently find the days and nights passing

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*Although the present author has since become sufficiently acquainted with the literature to realize that much that he learned from Fock's papers could have been learned from papers by White, Pekeris, Pryce, Furry, and others, the author recognizes the fact that the clarity of presentation and the thoroughness of the investigations which were found in Fock's papers were such that he can best describe Fock's influence upon his own work by asserting that Fock was his "teacher."

**Air Force Cambridge Research Center, Diffraction, Refraction, and Reflection of Radio Waves: Thirteen Papers by V. A. Fock, N.A. Logan and P. Blacksmith, eds., Report AFCRC-TN-57-102, ASTIA Document No. AD117276, Bedford, Mass., June, 1957

in such swift processions that one has to sometimes face reality and admit that the goal which one has set for oneself is too ambitious. Anyway, it was a great disappointment to the present author when in mid-1965 the long-promised Pergamon Press volume appeared in print.* This monograph is nothing more than 17 papers by Fock (with several having been written with co-authors) which (except for minor corrections) appear in their original form. Except for the inclusion of papers published since 1957, the book which is now available is essentially the same collection of "raw" reference material which the present author and his associates had sought to provide to scientists in the United States when the 391-page paperbound AFCRC report was distributed in 1957. The author strongly urges the serious student of diffraction theory to acquaint himself with these papers by Fock. However, since these are research papers and the material has not been put together in "textbook style," the reader may find Fock's book a difficult source from which to acquire a basic knowledge and background upon which to build the ability to read the hundreds of papers which are available in this subject area. Furthermore, since no effort is made to relate the material in Fock's book to the more recent publications of other authors, the reader will have to "forage" for himself to learn what other results are available to him. Making a literature survey requires considerable "detective work" since a number of the modern authors have worked independently of one another and cross-references to each other's works are often conspicuous by their absence.

There are sufficient gaps in our present state of knowledge in

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*V. A. Fock, Electromagnetic Diffraction and Propagation Problems, New York, Pergamon Press, 1965

this subject area that any book or monograph published today would be virtually out-of-date by the time it appeared in print. This is particularly true at the present time when the availability of the electronic computer is making it possible for us to attempt to obtain numerical results which were "impossible" during World War II when the widespread use of high frequency radio waves created a need for results from diffraction theory for those cases in which classical geometrical optics failed to provide an explanation of the observed phenomena. As an example of the progress which has been made in the computer field, it is interesting to cite the fact that Aiken* made the observation in 1945 that "...if it had been possible to run the computation without interruption forty-five days would have been sfficient time to complete the tables..." of Airy functions which Furry and other scientists needed in their studies of wave propagation. One of the FORTRAN IV programs which is given in this report could be used on the UNIVAC 1107 to produce these identical tables in approximately thirty minutes. The author can also cite a more recent example based upon his own experience. Upon some output sheets from the IBM-650 which were used to prepare the curves for the scattering properties of perfectly conducting spheres which appear in the appendix of the monograph of King and Wu** there appear some

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*The Staff of the Computation Laboratory, Tables of the Modified Hankel Functions of Order One-Third and of Their Derivatives, Cambridge, Mass., Harvard University Press, 1945

**R. W. P. King and T. T. Wu, The Scattering and Diffraction of Waves, Cambridge, Mass., Harvard University Press, 1959. (Graphs in appendix illustrate current distributions and bistatic cross sections for perfectly conducting spheres for ka = 1.1(0.6)9.5, 10, and 20.)

pencilled notes to indicate that these computations required forty-five minutes for a set of E-plane and H-plane curves for a single value of the parameter ka. It is now possible to generate this same data on a computer such as the UNIVAC 1107 in less than forty-five seconds.

However, as the present author looks back at his computing experience, his respect for the amazing speed of the modern electronic computer does not leave him standing in awe as much as does the realization that programming languages such as FORTRAN IV offer the modern scientist a powerful tool which was virtually beyond his wildest dream less than 20 years ago. One has to have grown-up with the "computer age" to really appreciate what a change has been made possible by the development of mathematically-oriented programming languages which en-enable a scientist who is not a "computer specialist" to perform millions of mathematical manipulations at the rate of thousands per second.

The author's first major computation job involved some diffraction patterns for a circular cylinder and the work was carried out in the late 1940's by a group of ladies who worked for months at Friden and Marchant desk machines. Even when two ladies would work in tandem on worksheets which were laid out in identical manners, the detection of errors was a very frustrating task. With human hands and minds at work, it was virtually impossible to eliminate blunders caused by an error in lookup in a table, in transfering a number into the keyboard, copying the number from the register onto the work sheets, an error in placing a decimal point, and, ABOVE ALL, one had to contend with frequent replacement of the computer personnel so that with even a moderate size group there was usually always a "new girl." Anyway, much of this was drastically changed a few years later when the author

set outto compute the scattering diagrams which are contained in the appendix of the monograph by King and Wu. The author had started some of these computations on desk machines of the Friden and Marchant variety, but he was bogged down in trying to detect the errors so as to be able to construct the scattering patterns. At this point, the services of G. E. Reynolds and M. R. Hoes were enlisted and they came up with a set of instructions which made it possible to obtain the desired computations-<u>free of human error</u>- on the IBM-650 computer. Although the manner in which these programmers managed to get the computer to perform was, to the author, a sort of "black magic," the author does recall the excessive handling of thousands of IBM cards as different portions of the computation were handled by the limited "memory" of the computer and the final results obtained by having hundreds of cards read back into the machine and the final answers assembled. This was a far cry from the tedious and frustrating manner of trying to obtain the results by hand, but the specialists trained in programming the computer were an essential part of the team.

The development of the FORTRAN, ALGOL, MAD, and similar "languages" has made it possible for the scientist, if he so desires (and budget conditions can sometimes make desire into a necessity), to dispense with the programmer for many of the day-to-day type of calculations. The introduction of these languages have had an important effect in that they are generally languages which are independent of a particular computer and hence a scientist who has developed a program for his computer (which might be a UNIVAC 1107, for example) may be able to share the program with another scientist who has acess to a totally different computer (such as an IBM 7094). This is a remarkable new aspect of cooperation between scientists and makes it possible for one to build upon the experience of other workers. In the days of hand computation

many of the present concepts of programming were in use. A worksheet properly laid out with all the instructions explicitly displayed at the headings of the proper columns constituted an algorithm (a set of explicit rules for carrying out a computattion) for the numerical problem which was under consideration. However, except for the layout for certain problems such as an harmonic analysis of a set of data, it was very uncommon for one scientist to pass on to another his "set-up" for the arriving at the numbers which he presented in his tables and graphs.

The situation has now markedly changed with the development of the new programming languages. The Communications of the Association for Computing Machinery have been supplying a medium for the publication of algorithms written in the ALGOL language for the past several years. However, the largest cooperative effort in making programs available for use by persons or groups other than the initiators is the IBM SHARE system. The rapid advances in computers in the past several years has made it almost mandatory that a program which is to be effectively shared with another user be written in a standard language. For example, the use of a machine-oriented language should be avoided whenever possible. The present author knows of one extremely large program which was coded in the FORTRAN ASSEMBLY PROGRAM (FAP) language and the need has now arisen to convert this program to be used on a computer which uses a different assembly language. The net result is that a team of programmers are engaged in translating a FAP-coded program into FORTRAN IV so as to be able to have the program compile upon other computers than the IBM-7090-7094 for which it was originally prepared. Because compilers can be expected to change relatively rapidly over the next few years, the use of a standard language such as FORTRAN IV is highly recommended. Although the different compilers may accept only certain "dialects" of FORTRAN IV, the changes which must be made in going from one compiler to another are rather minor if one restricts oneself to

the use of features which are available on the widest class of compilers. A useful reference for FORTRAN IV is the report* of the American Standards Association (ASA) FORTRAN Working Group. Their report lists side-by-side a set of standards for FORTRAN (which is FORTRAN IV) and BASIC FORTRAN (which is FORTRAN II). McCracken** has attempted to collect in tabular form some of the characterisitics of over two dozen programming languages which are being referred to as FORTRAN IV. Unfortunately, in many compilers FORTRAN II is not solely a subset of FORTRAN IV and hence not all FORTRAN II programs can be compiled on software which compiles FORTRAN IV programs. The language situation promises to be a bit more confused in mid-1966 when the manuals and the compilers will be available for IBM Operating System/360. Although this new language has been called FORTRAN VI, NPL (New Programming Language), and MPPL (Multi-Purpose Programming Language), the most recent announcements apparently refer to it as PL/I (Programming Language/One). However, some of the early indications are that FORTRAN IV as it is now defined will most likely be a subset of the more versatile IBM programming language.

The carrying out of certain portions of the work of the present contract has been hampered by the fact that much of the work prior to this contract had involved the use of a set of FAP-coded SHARE routines known as NPREC. The conventional FORTRAN II compilers did not provide means for the input and output of double precision numbers, and the floating point numbers were required to lie within the range 10^{-38} and 10^{38}. We had not only

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*W. P. Heising, "History and Summary of FORTRAN Standardization Development for the ASA," Comm. Assoc. Comput. Mach., Vol. 7 Oct., 1964, pp. 590-625

**D. D. McCracken, "How to Tell If It's FORTRAN IV," Datamation, Vol. 11, Oct., 1965, pp. 38-41

the need, in certain algorithms, to greatly exceed the range of the numbers permitted in FORTRAN, but we also wanted to have the ability to input and output more than the 8-digits that were maximum number of digits which the FORTRAN compiler would handle. The author was confronted with several problems where the cancellation of numbers of comparable size was so complete that answers to even several significant figures could not be obtained with the relatively simple algorithm unless many significant figures could be manipulated. Nearly a decade ago the author showed that the use of high precision could be used to sum certain divergent series in diffraction theory and this led to the development at the Air Force Cambridge Research Center of the capability of working with quadruple precision numbers which consisted of floating point numbers of 37 significant figures.* However, the capability developed at AFCRC was limited to the computer available to them because this was a machine-language coded program. It was not until 1961 that the author learned of the existence of a FAP-coded SHARE program which would enable a FORTRAN programmer to compute in double precision floating point arithmetic. This very ingenious program had been prepared by R. I. Berggren, J. C. Cysbers, R. Hafner, and L. Sonneville of the Rocketdyne Division of North American Aviation, Inc. The prograr sed three consecutive 36-bit words to store a single number, wich one word being used for the exponent and two words for the fraction. This arrangement mads it possible to work with floating point numbers which consisted of more than 21 decimal digits, and with exponents whose absolute value could not exceed $(2^{35}-1) \approx 3.4\times10^{10}$. Although this set of subroutines could be employed in a FORTRAN II program, its use was rather clumsy. When it was announced in the Fall of 1964 that the UNIVAC 1107 which was to be installed at Lockheed in the Spring of 1965 was to have a double-precision capability of

*Air Force Cambridge Research Center, Cambridge Computer Interpretive Routine for Quadruple Precision Numbers, Report TN-59-155, 1959

21-significant decimal digits, the author resolved to attempt to convert to FORTRAN IV for the UNIVAC 1107 and carry out much of the previous work which had involved the use of the NPREC subroutines by employing the relatively simple double-precision of the new computer. However, upon delivery it was found that the UNIVAC 1107 performed double-precision by software operations which limited the precision of the numbers to 17 decimal digits. Nevertheless, the greater simplicity of the use of FORTRAN IV as compared with the use of FORTRAN II supplemented by the NPREC subroutines (together with the constant anxiety to the effect that a systems change on the IBM 7094 might result in the NPREC subroutine, coded in FAP, no longer being compatible with the system tapes being installed in the more advanced system monitors being made available by IBM) led the author to seek to continue to strive towards the use of the FORTRAN IV compiler for the UNIVAC 1107. Although the loss of 4 to 5 significant figures when the FORTRAN IV program was coded to replace the FORTRAN II-NPREC program proved to limit a few of the problems that could be handled, two far more important limitations appeared. Since the manipulations had been almost solely performed internally in the computer and the magnitude of the numbers involved in many of the algorithms had not been known to the author, it came as quite a shock to realize that many of the algorithms which had worked successfully when using the NPREC subroutines could not be used at all on the UNIVAC 1107 (or, on the FORTRAN IV compiler for the IBM 7094) because the exponents exceeded the magnitude of 10^{38} which is an inherent property of both the IBM and the UNIVAC compilers. Most compiler designers have apparently decided that the need for the manipulation of such large numbers is restricted to such special applications that only the compilers for the Control Data Corporation (CDC 3000 and 6000 series) and Philco (2000 series) computers take into consideration these needs, and even here the restrictions are to 10^{308} and 10^{616}, respectively.

A second surprise that came with the attempts to use the more sophisticated UNIVAC 1107 computer was the receipt from the compiler of messages reading "...UNRESOLVABLE AMBIGUITY CAUSED BY SOURCE LANGUAGE ERROR..." when attempts were made to run certain FORTRAN II-NPREC programs on the UNIVAC 1107 after the conversion process (consisting mainly of changing read and write statements and the making of allowances for some minor changes such as the replacement of COSF(X) by COS(X) for single-precision cosines and DPCOS(X) by DCOS(X) for double-precision cosines) had been carried out. The difficulty can be traced to several new sophistications of the newer software and hardware of the new generation of electronic computer of which the UNIVAC 1107 is a typical example. With the newer compilers, one is dealing with both new hardware and new software. One of the hardware difficulties apparently arises from the fact that greatly increased operation speed is achieved by using a set of film registers to store the subscripts which are used during the execution of a DO-loop. This means that for most programs a very significant saving in computer time is saved since the subscripts are not computed, stored in memory, and then retrieved as was the case in the earlier hardware. However, if one has made excessive use of subscripts in a DO-loop, the system apparently becomes saturated and informs the programmer that an unresolvable ambiguity has been encountered. Therefore, in such cases the original algorithm has to be recoded and broken down into smaller units. A not-too-dissimilar situation can occur with rather long arithmetic expressions that involve a great deal of multiplying, dividing, raising to a power, etc., and the user has been lax in the use of parentheses. The compilers have apparently been designed to try to "look ahead" and optimize an expression and it is possible that a lengthy expression which was compiled by "brute force" on an earlier FORTRAN II compiler may result in a message regarding an unresolvability in the source language

when submitted to a more sophisticated compiler. This latter type of difficulty is readily avoided by using parenthesis wherever they help the "human reader" to understand what is required and the result will generally be that the "inhuman compiler" will also encounter less difficulty in understanding what it is that the programmer wishes to have the computer do with his lengthy set of instructions. Although it is discouraging to find that a program that already works on a less sophisticated compiler and computer may need to be drastically modified to be acceptable upon the newer equipment, the greater flexibility of the FORTRAN IV language and the greater speed of compilation and of execution of the generation of computers more than offsets the inconveniences.

However, the difficulties associated with the fact that the number of decimal digits in a double-precision number is 16 for the IBM 7094 and 17 for the UNIVAC 1107, while the magnitude of the numbers may not exceed 10^{38}, remains a serious limitation for a number of computing algorithms. In spite of the fact that its use is somewhat clumsy, the author would like to see the old NPREC subroutines revised so as to be compatible with FORTRAN IV and be acceptable to the "systems tape" on which is "located" the FORTRAN IV compiler. Although the author knows of no efforts which will lead to this capability for the UNIVAC 1107 or 1108, the author knows that a group at the Ames Research Center of the National Aeronautics and Space Administration are in the process of preparing a FORTRAN IV version of the NPREC subroutines which will go several steps beyond the original NPREC subroutines and which may permit the use of triple- and quadruple-precision. When this set of subroutines has been debugged and submitted to the IBM SHARE library, the author looks foward to employing it for many problems which have been currently stymied because the algorithm known to the author would require drastic revision before the quantity can be computed with the normal range of precision and magnitude that the FORTRAN compilers have been designed to

cope with in their everyday use. That this class of problems is not purely academic and confined to only a few problems such as those encountered by the present author can be seen by observing that there have recently appeared several papers that deal with "arbitrary precision" floating-point computations and the associated input-output conversion techniques.*

Further evidence of the fact that this type of increased facility for the electronic computer is more widespread than one might imagine at a first consideration of the problem can be seen from the fact that R. K. Holmberg of the National Bureau of Standards, Boulder, Colorado, has prepared a special set of IBM 7090 subroutines which are known as the EPREC or "Extended Precision Arithmetic Package." Whereas the NPREC package from North American Aviation used two words for the fraction and one word for the exponent, the EPREC package requires just one word for the fraction and one word for the exponent. The result is that the precision of the number is essentially the same as in conventional FORTRAN, but overflow and underflow problems are postponed until much larger numbers are encountered since the routine will handle numbers as large as 10^{4920}.

The drawback of the use of software such as the NPREC and EPREC subroutines is that a considerable amount of the core storage (it takes nearly 4000 "words" to store the NPREC subroutines) is consumed in making the subroutines available. The author hopes that in the near future that the computer manufacturers

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*A. H. Stroud and D. Secrest, "A Multiple-Precision Floating-Point Interpretive Program for the Control Data 1604," <u>Computer J.</u>, Vol. 7, Jan., 1964, pp. 62-66

Y. Ikebe, "Note on Triple-Precision Floating Point Arithmetic with 132-Bit Numbers," <u>Comm. Assoc. Comput. Mach.</u>, Vol. 8, March, 1965, pp. 175-177

will become more aware of the special needs of scientific computation and build in the ability to handle numbers of greater precision and larger range in the hardware itself. At least one manufacturer seems to have made a good start in this direction,namely the Honeywell Series 200. The programmer can specify by a parameter card at the beginning of the source program an integer N which will determine the number of "characters" to be employed for each quantity which is the appear in the program that follows. Each "character" consists of 6 binary bits and hence the fact that any number of "characters" between 3 to 12 can be specified means that the programmer can specify by means of a control card whether the program is to use floating point numbers that range from an accuracy of 5 digits to 20 digits. The range of normalized floating point values is given as 10^{-616} to 10^{+616} in a recent brochure on the Honeywell Model 4200. It is interesting to observe that because of the variable word length made available to the programmer, that the "dialect" of FORTRAN IV which is used on this Honeywell computer does not include the type declarations COMPLEX and DOUBLE PRECISION. It is obvious that the need for double precision has been eliminated because of the choice of word length that is made available to the programmer, but the denial of the complex arithmetic ability found in other FORTRAN compilers means that the programmer must work out the real and imaginary parts of all quantities. However, the author can forsee the day when a later generation of this compiler will have the complex arithmetic built into the compiler. The author hopes that other compiler designers will see the advantages to the scientific computer of having the ability to control the precision by means of a "control card" and that the need for clumsy subroutines (written in assembly languages such as FAP for the IBM-7094 and in SLEUTH for the UNIVAC 1107) to perform calculations of high precision with "large" numbers will be eliminated in a future generation of FORTRAN IV compilers.

The time "ran out" on the present contract before the author had the opportunity to employ the Stromberg-Carlson 4020 film plotter which is available as part of the computer facilities at LMSC. A subroutine has been prepared by M. H. Seigel* for a customer on another contract who wished to make contour plots (similar to the familiar geological contour maps) of the radar reflectivity of certain targets as a function of the polar angles θ and the azimutal angles φ, when the angles are treated in the contour plot as a number pair (θ, φ) in a complex plane. It is hoped that in future work that this mode of presentation will be frequently employed in the author's efforts to depict to his readers the behavior of diffraction phenomena as a function of the complex-valued impedance parameter. This new plotter opens up a broad new field of applications for the visual presentation of data. The reader is urged to consult Fig. 1 of a recent paper by Chezem** where it is shown that the "beautiful" three-dimensional presentations of the behavior of functions which have long intrigued physicists who thumb through the pages of Jahnke and Emde have now been brought within the reach of any programmer who has access to a versatile electronic plotter such as the SC 4020.

The author hopes to continue the work which is reported upon in this document. He looks foward to the day when sufficient knowledge of the means by which the required functions can be evaluated will have been acquired so that the user need only follow a "cook-book" type of set of instructions to obtain numerical results for a practical problem. Just as the housewife seldom

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*Lockheed Missiles and Space Company, Digital Contour Plotting Techniques, by M. H. Seigel, Report ISSA-507, Sunnyvale, Calif., June, 1965

**C. G. Chezem, "Note on "Three-Dimensional" Plotting As a Technique for Finding the Zeros of Functions in the Complex Plane," Computer J., Vol. 8, Oct., 1965, p. 288

has any knowledge about the complicated chemistry taking place in her cooking utensils, so also the electronics engineer seldom has a need to actually know the fine details of asymptotic expansions, Airy functions, etc., in order to arrive at the guidelines which he requires from theory provided he can have before him the proper "cook-book" type of instructions as to how to obtain the numerical information which will describe his physical phenomena. There has already been an enormous amount of work reported upon by investigators who have found diffraction theory a fertile area from which to pluck a subject to develop into a paper for publication. However, in spite of the large amount of tables and graphs which are available, the investigator who seeks to obtain some results "...for an engineering study which must be completed by a week from Friday..." will usually find his situation quite hopeless. When he goes to the library to do his literature survey the chances are quite good that he will find scores of papers which are quite confusing because although the authors are treating essentially the same problems by essentially the same methods, they persist in using their own personal preferences for the notation for the various functions involved and hence a comparison of the results of the papers will be quite difficult. Then, after he has mastered the papers (long after the "due date" for the engineering study) he will almost certainly find that certain difficult aspects of the theory have been glossed over and that the filling in of these gaps requires considerable experience, mathematical ingenuity, and competence in numerical analysis and computer programming.

The author has worked at this subject area for long enough, and he is aware of his own personal limitations, and therefore his dreams of providing a "relatively simple" set of rules and programs for the practical engineer have become less ambitious with the passing of several years. However, the dream persists and the author hopes that his next reports will come closer to helping his engineering colleagues than does this present report.

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Section 1
THE DIFFRACTION FUNCTIONS

This research program is concerned with the theory and application of the Green's function $D(x, y, y_o, q)$ defined by

$$\text{Differential Equation} \quad \frac{\partial^2 D}{\partial y^2} + i\frac{\partial D}{\partial x} + yD = -\delta(x)\,\delta(y - y_o) \qquad \begin{matrix} 0 \le y < \infty \\ 0 < x < \infty \end{matrix} \tag{1.1}$$

$$\text{Boundary Condition} \quad \left[\frac{\partial D}{\partial y} + qD\right]_{y=o} = 0 \tag{1.2}$$

$$\text{Radiation Condition} \quad \lim_{y \to \infty}\left[\frac{1}{\sqrt[4]{y}}\left(\frac{\partial D}{\partial y} - i\sqrt{y}\,D\right)\right] = 0 \tag{1.3}$$

The solution of this problem involves the Airy functions

$$\left(\frac{d^2}{dt^2} - t\right) w_{1,2}(t) = 0 \tag{1.4}$$

$$w_{1,2}(t) = \sqrt{\pi}\left[Bi(t) \pm i\,Ai(t)\right] = \frac{1}{\sqrt{\pi}}\int_0^\infty \exp\left(-\frac{1}{3}x^3 + xt\right)dx$$

$$\pm \frac{i}{\sqrt{\pi}}\int_0^\infty \exp\left[-i\left(\frac{1}{3}x^3 + xt\right)\right]dx \tag{1.5}$$

$$v(t) = \frac{1}{2i}\left[w_1(t) - w_2(t)\right] = \sqrt{\pi}\,Ai(t) = \frac{1}{\sqrt{\pi}}\int_0^\infty \cos\left(\frac{1}{3}x^3 + xt\right)dx \tag{1.6}$$

It is possible to express this Green's function in the form of a Fourier integral or in the form of a "normal mode" expansion which involves roots t_s which are solutions of (1.8).

$$D(x, y, y_0, q) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ixt) \Big[w_1(t - y_>)\, v(t - y_<)$$

$$- \frac{v'(t) - q\,v(t)}{w'_1(t) - q w_1(t)}\, w_1(t - y_0)\, w_1(t - y) \Big] dt$$

$$= -i \sum_{s=1}^{\infty} \frac{\exp(ixt_s)\, w_1(t_s - y_0)\, w_1(t_s - y)}{\left[w'_1(t_s)\right]^2 - t_s \left[w_1(t_s)\right]^2} \tag{1.7}$$

where

$$w'_1(t_s) - q\, w_1(t_s) = 0 \tag{1.8}$$

In some applications, it is convenient to renormalize $D(x,y,y_0,q)$ in order to employ a function $V(x,y,y_0,q)$ which has the property $V(0,0,0,0) = 2$.

$$V(x, y, y_0, q) = 2\sqrt{-i\pi x}\; D(x, y, y_0, q) \tag{1.9}$$

$$\frac{\partial^2 V}{\partial y^2} + i\frac{\partial V}{\partial x} + \left(y - \frac{i}{2x}\right) V = 0 \tag{1.10}$$

$$\left(\frac{\partial V}{\partial y} + qV\right)_{y=0} = 0 \tag{1.11}$$

When $y_0 \to \infty$, it is convenient to express $D(x,y,y_0,q)$ in the form

$$D(x, y, y_0, q) \xrightarrow[y_0 \to \infty]{} \frac{\exp\left(i\,\frac{2}{3}\, y_0^{3/2} + i\,\frac{\pi}{4}\right)}{2\sqrt{\pi}\;\sqrt[4]{y_0}}\, E(x - \sqrt{y_0}\,, y, q) \tag{1.12}$$

$$E(\xi, y, q) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(i\xi t) \left[v(t - y) - \frac{v'(t) - qv(t)}{w'_1(t) - qw_1(t)}\; w_1(t - y) \right] dt \tag{1.13}$$

$$\frac{\partial^2 E}{\partial y^2} + i\,\frac{\partial E}{\partial \xi} + y E = 0 \quad , \quad \left(\frac{\partial E}{\partial y} + q E\right)_{y=0} = 0 \tag{1.14}$$

It is often convenient to decompose $E(\xi,y,q)$ into a "plane wave" field $E_o(\xi,y)$ and a reflected field $P(\xi,y,q)$.

$$E_o(\xi, y) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(i\xi t)\, v(t-y)\, dt$$

$$= \exp\left(i\xi y - i\frac{\xi^3}{3}\right) \tag{1.15}$$

$$P(\xi, y, q) = -\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(i\xi t)\, \frac{v'(t) - q v(t)}{w_1'(t) - q w_1(t)}\, w_1(t-y)\, dt \tag{1.16}$$

1.1 Applications

The function $V(x,y,y_o,q)$ is well known in the theory of the propagation of the ground wave around the earth's surface under the conditions of "normal refraction" (i.e., in the absence of ducts and atmospheric inhomogeneities). As an example, we cite the problem of finding the Hertz potential for a vertically-directed electric dipole located at $z = r_o$ on the polar axis in the region exterior to a sphere of radius $\underline{a}$. Let the Hertz potential in free space be described by

$$U_o = \exp(ikR)/(4\pi R) \quad , \quad r = \sqrt{\rho^2 + (z - r_o)^2} \tag{1.17}$$

The field in the presence of the sphere must satisfy the equations

$$\left(\nabla^2 + k^2\right) U = -\frac{\delta(\Theta)\, \delta\left(r - r_o\right)}{2\pi r_o^2 \sin\Theta} \tag{1.18}$$

$$\frac{\partial U}{\partial r} + k\left[\left(\frac{2}{ka}\right)^{1/3} q\right] U = 0 \quad , \quad r = a \tag{1.19}$$

The solution is generally given in the form

$$U \sim \frac{e^{iks}}{4\pi s}\sqrt{\frac{\theta}{\sin\theta}}\; V(x, y, y_o, q) \tag{1.20}$$

$$x = \left(\frac{k}{2a^2}\right)^{1/3} s, \quad y = \left(\frac{2k^2}{a}\right)^{1/3} h, \quad y_o = \left(\frac{2k^2}{a}\right)^{1/3} h_o \tag{1.21}$$

$$s = a\theta \quad , \qquad h = r - a \quad , \quad h_o = r_o - a \tag{1.22}$$

The attenuation function $V(x,y,y_o,q)$ is often written in a form which employs a so-called "height-gain" function $f_s(y)$

$$V(x, y, y_o, q) = 2\sqrt{i\pi x}\sum_{s=1}^{\infty} \frac{\exp(ixt_s)}{t_s - q^2}\, f_s(y)\; f_s(y_o) \tag{1.23}$$

$$f_s(y) = \frac{w_1(t_s - y)}{w_1(t_s)} = 1 - qy + \frac{1}{2} t_s y^2 + \cdots \tag{1.24}$$

$$f_s(y) \xrightarrow[y \gg |t_s|]{} \frac{\exp\left[i\left(\frac{2}{3}y^{3/2} - \sqrt{y}\, t_s - \frac{\pi}{4}\right)\right]}{\sqrt[4]{y}\; w_1(t_s)} \left\{1 + i\,\frac{t^2}{4\sqrt{y}} + \cdots\right\} \tag{1.25}$$

In the pioneering work of van der Pol and Bremmer (Ref. 1) in the late 1930's, the attenuation function $V(x,y,y_o,q)$ was quite adequate because of the low antenna heights and the relatively long radio wave lengths which were of interest . However, after the development of radar in the early 1940's it became very important to treat the case of very short wavelengths and very great heights. The calculations which were made by van der Pol and Bremmer from regions deep in the shadow to just within the lighted region had been made on the basis of (1.7). Fig. 1-1 is taken from one of the classic papers by these authors and shows how by using an increasing number of terms that the variation of the field can be followed from its region of deep shadow below the horizon to points within the lighted region where the dashed lines in Fig. 1-1 show the envelop as predicted by optical

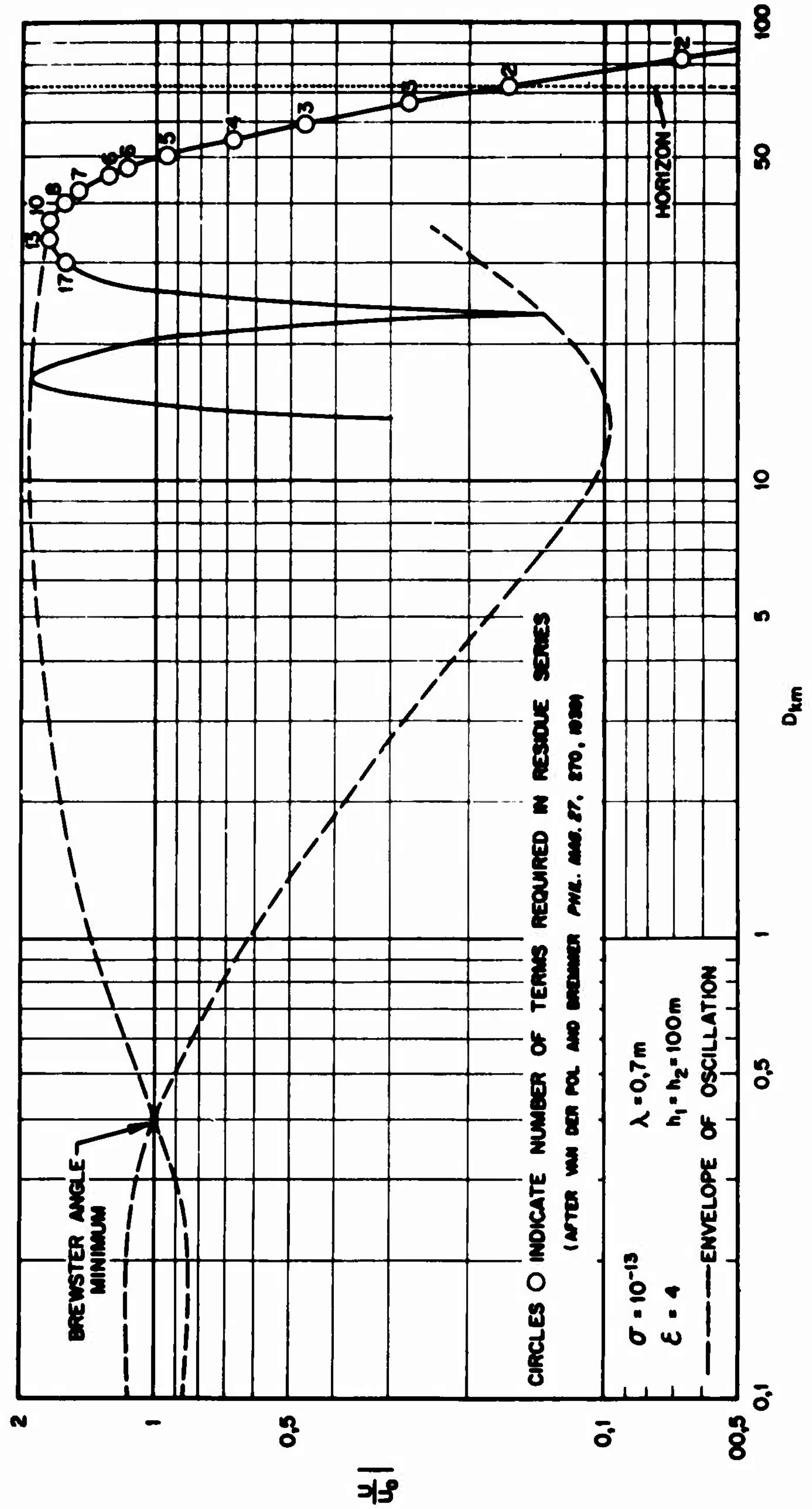

Fig. 1-1 Variation of Field Strength Near the Horizon

theory. When the asymptotic form for the height-gain factor given by (1.25) was introduced into (1.23), the residue series, or "normal mode" expansion, failed to converge except for points below the optical horizon. Since this situation corresponded to the very practical problem of predicting the high altitude coverage of ground-based radars, the problem was very vigorously attacked by scientists in the United States, England, and in Russia. In 1941, Burrows and Gray (Ref. 2) (in a notation which is quite different from that which we are using) expressed the results of the van der Pol-Bremmer theory for this case in the form

$$U \sim \frac{\exp(i\,ka\,\theta)}{4\pi a\Theta}\sqrt{\frac{\Theta}{\sin\Theta}}\exp\left(i\,\frac{2}{3}\,y^{3/2}\right)V_1(x-\sqrt{y}\,,\,q) \tag{1.26}$$

$$x = \left(\frac{ka}{2}\right)^{1/3}\theta\,,\; y = \left(\frac{2}{ka}\right)^{1/3}k(r-a)\,,\, q = i\left(\frac{ka}{2}\right)^{1/3}\frac{k}{k_2}\sqrt{1-\frac{k^2}{k_2^2}} \tag{1.27}$$

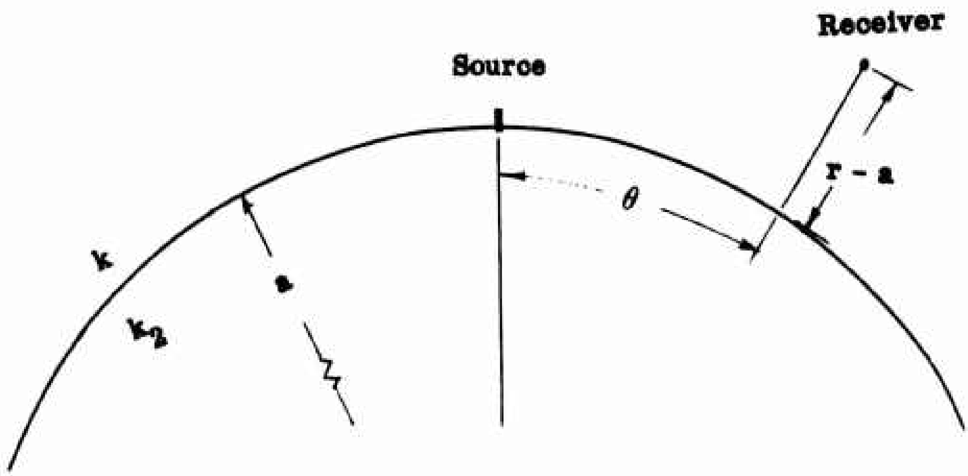

Fig. 1-2 Source on the Surface of a Spherical Interface

In 1945, Fock (Ref. 3) published a similar result (in the same notation we have used) and gave a table of $V_1(\xi,0)$ for $\xi = 4.5$ $(0.1)4.5$. It was long overlooked that Burrows and Gray (Ref. 2) gave a curve for $V_1(\xi,0)$ and $V_1(\xi,\infty)$ for $\xi>0$ in their original paper. The quantity plotted in their graph was defined by

$$F_L(q) = \left| \frac{\left(t_1 - q^2\right) w_1(t_1)}{\sqrt[3]{4}\; w'_1\left(t_1^\infty\right)} \, V_1\left(\frac{\left|t_1^\infty\right|}{\left|t_1\right|} \frac{L}{\sqrt[3]{2}}, q\right) \right| \tag{1.28}$$

The curves given are for $q = 0$ and $q = \infty$, with $L \geq 0$.

These results of Burrows-Gray and Fock are only valid for natural units of height y of the order of two to five. For larger heights, it is desirable to use the reciprocity principle and use the fields on the surface which are induced by a plane wave. This problem was studied by Fock (Ref. 3). Fock described his results as "The Principle of the Local Field in the Penumbra Region." According to this principle: "The transition from light to shadow on the surface of the body takes place in a narrow strip along the boundary of the geometrical shadow. The width of this strip is of the order of $(\lambda a^2/\pi)^{\frac{1}{3}}$, where a is the radius of curvature of the normal section of the body in the plane of incidence. If the geometry is as illustrated in Fig. 1-3, then the principle

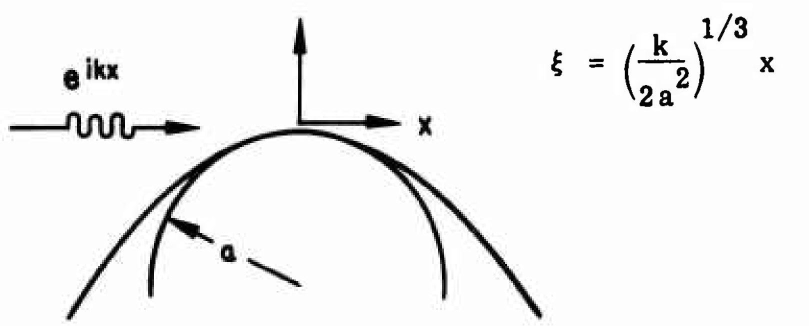

Fig. 1-3 Plane Wave Incident Upon a Convex Surface

is expressed in the following formulae.

$$H_y = H_y^o \,[\exp(ikx + \frac{\xi^3}{3})]\, V_1(\xi,q) \qquad (1.29)$$

$$H_x = i(ka/2)^{-\frac{1}{3}}\, H_z^o \,[\exp(ikx + \frac{\xi^3}{3})]\, f(\xi) \qquad (1.30)$$

The current distribution functions $V_1(\xi,q)$ and $f(\xi)$ are defined as limiting forms of Eq. (1-23) which results from a use of Eq. (1-25)

$$V_1(\xi, q) = \lim_{y_o \to \infty} \left\{ \frac{\sqrt[4]{y_o}}{\sqrt{\xi + \sqrt{y_o}}} \exp\left(- i \frac{2}{3} y_o^{3/2}\right) V(\xi + \sqrt{y_o}, 0, y_o, q) \right\}$$

$$= i 2\sqrt{\pi} \sum_{s=1}^{\infty} \frac{\exp(i\xi t_s)}{(t_s - q^2) w_1(t_s)}$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(i\xi t) \frac{1}{w_1'(t) - q w_1(t)} dt \qquad (1.31)$$

and

$$f(\xi) = \left\{ \lim_{q\to\infty} - q V_1(\xi,q) \right\} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\exp(i\xi t)}{w_1(t)} dt \qquad (1.32)$$

For $x < 0$ these formulae transform into the equations of geometric optics. This property can be readily verified by noting that for ξ tending to large negative values that the special functions have the behavior

$$V_1(\xi,q) \to \frac{2 \exp(-i\xi^3/3)}{1 + i(q/\xi)}$$

$$f(\xi) \to i2\xi \exp(-i\xi^3/3)$$

For the special case $\xi = 0$ (i.e., the horizon) we know that $V_1(0,0) = 1.399$ and $f(0) = 0.766 \exp(-i\pi/3)$. The behavior for the for other values of q is depicted in Fig. 1-4.

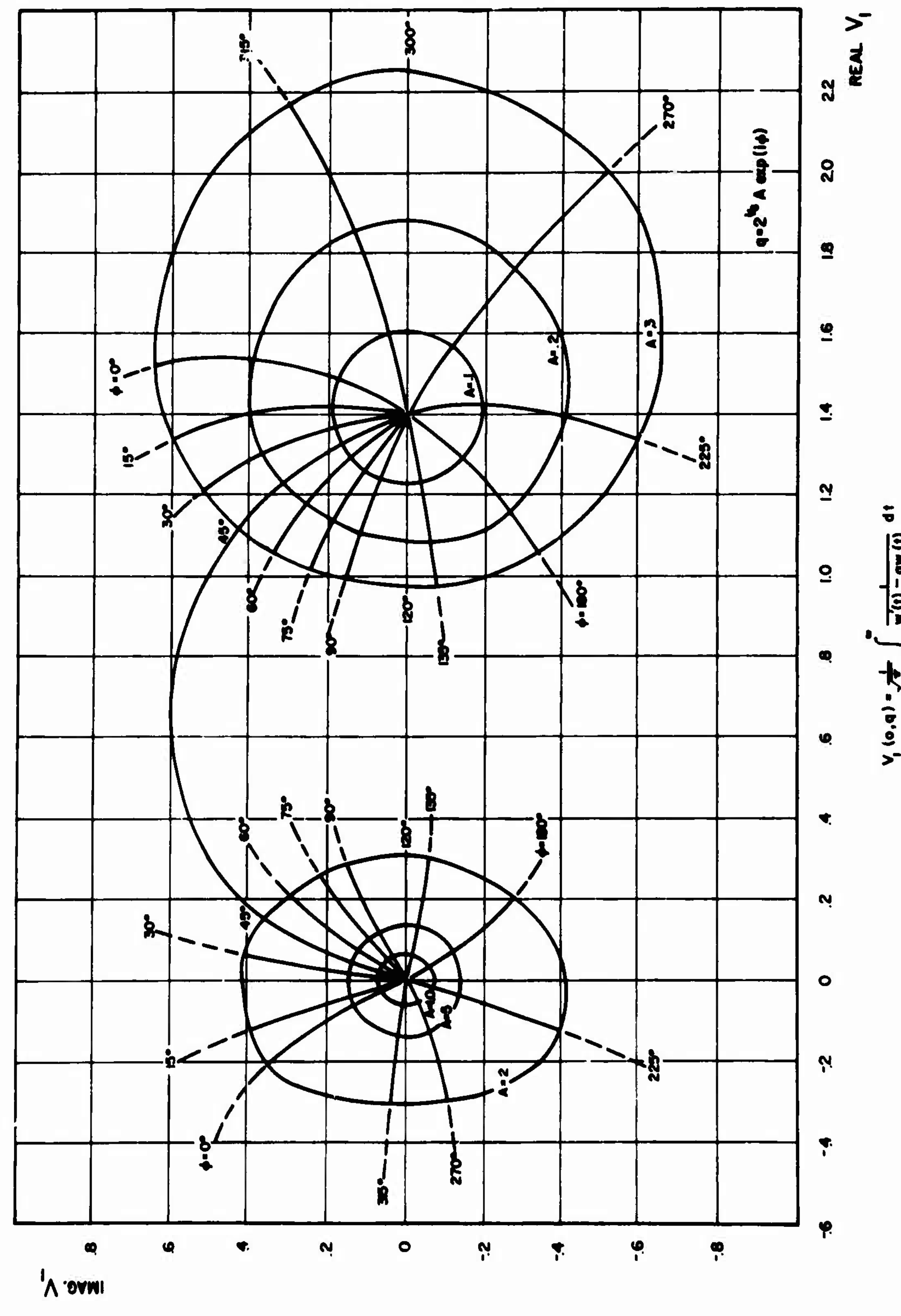

Fig. 1-4 Behavior of $V_1(0,q)$

The principle which was enunciated by Fock in 1945, was extended to the case when both the source and the receiver were above the surface by the papers of Pekeris (Ref. 4) in 1947 and Fock (1951). However, let us first review the manner in which the elevated source and receiver were handled in the work of van der Pol and Bremmer (Ref. 1).

The starting point for the discussion will be the exact solution for the Hertz potential for a vertical electric dipole. The problem can be discribed as follows:

Differential Equation

$$\nabla^2 U + k^2 U = -\frac{\delta(r - r_o)\,\delta(\Theta)}{2\pi r^2 \sin\Theta} \tag{1.33}$$

$$k^2 = \omega^2 \epsilon_o \mu_o \text{ for } r>a \;; \qquad k_1^2 = \omega^2 \epsilon_1 \mu_o + i\,\omega\,\mu_o \sigma_1 \text{ for } r<a \tag{1.34}$$

Boundary Condition

$$\frac{\partial}{\partial r}(r\,U) \text{ and } k^2 U \text{ continuous across } r = a \tag{1.35}$$

Periodic Condition

$$U(\Theta \pm 2m\pi) = U(\Theta) \tag{1.36}$$

Radiation Condition

$$\lim_{r \to \infty} \left\{ r \left[\frac{\partial U}{\partial r} - i\,k\,U \right] \right\} = 0 \tag{1.37}$$

The classical methods for the solution of a problem of this type involves the expansion of the field in a series of spherical harmonics. The resulting representation for $U(\theta, r, r_o, \Gamma)$ involves a complex combination of spherical Hankel functions. If we let $r_>$ and $r_<$ represent, respectively, the larger and the smaller of the two radial distances r and r_o, then the exact solutions take the form of the following series:

$$U(\Theta, r, r_o, \Gamma) = \frac{ik}{8\pi} \frac{1}{(kr)(kr_o)} \sum_{n=0}^{\infty} (2n+1) \left| \zeta_n^{(1)}(kr_>) \zeta_n^{(2)}(kr_<) - \frac{\zeta_n^{(2)'}(ka) + \Gamma \zeta_n^{(2)}(ka)}{\zeta_n^{(1)'}(ka) + \Gamma \zeta_n^{(1)}(ka)} \zeta_n^{(1)}(kr) \zeta_n^{(1)}(kr_o) \right| P_n(\cos\Theta) \tag{1.38}$$

$$\zeta_n^{(1,2)}(x) = \sqrt{\frac{\pi x}{2}} H_{n+1/2}^{(1,2)}(x) \qquad \psi_n(x) = \sqrt{\frac{\pi x}{2}} J_{n+1/2}(x) \tag{1.39}$$

$$\Gamma = -\frac{k}{k_1} \frac{\psi'_n(k_1 a)}{\psi_n(k_1 a)} \sim i \frac{k}{k_1} \sqrt{1 - \left(\frac{k}{k_1}\right)^2} \tag{1.40}$$

The classical procedure involves the transformation of the sum into an integral. From the integral, one extracts an integral which can be evaluated by the method of stationary phase to yield the reflected wave. Before we consider the form taken by the result, we must make some geometrical definitions by introducing Fig. 1-5.

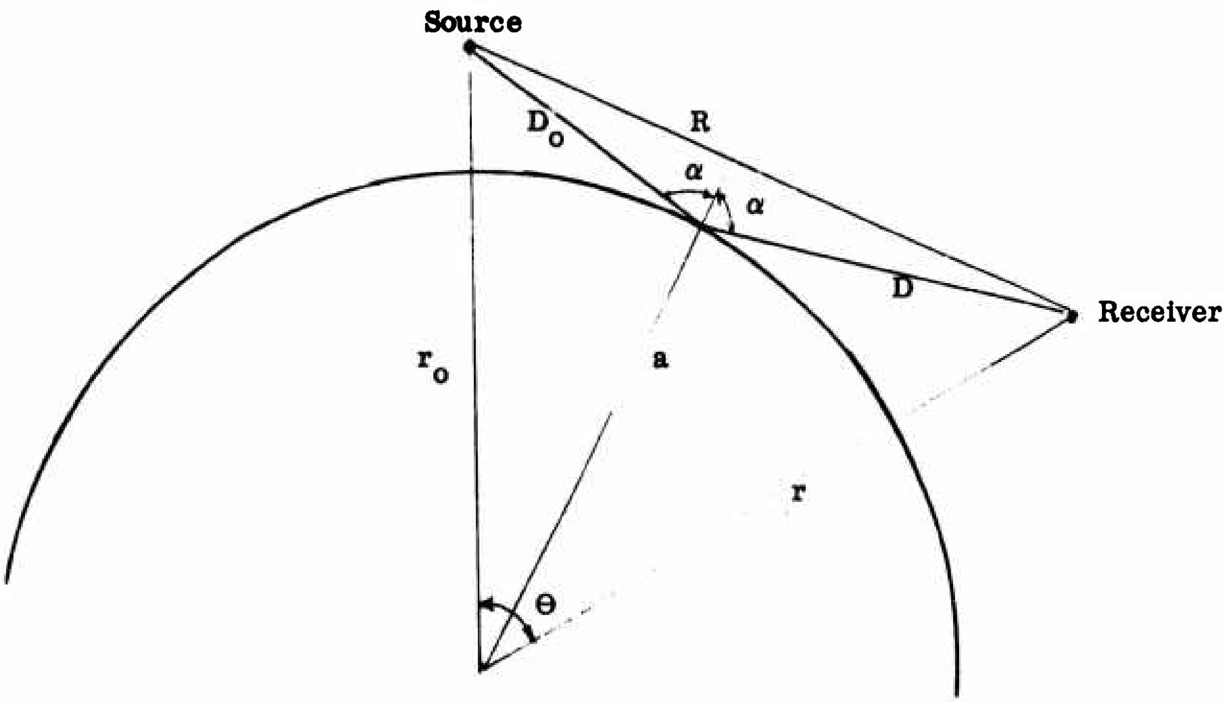

Fig. 1-5 Reflection from a Convex Surface

$$\mho(\Theta, r, r_o, \Gamma) \sim \frac{\exp(ikR)}{4\pi R} - \frac{ik}{4\pi}\frac{1}{(kr)(kr_o)}\int_{-\infty+io}^{\infty+io} (n+1/2)\frac{\zeta_n^{(2)'}(ka) + \Gamma\zeta_n^{(2)}(ka)}{\zeta_n^{(1)'}(ka) + \Gamma\zeta_n^{(1)}(ka)}$$

$$\zeta_n^{(1)}(kr)\,\zeta_n^{(1)}(kr_o)\left[P_n(\cos\Theta) - i\frac{2}{\pi}Q_n(\cos\Theta)\right] dn$$

$$\sim \frac{\exp(ikR)}{4\pi R} + \frac{1}{4\pi}\left\{\sqrt{\frac{a\sin\alpha}{rr_o\sin\Theta}}\;\frac{\cos\alpha + i\Gamma}{\cos\alpha - i\Gamma}\right.$$

$$\left.\sqrt{\frac{a\cos\alpha}{2R_oR_1 + a(R_o+R_1)\cos\alpha}}\right\}\exp\left[ik(R_o + R_1)\right] \qquad (1\text{-}41)$$

Van der Pol and Bremmer showed that this formula agrees with the approximation based upon the attenuation function Eq. (1-20) provided the comparison is made in the vicinity of the first and second lobe just above the horizon. However, for points well above the horizon, the attentuation function fails to agree with the stationary phase (geometrical optics) result.

It would be interesting to seek a formula which agrees with the classic result involving $V(x,y,y_o,q)$ for points near the horizon, but which is in better agreement with the stationary phase result for points well above the horizon. Such a formula has been found in the course of the studies made by the present author. It involves the reflected field $R(x,y,y_o,q)$ which can be defined by separating the Green's function $D(x,y,y_o,q)$ into a free space field $D_o(x,y,y_o)$ and a secondary field which for points above the horizon can be interpreted as a reflected field. The decomposition is made as follows:

$$D(x,y,y_o,q) = D_o(x,y,y_o) + R(x,y,y_o,q) \qquad (1\text{-}41)$$

$$D_0(x,y,y_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(ixt)\, w_1(t-y)\, v(t-y_0)\, dt$$

$$= \frac{\exp\left(i\frac{\pi}{4}\right)}{2\sqrt{\pi x}} \exp\left[-i\frac{x^3}{12} + i\frac{x}{2}(y+y_0) + i\frac{(y-y_0)^2}{4x}\right] \qquad (1\text{-}42)$$

$$R(x,y,y_0,q) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(ixt)\, \frac{v'(t) - q\,v(t)}{w_1'(t) - qw_1(t)}\, w_1(t-y)\, w_1(t-y_0)\, dt \qquad (1\text{-}43)$$

The method of stationary phase can be used to show that

$$R(\zeta+\zeta_0,\ \zeta^2 + 2\mu\zeta,\ \zeta_0^2 + 2\mu\zeta_0,\ q) \xrightarrow[\mu\gg 1]{}$$

$$-\frac{\exp(i\pi/4)}{2\sqrt{\pi}}\, \frac{q - i\mu}{q + i\mu} \sqrt{\frac{\mu}{2\zeta\zeta_0 + \mu(\zeta+\zeta_0)}}\, \exp(i\omega^*) \qquad (1\text{-}44)$$

where

$$\omega^* = \frac{2}{3}(\zeta^3 + \zeta_0^3) + 2\mu(\zeta^2 + \zeta_0^2) + \mu^2(\zeta + \zeta_0) \qquad (1\text{-}45)$$

It will often be more convenient to absorb the phase factor in Eq. (1-45) into the definition of the reflection function and express the results in terms of the modified reflection function $S(\mu,\zeta,\zeta_0,q)$

$$S(\mu,\zeta,\zeta_0,q) = \exp(-i\omega^*)\ R(\zeta+\zeta_0,\ \zeta^2 + 2\mu\zeta,\ \zeta_0^2 + 2\mu\zeta_0,\ q)$$

$$\xrightarrow[\mu\to\infty]{} -\frac{1}{2}\sqrt{\frac{1}{\pi}}\ \frac{q - i\mu}{q + i\mu} \sqrt{\frac{\mu}{2\zeta\zeta_0 + \mu(\zeta+\zeta_0)}} \qquad (1\text{-}46)$$

The new formula for the region of direct visibility then takes the form

$$U(\Theta, r, r_0, \Gamma) = \frac{\exp(ikR)}{4\pi R}$$

$$+\sqrt{\frac{-ik}{8\pi}}\left(\frac{2}{ka}\right)^{1/3} \sqrt{\frac{a \sin\alpha}{rr_0 \sin\Theta}}\, \exp\left|ik(D_0 + D)\right|\ S(\mu,\zeta,\zeta_0,q) \qquad (1\text{-}47)$$

where

$$\zeta = \left(\frac{ka}{2}\right)^{1/3} \frac{D}{a} \qquad \mu = \left(\frac{ka}{2}\right)^{1/3} \cos\alpha$$

$$\zeta_o = \left(\frac{ka}{2}\right)^{1/3} \frac{D_o}{a} \qquad q = \left(\frac{ka}{2}\right)^{1/3} \Gamma \qquad (1\text{-}48)$$

For points well above the horizon, $\mu \to \infty$ as $ka \to \infty$ since $\cos\alpha > 0$. Therefore, we observe that with the above definitions of ζ_o, ζ, and μ our new formulation agress with the classical results obtained from a stationary phase evaluation of the exact solution. The new formulation has the advantage, however, of leading to results which are useful up to the horizon. In fact, it can even be used in the region just beyond the horizon.

The asymptotic forms that appear in Eq. (1-44) and Eq. (1-46) are only valid provided the parameter μ is large. From Eq. (1-48) we see that since α tends to 90° as one approaches the horizon that the value of μ may become small (and even vanish) in this region regardless of how large the radius of curvature may be in comparison with the wavelength (i.e., no matter how large $ka = 2\pi a/\lambda$ may be). Therefore, if we continue to require that both ζ and ζ_o be very large, but leave μ unrestricted, we will find that the asymptotic properties of the field will have to be described in terms of a function $V_{11}(\xi,q)$ which we will refer to as the reflection coefficient function. If we use the leading term in the asymptotic expansion described by Eq. (1-25) we find that

$$V_{11}(\xi,q) = \lim_{\substack{y_o \to \infty \\ y \to \infty}} \left\{ \frac{\sqrt[4]{y_o y}}{\sqrt{\xi + \sqrt{y} + \sqrt{y_o}}} \exp\left[-i\,\frac{2}{3}\left(y^{3/2} + y_o^{\,3/2}\right)\right] V\left(\xi + \sqrt{y} + \sqrt{y_o},\, y,\, y_o,\, q\right) \right\} \qquad (1\text{-}49)$$

We can represent $V_{11}(\xi,q)$ by means of a residue series (involving the roots t_s defined by Eq. (1-8)) or by means of a Fourier integral.

$$V_{11}(\xi,q) = \exp\left(i\,\frac{3\pi}{4}\right) 2\sqrt{\pi}\sum_{s=1}^{\infty}\frac{\exp(i\xi t_s)}{\left(t_s - q^2\right)\left[w_1(t_s)\right]^2}$$

$$= -\frac{\exp(i\pi/4)}{\sqrt{\pi}}\int_{-\infty}^{\infty}\exp(i\xi t)\,\frac{v'(t) - q\,v(t)}{w_1'(t) - q\,w_1(t)}\,dt \qquad (1\text{-}50)$$

However, $V_{11}(\xi,q)$ is singular when $\xi \to 0$ and hence it is convenient to define a closely related function $V_2(\xi,q)$ which is an entire function of ξ.

$$V_2(\xi,q) = V_{11}(\xi,q) - (2\sqrt{\pi}\,\xi)^{-1}\exp(i\pi/4) \qquad (1\text{-}51)$$

If we use the properties which we have just cited, we can show that the function $S(\mu,\zeta,\zeta_o,q)$ defined by Eq. (1-46) has the property

$$S(\mu,\zeta,\zeta_o,q) \xrightarrow[\substack{\zeta>1\\ \zeta_o>1\\ \mu>0}]{} \sqrt{\frac{i}{2\pi}}\,\frac{\exp\left(-i\,\frac{2}{3}\,\mu^3\right)}{\sqrt{2\zeta\zeta_o + \mu(\zeta+\zeta_o)}}\,V_{11}(-2\mu,q) \qquad (1\text{-}52)$$

If we reconsider the function $R(x,y,y_o,q)$, we see that these properties lead to

$$R\left(\zeta+\zeta_o,\ \zeta^2 + 2\mu\zeta,\ \zeta_o^2 + 2\mu\zeta_o,\ q\right) \xrightarrow[\zeta,\ \zeta_o \gg 1]{}$$

$$\frac{\exp\left(i\,\frac{\pi}{4}\right)}{\sqrt{2\pi}\sqrt{2\zeta\zeta_o + \mu(\zeta+\zeta_o)}}\,V_{11}(-2\mu,q)\exp\left(i\omega^* - i\,\frac{2}{3}\mu^3\right) \qquad (1\text{-}53)$$

For points well above the horizon (i.e., for large positive values of μ, the reflection coefficient function $V_{11}(-2\mu,q)$ has the asymptotic behavior

$$V_{11}(-2\mu,\, q) \xrightarrow[\mu \gg 1]{} -\frac{q - i\mu}{q + i\mu} \sqrt{\frac{\mu}{2}}\, \exp\left(i\,\frac{2}{3}\,\mu^3\right) \tag{1-54}$$

$$V_{11}(-2\mu,\, q) \xrightarrow[\substack{\mu \gg 1 \\ q \to \infty}]{} -\sqrt{\frac{\mu}{2}}\, \exp\left[i\left(\frac{2}{3}\,\mu^3 - \frac{2\mu}{q}\right)\right] \tag{1-55}$$

When the asymptotic estimates for $V_{11}(-2\mu,q)$ are used in Eq. (1-52) and Eq. (1-47), we are led back to the result which was given in Eq. (1-41). However, Eq. (1-41) was valid only for points well above the horizon and we have now laid the groundwork for an asymptotic estimate which agrees with the predictions of Eq. (1-41) when the parameter μ is very large and positive but which has the advantage of holding for points in the vicinity of the horizon. Before we proceed to discuss the behavior on the horizon itself, let us first obtain a result which is valid in the vicinity of the horizon and which will contain the familiar Fresnel integral which we should expect to encounter in this region on the basis of the considerations of physical optics. For our purposes it is convenient to introduce the modified Fresnel integral $K(\tau)$ which was employed by Fock (Ref. 5).

$$\begin{aligned} K(\tau) &= \exp\left(-i\,\tau^2 - i\,\frac{\pi}{4}\right)\frac{1}{\sqrt{\pi}} \int_{\tau}^{\infty} \exp(i\,s^2)\,ds \\ &\xrightarrow[\tau \to \infty]{} \frac{\exp\left(i\,\frac{\pi}{4}\right)}{2\sqrt{\pi}\,\tau} \end{aligned} \tag{1-56}$$

From the property described by Eq. (1-51), we observe that

$$\mu\, K(\mu\xi) + V_2(\xi,\, q) \xrightarrow[\substack{\mu \to \infty \\ \xi > 0}]{} V_{11}(\xi,\, q) \tag{1-57}$$

We will define the "horizon" to be the locus of points for which

$x = (\sqrt{y} + \sqrt{y_o})$ and we will introduce the parameter

$$\xi = x - \sqrt{y} - \sqrt{y_o} \tag{1-58}$$

which will be positive below the horizon, zero on the horizon, and negative above the horizon. We also define the parameters μ and τ

$$\mu = \sqrt{\frac{\sqrt{y}\,\sqrt{y_o}}{\sqrt{y} + \sqrt{y_o}}} \quad , \quad \tau = \mu\xi \tag{1-59}$$

Let $F(x,y,y_o)$ be the Fresnel diffraction field

$$F(x,y,y_o) = \sqrt[4]{\frac{x^2}{y\,y_o}}\, \exp\left[i\,\frac{2}{3}\left(y^{3/2} + y_o^{3/2}\right)\right] \mu\, K(\tau), \qquad \xi > 0 \quad \text{(1-60a)}$$

$$= \sqrt[4]{\frac{x^2}{y\,y_o}}\, \exp\left[i\,\frac{2}{3}\left(y^{3/2} + y_o^{3/2}\right)\right]\left[-\mu\, K(-\tau)\right]$$

$$+ \exp\left[i\left(-\frac{x^3}{12} + \frac{x}{2}\,(y + y_o) + \frac{(y - y_o)^2}{4x}\right)\right] \qquad \xi < 0 \quad \text{(1-60b)}$$

We can then show that

$$V(x, y_o y, q) \xrightarrow[\substack{y \to \infty \\ y_o \to \infty}]{} F(x,y,y_o) + \sqrt[4]{\frac{x^2}{y\,y_o}}\, \exp\left[i\,\frac{2}{3}\left(y^{3/2} + y_o^{3/2}\right)\right] V_2(\xi, q) \tag{1-61}$$

If we let μ tend to infinity we obtain from (1-61) the result

$$V(x,y,y_o,q) - \frac{\sqrt{x}}{\sqrt{\sqrt{y}\,\sqrt{y_o}}}\, V_{11}(\xi,q) \xrightarrow[\substack{y_o \to \infty \\ y \to \infty \\ \mu \to \infty}]{} G(x,y,y_o) \tag{1-61}$$

where $G(x,y,y_o) = 0$ when $\xi > 0$ and

$$G(x,y,y_o) = \exp[i(-\frac{x^3}{12} + \frac{x}{2}(y + y_o) + \frac{(y - y_o)^2}{4x})]$$

when $\xi < 0$. The result in Eq. (1-61) reveals that the function $V_{11}(\xi,q)$ describes the reflected field in the lighted region (i.e., $\xi < 0$) and the same function describes the diffracted field in the shadow region (i.e., $\xi > 0$). In the vicinity of the horizon we need to employ the function $V_2(\xi,q)$. On the horizon (i.e., $\xi = 0$) we find that

$$V(\sqrt{y} + \sqrt{y_o}, y, y_o, q) \xrightarrow[\substack{y \to \infty \\ y_o \to \infty}]{} [\frac{1}{2} + \frac{1}{\mu} V_2(0,q)]\exp(i\Lambda) \tag{1-62}$$

where

$$\Lambda = \frac{2}{3}(y^{3/2} + y_o^{3/2})$$

In Fig. 1-6 we present an illustration which shows the behavior*

*Fig. 1-6 was previously presented by the author in Ref.[6]. Data has been generated and a much more detailed plot has been begun. However, the author became "side-tracked" on this task when it was found that as $q \to Q$, where Q is real and positive, that the data revealed an irregular behavior. The author became intrigued with this phenomena and investigated the behavior of the analogue of $V_2(0,q)$ in the problem of diffraction by a circular cylinder. Computations made with the exact solution also showed similar results to those found with the data for $V_2(0,q)$. The author strongly suspects that the erratic behavior is caused by errors introduced into the numerical work because of the fact that the expression $w_1'(t) - qw_1(t)$ which appears in the denominator of the integral defining $V_2(\xi,q)$ vanishes for a point close to the contour of integration. The author hopes to be able to devote time in the immediate future to the task of completing this very detailed contour plot of $V_2(0,q)$.

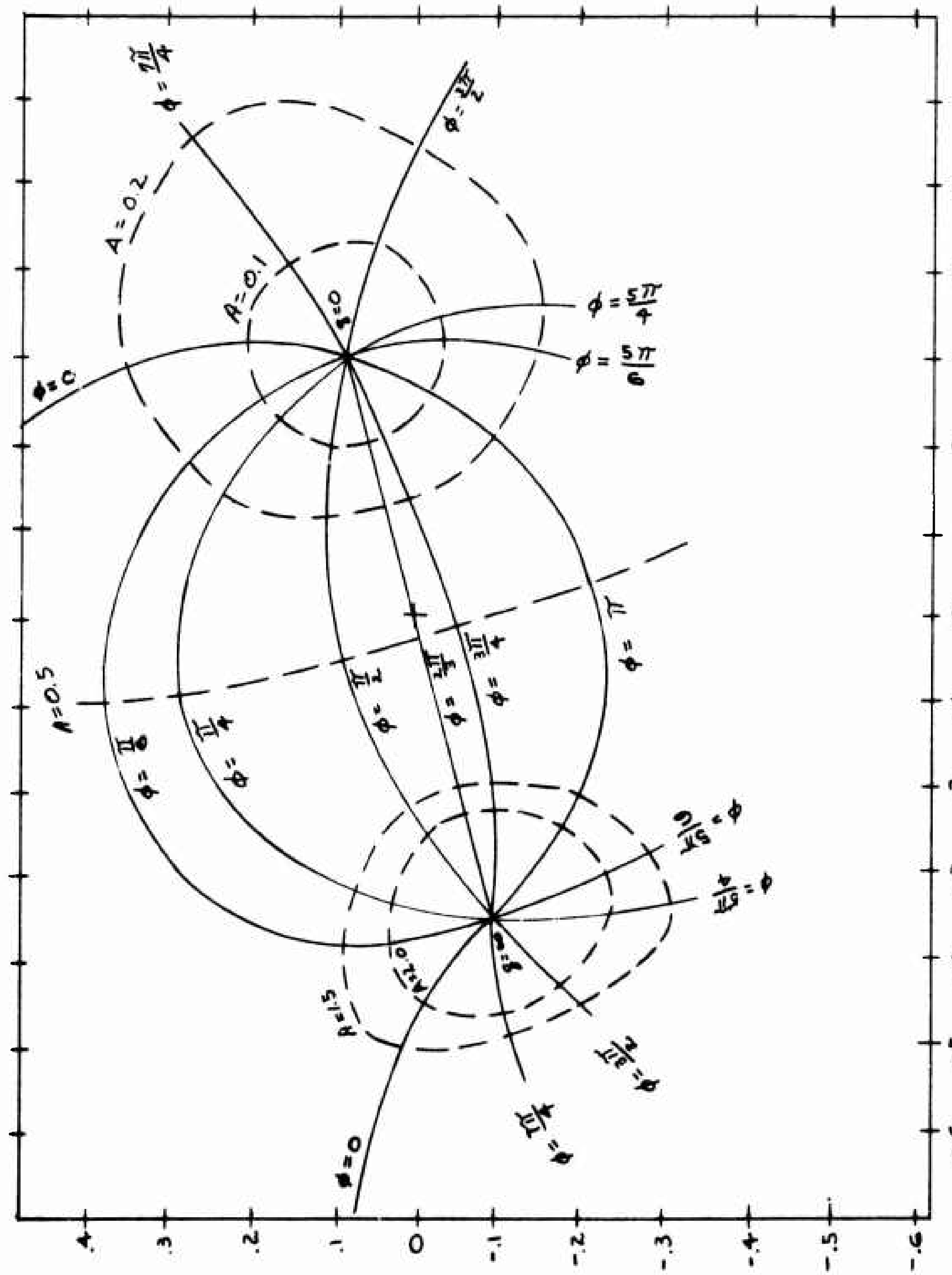

Fig. 1-6 Behavior of $V_2(0,q) = \lim_{\xi\to 0}\left\{-\frac{\exp(i\,\pi/4)}{\sqrt{\pi}}\int_{-\infty}^{\infty}\exp(i\xi t)\frac{v'(t)-qv(t)}{w_1'(t)-qw_1(t)}dt - \frac{\exp(i\,\pi/4)}{2\sqrt{\pi}\,\xi}\right\}$

of $V_2(0,q)$ as a function of q. The locus for which $\varphi = \arg(q) = 45^\circ$ was previously presented in a paper by Wait and Conda (Ref. 7). The reader will observe that the form taken by q in Fig. 1-6 has been taken to be

$$q = 2^{\frac{1}{3}}A \exp(i\varphi) \tag{1-63}$$

This form was chosen in order that the curves might supplement the curve presented by Wait and Conda who also used A in the manner displayed in Eq. (1-63).

If we apply the result in Eq. (1.61) to the function $S(\mu,\zeta,\zeta_o,q)$ defined in Eq. (1-46) we find that

$$S(\mu,\zeta,\zeta_o,q) \underset{\substack{\zeta\to\infty \\ \zeta_o\to\infty \\ \mu\to 0}}{\longrightarrow} \frac{1}{2}\sqrt{\frac{i}{\pi}}\ \frac{1}{\sqrt{\zeta\zeta_o}} \exp\left(-i\,\frac{2}{3}\,\mu^3\right)\left[-\eta\, K(-\tau) + V_2(-2\mu,\, q)\right] \tag{1-64}$$

where

$$\tau = -2\mu\eta \quad , \quad \eta = \sqrt{\frac{\zeta\zeta_o}{\zeta+\zeta_o}} \tag{1-65}$$

If Eq. (1-64) is used in Eq. (1-47) we obtain a useful extension of the classical reflection formulae. On the horizon, $\mu = 0$, and the theory reveals that we have the approximation

$$U(\theta,r,r_o,\Gamma) \approx \frac{\exp(ikR)}{4\pi R} + M(\theta,r,r_o)\ V_2(0,q) \tag{1-66a}$$

where

$$M(\theta,r,r_o) = \frac{1}{2\pi}\left(\frac{ka}{2}\right)^{1/3} \frac{1}{\sqrt{2kDD_o}} \sqrt{\frac{a}{rr_o \sin\theta}} \exp\left[ik(D+D_o)\right] \tag{1-66b}$$

If we let r and r_o tend to infinity, we obtain the result

$$U(\theta,r,r_o,\Gamma) \underset{\substack{r_o\to\infty \\ r\to\infty}}{\longrightarrow} \left\{\frac{1}{2} + 2\left(\frac{ka}{2}\right)^{1/3}\sqrt{\frac{D+D_o}{2kDD_o}}\ V_2(0,\, q)\right\} \frac{\exp(ikR)}{4\pi R} \tag{1-67}$$

The factor "1/2" which appears as the first term inside the braces of Eq. (1-67) is the familiar Fresnel field factor for the field on the shadow boundary. The effects of the curvature of the surface which is tangent to the line joining the source and the receiver is contained in the term which contains the function $V_2(0,q)$.

The function $U(\theta,r,r_o,\Gamma)$ arose in the problem of an axially symmetric source above a spherical surface. We can show that very similar results can be obtained when the diffracting surface is a circular cylinder. For this case, we can again use Fiq. 1-5 and define μ, ζ,ζ_o, and q as in the case of the sphere. Let us denote the Green's function for the cylinder problem by $C(\theta,r,r_o,\Gamma)$. The optical results for this function are of the form

$$C(\theta,r,r_o,\Gamma) \xrightarrow[\mu\to\infty]{} \frac{\exp[i(kR+\pi/4)]}{\sqrt{8\pi kR}} + \left\{\sqrt{\frac{i}{8\pi k}}\ \frac{\cos\alpha + i\Gamma}{\cos\alpha - i\Gamma} \sqrt{\frac{a\cos\alpha}{2DD_o + a(D+D_o)\cos\alpha}}\ \exp\left[ik(D+D_o)\right]\right\} \quad (1\text{-}68)$$

For points near the horizon, we can use use the same arguments which led to Eq. (1-67) to show that

$$C(\theta,r,r_o,\Gamma) \xrightarrow[\substack{\mu\to o\\ r\to\infty\\ r_o\to\infty}]{} \left\{\frac{1}{2} + 2\left(\frac{ka}{2}\right)^{1/3}\sqrt{\frac{D+D_o}{2kDD_o}}\ V_2(0,q)\right\}\frac{\exp[i(kR+\pi/4)]}{\sqrt{8\pi kR}} \quad (1\text{-}69)$$

The case $r_o \to \infty$ corresponds to the problem of reflection of a plane wave from a circular cylinder. This problem follows from the results given in Eqs. (1-68) and (1-69) is we use the approximation

$$\frac{\exp(ikR)}{\sqrt{8\pi kR}} \sim \frac{\exp(ikr_o)}{\sqrt{8\pi kr_o}}\ \exp(-ikr\cos\theta) \quad (1\text{-}70)$$

and factor out the term $\exp(ikr_o)/(\sqrt{8\pi kr_o})$. We discuss the case of a cylinder illuminated by a plane wave in more detail in

Section 3.

Let us turn now to consider the two parameter functions which are the limiting forms of $V(x,y,y_0,q)$. These have been discussed in considerable detail by Logan (Ref. 8).

When $y = y_0 = 0$ we obtain the function $V_0(\xi,q)$ which possesses the representations

$$V_0(\xi\,,q) = \frac{1}{2}\,V(\xi,\,0,\,0,\,q)$$

$$= \sqrt{\pi\xi}\;\exp(i\,\pi/4)\sum_{s=1}^{\infty}\frac{\exp(i\xi t_s)}{t_s - q^2}$$

$$= \frac{\exp(-i\,\pi/4)}{2}\sqrt{\frac{\xi}{\pi}}\int_{-\infty}^{\infty}\exp(i\xi t)\;\frac{w_1(t)}{w_1'(t) - q\,w_1(t)}\;dt \qquad (1\text{-}71)$$

For $\xi > 0$ the residue series in Eq. (1-71) converges fast enough to be a practical means of computing $V_0(\xi,q)$. For $\xi \to 0$ it becomes completely impractical to sum the residue series and it becomes more attractive to follow Pekeris (Ref. 4) and express the integral in Eq. (1-71) in the form

$$V_0(\xi,q) = \frac{1}{2}\sqrt{\frac{-i\xi}{\pi}}\int_{\infty\exp(i\frac{2}{3}\pi)}^{\infty}\exp(i\xi t)\;\frac{w_1(t)}{w_1'(t) - qw_1(t)}\;dt \qquad (1\text{-}72)$$

Pekeris then used an asymptotic representation for the integrand which, in our notation, is of the form

$$\frac{w_1(t)}{w_1'(t) - qw_1(t)} \approx \frac{1}{\sqrt{t} - q} \qquad (1\text{-}73)$$

to suggest that the integrand should be broken into two components as in the following equation:

$$\frac{w_1(t)}{w_1'(t) - qw_1(t)} = \frac{1}{\sqrt{t} - q} + \frac{\sqrt{t}\, w_1(t) - w_1'(t)}{(\sqrt{t} - q)[w_1'(t) - qw_1(t)]} \tag{1-74}$$

We can then use the property of the Airy function

$$w_1[t\exp(i\tfrac{2}{3}\pi)] = \exp(i\tfrac{1}{3}\pi)w_2(t) \tag{1-75}$$

to express $V_0(\xi,q)$ in the form

$$V_0(\xi,q) = \frac{1}{2}\sqrt{\frac{-i\xi}{\pi}} \int_{\infty\exp(i\frac{2}{3}\pi)}^{\infty} \frac{\exp(i\xi t)}{\sqrt{t} - q}\, dt$$

$$+ \frac{1}{2}\sqrt{\frac{-i\xi}{\pi}} \int_0^{\infty} \exp(i\xi t)\, F_1(t,q)dt$$

$$+ \frac{1}{2}\sqrt{\frac{-i\xi}{\pi}} \int_0^{\infty} \exp(-\frac{\sqrt{3} + i}{2}\,\xi t)\, F_2(t,q^*)dt \tag{1-76}$$

where

$$F_1(t,q) = \frac{\sqrt{t}\, w_1(t) - w_1'(t)}{(\sqrt{t} - q)[w_1'(t) - qw_1(t)]} \tag{1-77a}$$

$$F_2(t,q) = \exp(i\tfrac{1}{3}\pi)\frac{\sqrt{t}\, w_2(t) + w_2'(t)}{(\sqrt{t} + q^*)[w_2'(t) - q^* w_2(t)]} \tag{1-77b}$$

Pekeris observed that the first integral in Eq. (1-76) leads to the Weyl-van der Pol formula, i.e.,

$$\frac{1}{2}\sqrt{\frac{-i\xi}{\pi}}\int_{\infty\exp(i\frac{2}{3}\pi)}^{\infty}\frac{\exp(i\xi t)}{\sqrt{t}-q}\,dt = 1 - 2\sigma\exp(-\sigma^2)\int_{i\infty}^{\sigma}\exp(\alpha^2)\,d\alpha \qquad (1\text{-}78)$$

where σ is the Sommerfeld "numerical distance"

$$\sigma = q\sqrt{-i\xi} \qquad (1\text{-}79)$$

Pekeris evaluated the integrals which contain the factors $F_1(t,q)$ and $F_2(t,q)$ by means of numerical integration. The author knows of no later writers who have employed these techniques to evaluate Eq. (1-76). The technique of numerical integration is well known for the case of the functions $V_1(\xi,q)$ and $V_2(\xi,q)$, but the decomposition discovered by Pekeris has escaped the attention of those writers whose papers the author has consulted. The integral in Eq. (1-78) leads to the error function of complex argument.

The function $V_o(\xi,q)$ can be shown to be a solution of the integral equation

$$V_o(\xi,q) = \exp\left(-i\frac{1}{12}\xi^3\right) - \frac{\exp\left(-i\frac{\pi}{4}\right)}{2}\sqrt{\frac{\xi}{\pi}}\int_0^{\xi} V_o(x,q)\exp\left[-i\frac{1}{12}(\xi-x)^3\right]$$

$$\left[(\xi - x) - 2iq\right]\frac{dx}{\sqrt{x(\xi-x)}} \qquad (1\text{-}80)$$

In Ref. 8 the author has discussed representations for the two limiting cases, namely

$$v(\xi) = V_o(\xi,0) = \frac{\exp(-i\,\pi/4)}{2}\sqrt{\frac{\xi}{\pi}}\int_{-\infty}^{\infty}\exp(i\xi t)\,\frac{w_1(t)}{w_1'(t)}\,dt \qquad (1\text{-}81)$$

$$u(x) = \lim_{q\to\infty}\left|2i\xi q^2 V_o(\xi,q)\right| = \frac{\exp(-i\,3\pi/4)}{\sqrt{\pi}}\,\xi^{3/2}\int_{-\infty}^{\infty}\exp(i\xi t)\,\frac{w_1'(t)}{w_1(t)}\,dt \qquad (1\text{-}82)$$

In Fig. 1.7 we illustrate the behavior of the real and the

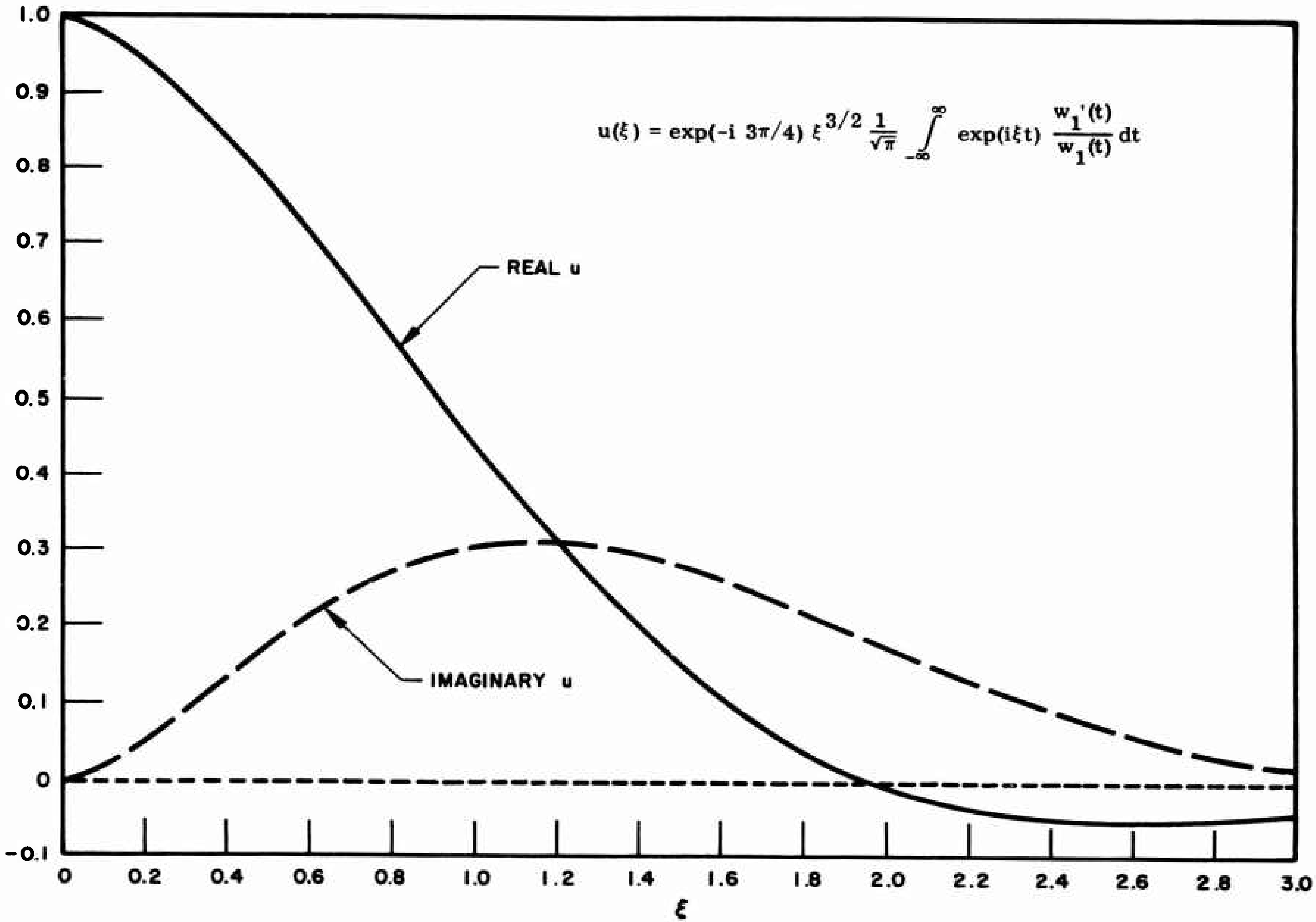

Fig. 1.7 Attenuation Function $u(\xi)$ (Real and Imaginary Parts)

imaginary parts of $u(\xi)$. In Fig. 1.8 we illustrate the behavior of the modulus of $u(\xi)$.

In Fig. 1.9 we show the behavior of the real and imaginary parts of $v(\xi)$. In Fig. 1.10 we illustrate the behavior of the modulus of $v(\xi)$.

For small values of ξ we can compute $u(\xi)$ and $v(\xi)$ from series which progress in successive powers of $\xi^{3/2}$. The theory behind these representations has been developed in Ref. 3. The series are of the form

(Text continues on p. 1-28)

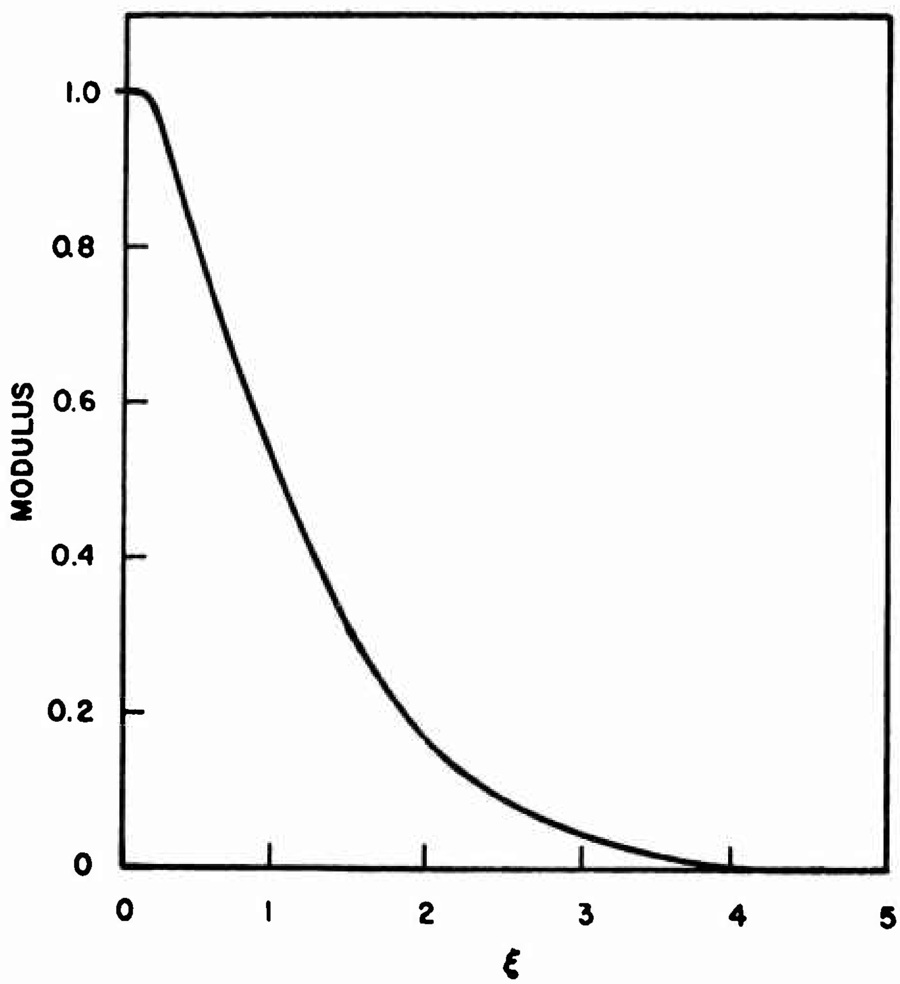

Fig. 1.8 Attenuation Function $u(\xi)$ (Modulus)

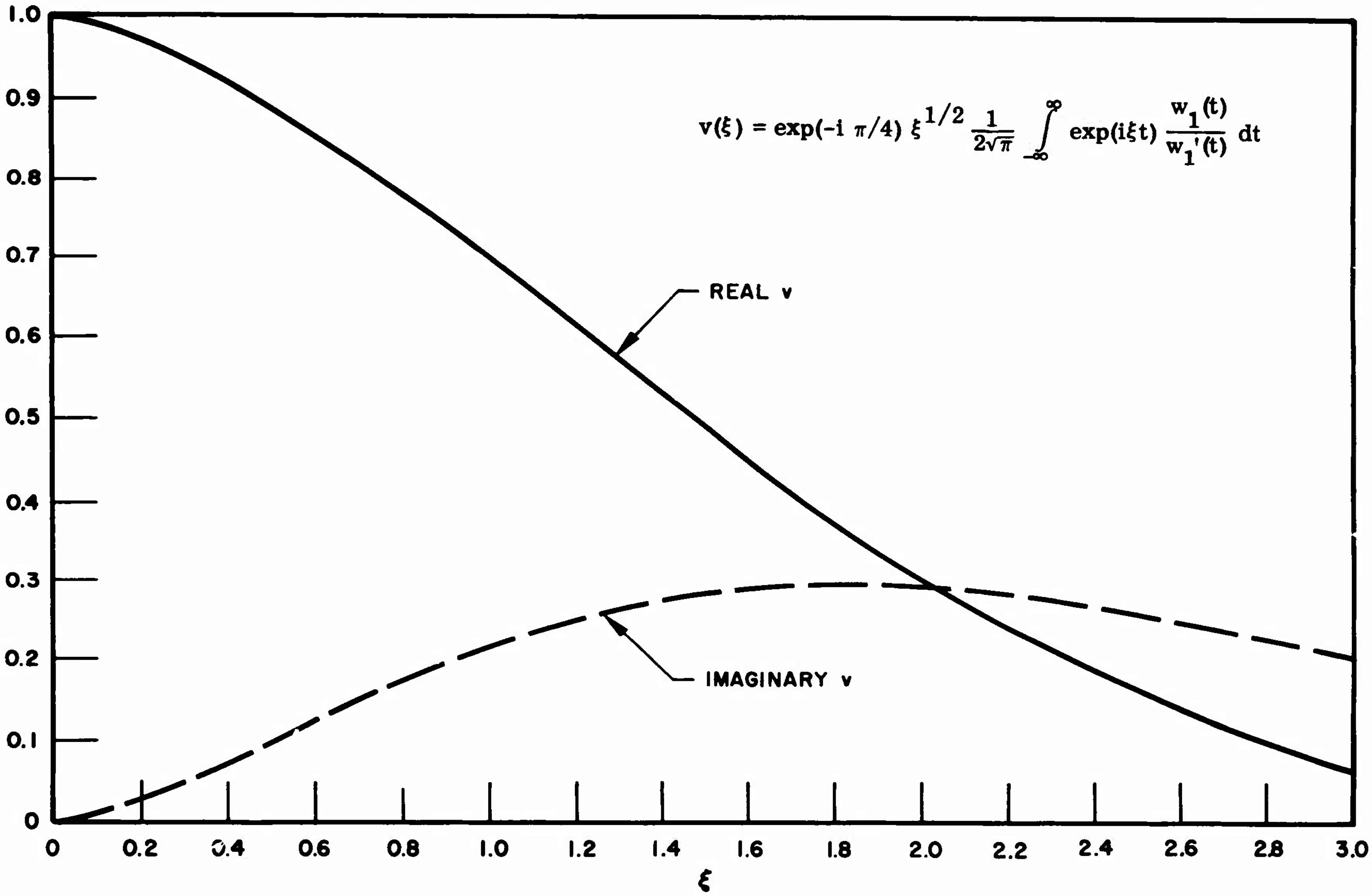

Fig. 1.9 Attenuation Function $v(\xi)$ (Real and Imaginary Parts)

$$u(\xi) = \left\{1 + \frac{1}{4}\frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma(2)} \exp\left(-i\frac{\pi}{4}\right)\xi^{3/2} - \frac{5}{32}\frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(\frac{5}{2}\right)} \exp\left(-i\frac{\pi}{2}\right)\xi^{3} + \ldots\ldots\right\}$$

$$= -2\sqrt{-i\pi\xi^{3}} \sum_{n=0}^{\infty} A_n \frac{\left[e^{-i\,\pi/6}\xi\right]^{\frac{3n-3}{2}}}{\Gamma\left(\frac{3n-1}{2}\right)} \qquad (1\text{-}83)$$

$$v(\xi) = \left\{1 - \frac{1}{4}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} e^{\left(-i\pi/4\right)}\xi^{3/2} + \frac{7}{32}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{2}\right)} \exp\left(-i\frac{\pi}{2}\right)\xi^{3} + \cdots\right\}$$

$$= -\exp\left(-i\frac{\pi}{3}\right) \sum_{n=0}^{\infty} B_n \frac{\left[e^{-i\pi/6}\xi\right]^{\frac{3n-1}{2}}}{\Gamma\left(\frac{3n+1}{2}\right)} \qquad (1\text{-}84)$$

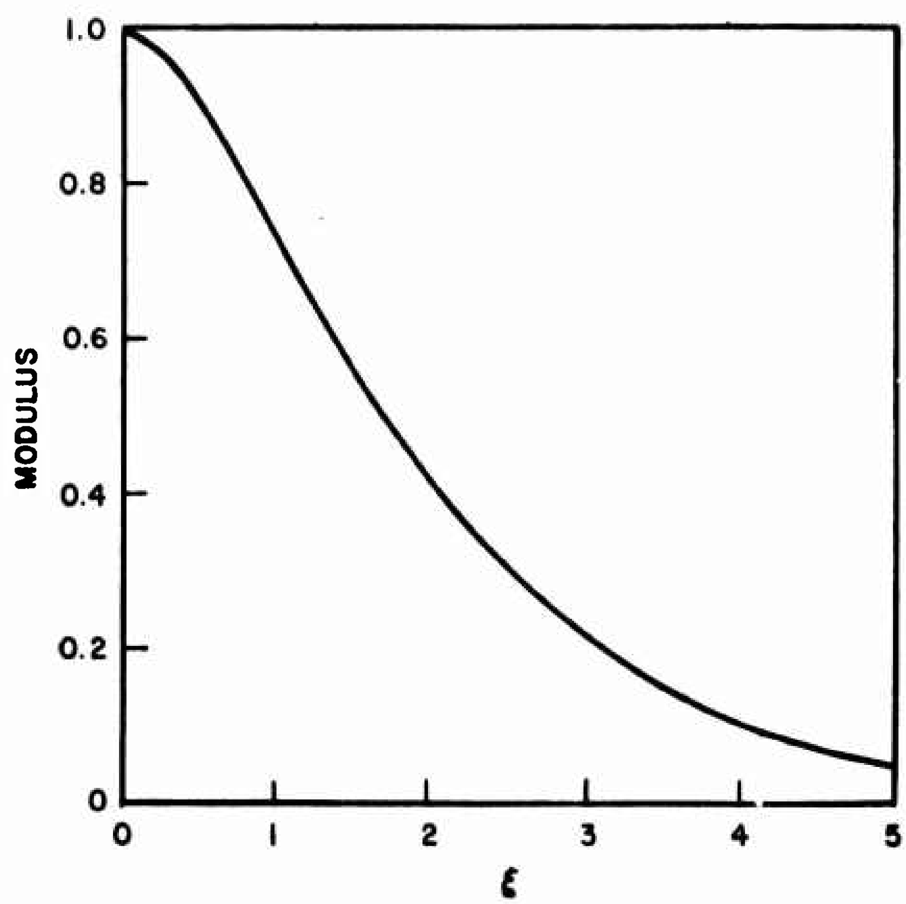

Fig. 1.10 Attenuation Function $v(\xi)$ (Modulus)

In Eq. (1-83) the coefficients A_n are the coefficients of $\alpha^{(-3n+\frac{1}{2})}$ in the asymptotic expansion of the logarithmic derivative of the Airy function

$$\frac{Ai'(\alpha)}{Ai\,(\alpha)} = \sum_{n=0}^{\infty} A_n \alpha^{(3n-1/2)} = -\sqrt{\alpha} - \frac{1}{4}\frac{1}{\alpha} + \frac{5}{32}\frac{1}{\alpha^{5/2}} - \frac{15}{64}\frac{1}{\alpha^4} + \frac{1105}{2048}\frac{1}{\alpha^{11/2}}$$

$$- \frac{1695}{1024}\frac{1}{\alpha^7} + \frac{414125}{65536}\frac{1}{\alpha^{17/2}} - \frac{59025}{2048}\frac{1}{\alpha^{10}}$$

$$+ \frac{12820\,31525}{8388608}\frac{1}{\alpha^{21/2}} - \frac{2421\,83775}{262144}\frac{1}{\alpha^{13}} + \cdots \qquad (1\text{-}85)$$

In Eq. (1-84) the coefficients B_n are the coefficients of $\alpha^{(-3n+\frac{1}{2})}$ in the asymptotic expansion of the logarithmic derivative of the derivative of the Airy function

$$\frac{d}{d\alpha} \log Ai'(\alpha) = \frac{Ai''(\alpha)}{Ai'\,(\alpha)} = \alpha\,\frac{Ai(\alpha)}{Ai'(\alpha)} = \sum_{n=0}^{\infty} B_n \frac{1}{\alpha^{\frac{3n-1}{2}}}$$

$$= -\sqrt{\alpha} + \frac{1}{4}\frac{1}{\alpha} - \frac{7}{32}\frac{1}{\alpha^{5/2}} + \frac{21}{64}\frac{1}{\alpha^4} - \frac{1463}{2048}\frac{1}{\alpha^{11/2}}$$

$$+ \frac{2121}{1024}\frac{1}{\alpha^7} - \frac{495271}{65536}\frac{1}{\alpha^{17/2}} + \frac{136479}{4096}\frac{1}{\alpha^{10}}$$

$$- \frac{14457\,13003}{8388608}\frac{1}{\alpha^{23/2}} + \frac{268122561}{262144}\frac{1}{\alpha^{13}} - \cdots \qquad (1\text{-}86)$$

In Table 1-1 we present numerical values for the coefficients A_n and B_n.

In Tables 1-2 and 1-3 we present numerical values for the functions $u(\xi)$ and $v(\xi)$.

The author is preparing a revision of Ref. 8 in which computer programs for these functions, as well as for $V_1(\xi,q)$ and $V_2(\xi,q)$, will be given.

Table 1-1

VALUE OF A_n AND B_n OCCURING IN ASYMPTOTIC EXPANSIONS OF THE LOGARITHMIC DERIVATIVES OF $Ai(\alpha)$ AND $Ai'(\alpha)$

n	A_n	B_n
0	-0.100000000000000000000D 01,	-0.100000000000000000000D 01,
1	-0.250000000000000000000D+00,	+0.250000000000000000000D+00,
2	+0.156250000000000000000D+00,	-0.218750000000000000000D+00,
3	-0.234375000000000000000D+00,	+0.328125000000000000000D+00,
4	+0.539550781250000000000D+00,	-0.714355468750000000000D+00,
5	-0.165527343750000000000D 01,	+0.207128906250000000000D 01,
6	+0.631904602050781250000D 01,	-0.755723571777343750000D 01,
7	-0.288208007812500000000D 02,	+0.333200683593750000000D 02,
8	+0.152830067276954650879D 03,	-0.172342419981956481933D 03,
9	-0.923857784271240234376D 03,	+0.102280640029907226562D 04,
10	+0.627145402831956744194D 04,	-0.684776647836342453957D 04,
11	-0.472420875728130340576D 05,	+0.510385042383670806885D 05,
12	+0.391093797395304136444D 06,	-0.419013417550075042527D 06,
13	-0.352962918848544359207D 07,	+0.375636973584705591202D 07,
14	+0.344923594225291310522D 08,	-0.365073309173042538587D 08,
15	-0.362859054167324211448D 09,	+0.382303664585640091449D 09,
16	+0.408874819536233990133D 10,	-0.429111919679893578455D 10,
17	-0.491329314661619090110D 11,	+0.513943542891301027641D 11,
18	+0.627198459188666032326D 12,	-0.654173445397702049560D 12,
19	-0.847611398458811941308D 13,	+0.881828408318901625950D 13,
20	+0.120897389055654125660D 15,	-0.125496177754663210762D 15,
21	-0.181496999003719169264D 16,	+0.188024788284105658508D 16,
22	+0.286071222790515967392D 17,	-0.295829251279511604806D 17,
23	-0.472335609717781730361D 18,	+0.487657445864043740510D 18,
24	+0.815277057364795720245D 19,	-0.840488648956753985073D 19,
25	-0.146831645154966902644D 21,	+0.151169864003790799746D 21,
26	+0.275449757197776501747D 22,	-0.283240986250299855021D 22,
27	-0.537378756984207398462D 23,	+0.551957349739728293224D 23,
28	+0.108866223659917805940D 25,	-0.111703709893164884145D 25,
29	-0.228710461424380230366D 26,	+0.234446449592475986762D 26,
30	+0.497629754088787008189D 27,	-0.509656139639392228718D 27,
31	-0.112005330641509715668D 29,	+0.114617212361596047251D 29,
32	+0.260496208841598317131D 30,	-0.266364897571689195131D 30,
33	-0.625379032469004638302D 31,	+0.639006344470836503258D 31,
34	+0.154824953634757043907D 33,	-0.158091618384036980004D 33,
35	-0.394908158161364723226D 34,	+0.402984156927023863685D 34,
36	+0.103689211790902314958D 36,	-0.105746442975289905250D 36,
37	-0.280026594509852636416D 37,	+0.285421533273123163778D 37,
38	+0.777246029472902746522D 38,	-0.791798924680840649638D 38,

Table 1-2

THE ATTENUATION FUNCTION $u(\xi)$

ξ	Real	Imag.	Modulus	Argument
0	1.0000	0	1.0000	0
0.1	0.9802	0.0192	0.9804	1.12
0.2	0.9440	0.0528	0.9455	3.20
0.3	0.8974	0.0922	0.9021	5.86
0.4	0.8429	0.1334	0.8534	9.00
0.5	0.7822	0.1738	0.8013	12.51
0.6	0.7156	0.2123	0.7465	16.52
0.7	0.6487	0.2434	0.6929	20.57
0.8	0.5792	0.2700	0.6390	24.97
0.9	0.5090	0.2902	0.5859	29.68
1.0	0.4403	0.3035	0.5348	34.58
1.1	0.3740	0.3102	0.4859	39.67
1.2	0.3112	0.3105	0.4396	44.94
1.3	0.2526	0.3050	0.3960	50.36
1.4	0.1991	0.2944	0.3554	55.93
1.5	0.1510	0.2796	0.3178	61.63
1.6	0.1086	0.2615	0.2831	67.45
1.7	0.0719	0.2409	0.2514	73.38
1.8	0.0409	0.2187	0.2225	79.40
1.9	0.0154	0.1957	0.1963	85.50
2.0	-0.0051	0.1726	0.1727	91.68
2.1	-0.0209	0.1501	0.1515	97.93
2.2	-0.0326	0.1285	0.1326	104.24
2.3	-0.0407	0.1083	0.1157	110.61
2.4	-0.0458	0.0898	0.1008	117.02
2.5	-0.0483	0.0730	0.0875	123.48
2.6	-0.0487	0.0581	0.0759	129.98
2.7	-0.0476	0.0452	0.0656	136.50
2.8	-0.0453	0.0341	0.0567	143.06
2.9	-0.0421	0.0247	0.0488	149.63
3.0	-0.0384	0.0169	0.0420	156.23
3.1	-0.0345	0.0106	0.0361	162.85
3.2	-0.0304	0.0056	0.0309	169.49
3.3	-0.0264	0.0018	0.0265	176.13
3.4	-0.0226	-0.0011	0.0226	182.79
3.5	-0.0190	-0.0032	0.0193	189.46
3.6	-0.0158	-0.0046	0.0165	196.13
3.7	-0.0129	-0.0054	0.0140	202.81
3.8	-0.0104	-0.0059	0.0119	209.50
3.9	-0.0082	-0.0060	0.0101	216.19
4.0	-0.0063	-0.0059	0.0086	222.88

Table 1-3

THE ATTENUATION FUNCTION $v(\xi)$

ξ	Real	Imag.	Modulus	Argument
0.1	0.9901	0.0098	0.9901	0.57
0.2	0.9720	0.0271	0.9724	1.59
0.3	0.9486	0.0484	0.9498	2.92
0.4	0.9210	0.0721	0.9238	4.47
0.5	0.8899	0.0969	0.8952	6.22
0.6	0.8559	0.1221	0.8646	8.12
0.7	0.8195	0.1469	0.8325	10.16
0.8	0.7810	0.170[illegible]	0.7995	12.33
0.9	0.7410	0.1929	0.7657	14.59
1.0	0.6999	0.2134	0.7317	16.96
1.1	0.6581	0.2317	0.6977	19.40
1.2	0.6159	0.2477	0.6638	21.91
1.3	0.5737	0.2612	0.6304	24.48
1.4	0.5317	0.2723	0.5976	27.11
1.5	0.4908	0.2809	0.5655	29.79
1.6	0.4506	0.2970	0.5342	32.50
1.7	0.4115	0.2908	0.5039	35.25
1.8	0.3739	0.2924	0.4747	38.03
1.9	0.3379	0.2920	0.4466	40.83
2.0	0.3036	0.2897	0.4196	43.66
2.1	0.2710	0.2857	0.3938	46.51
2.2	0.2404	0.2801	0.3692	49.37
2.3	0.2117	0.2733	0.3457	52.24
2.4	0.1850	0.2653	0.3265	55.12
2.5	0.1602	0.2564	0.3024	58.01
2.6	0.1373	0.2468	0.2824	60.91
2.7	0.1163	0.2365	0.2636	63.81
2.8	0.0972	0.2258	0.2458	66.72
2.9	0.0747	0.2148	0.2291	69.63
3.0	0.0640	0.2035	0.2134	72.55
3.1	0.0499	0.1923	0.1986	75.46
3.2	0.0372	0.1810	0.1848	78.38
3.3	0.0260	0.1698	0.1718	81.30
3.4	0.0161	0.1589	0.1597	84.22
3.5	0.0074	0.1482	0.1484	87.14
3.6	-0.0002	0.1378	0.1378	90.06
3.7	-0.0067	0.1277	0.1279	92.98
3.8	-0.0122	0.1180	0.1187	95.90
3.9	-0.01[illegible]	0.1088	0.1101	98.82
4.0	-0.0208	0.0999	0.1021	101.74

Table 1-3 (Cont'd)

ξ	Real	Imag.	Modulus	Argument
4.1	-0.0240	0.0915	0.0946	104.66
4.2	-0.0265	0.0836	0.0877	107.58
4.3	-0.0285	0.0761	0.0812	110.50
4.4	-0.0299	0.0690	0.0752	113.42
4.5	-0.0309	0.0624	0.0697	116.34
4.6	-0.0315	0.0562	0.0645	119.25
4.7	-0.0318	0.0505	0.0597	122.17
4.8	-0.0318	0.0452	0.0552	125.09
4.9	-0.0315	0.0402	0.0511	128.01
5.0	-0.0310	0.0357	0.0472	130.93
5.1	-0.0303	0.0315	0.0437	133.85
5.2	-0.0294	0.0276	0.0404	136.77
5.3	-0.0285	0.0241	0.0373	139.69
5.4	-0.0274	0.0209	0.0345	142.61
5.5	-0.0263	0.0180	0.0319	145.52
5.6	-0.0251	0.0154	0.0295	148.44
5.7	-0.0239	0.0130	0.0272	151.36
5.8	-0.0226	0.0109	0.0251	154.28
5.9	-0.0214	0.0090	0.0232	157.20
6.0	-0.0202	0.0073	0.0214	160.12
6.1	-0.0189	0.0058	0.0198	163.04
6.2	-0.0177	0.0044	0.0183	165.96
6.3	-0.0165	0.0033	0.0169	168.87
6.4	-0.0154	0.0022	0.0156	171.79
6.5	-0.0143	0.0013	0.0144	174.71
6.6	-0.0132	0.0005	0.0132	177.63
6.7	-0.0122	-0.0001	0.0122	180.55
6.8	-0.0113	-0.0007	0.0113	183.47
6.9	-0.0103	-0.0012	0.0104	186.39
7.0	-0.0095	-0.0015	0.0096	189.30
7.1	-0.0086	-0.0019	0.0088	192.22
7.2	-0.0079	-0.0021	0.0082	195.14
7.3	-0.0072	-0.0023	0.0075	198.06
7.4	-0.0065	-0.0025	0.0069	200.98
7.5	-0.0058	-0.0026	0.0064	203.90
7.6	-0.0053	-0.0027	0.0059	206.82
7.7	-0.0047	-0.0027	0.0054	209.73
7.8	-0.0042	-0.0027	0.0050	212.65
7.9	-0.0038	-0.0027	0.0046	215.57
8.0	-0.0033	-0.0026	0.0043	218.49

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Section 2

THE ROOTS t_s OF $w_1'(t_s) - qw_1(t_s) = 0$

2.1 Notation for the Root Defining Equation

In electromagnetic diffraction problems which involve the propagation of waves around a convex surface, one often has to solve for the roots $\underline{t}$ of the equation

$$w_1'(t) - qw_1(t) = 0 \tag{2-1}$$

where $w_1'(t) = dw_1(t)/dt$ and q is a given "constant" which can generally be related to the physical quantities in such a manner that we can speak of q as being related to the "surface impedance." The function $w_1(t)$ is the Airy function introduced by Fock (Ref. 1) and used extensively by him, by Wait* (Ref. 2), by Logan (ref. 3), by Logan and Yee (Ref. 4), and by others. The function $w_1(t)$ satisfies the Airy differential equation

$$\frac{d^2y}{dt^2} - ty = 0 \tag{2-2}$$

and can be defined by the integral representation

$$w_1(t) = \frac{1}{\pi}\int_{-\infty}^{\infty e^{i\pi/3}} \exp\left(\frac{1}{3}z^3 - zt\right) dz \tag{2-3}$$

For real values of $\underline{t}$ it is convenient to resolve $w_1(t)$ into its real and imaginary parts by defining

$$w_1(t) = u(t) + iv(t) \tag{2-4}$$

where, for real values of t, both u(t) and v(t) are real.

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*Wait denotes our $w_1(t)$ by $w_2(t)$, and our $w_2(t)$ by $w_1(t)$.

The integral definitions of u(t) and v(t) for real t can be taken to be

$$v(t) = \frac{1}{\sqrt{\pi}} \int_0^\infty \cos\left(\frac{x^3}{3} + tx\right) dx \tag{2-4a}$$

$$u(t) = \frac{1}{\sqrt{\pi}} \int_0^\infty \left[\sin\left(\frac{x^3}{3} + tx\right) + \exp\left(-\frac{x^3}{3} + xt\right)\right] dx \tag{2-4b}$$

The Airy functions are so fundamental in the mathematical discussion of a variety of physical phenomena that it is almost unexplainable that such an important function had not received a universally accepted notation prior to the beginning of World War II. The author has observed that this function is now finding its way into more and more textbooks, but the notation which seems to be "catching on" the most firmly is that which is due to Jeffreys and Miller (Ref. 5). The functions which were tabulated by Miller were denoted by Ai(x) and Bi(x) and defined by the integrals

$$Ai(x) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt \tag{2-5a}$$

$$Bi(x) = \frac{1}{\pi} \int_0^\infty \left[\sin\left(\frac{t^3}{3} + xt\right) + \exp\left(-\frac{t^3}{3} + xt\right)\right] dt \tag{2-5b}$$

Since Eqs. (2-4) and (2-5) differ by only a factor of the form $\sqrt{\pi}$, it is very easy to employ both these notations since the conversion between them is

$$v(t) = \sqrt{\pi}\, Ai(t) \quad , \quad u(t) = \sqrt{\pi}\, Bi(t)$$

$$w_1(t) = \sqrt{\pi}\left[Bi(t) + i\, Ai(t)\right] \quad , \quad w_2(t) = \sqrt{\pi}\left[Bi(t) - i\, Ai(t)\right]$$

The need for the Airy functions in the theory of radiowave propagation during World War II was so great that each research group adopted its own notation. In the classic work published just before the start of the War, van der Pol and Bremmer (Ref. 6) had employed approximations for the Hankel functions $H_{\nu}^{(1)}(kr)$ which involved the Hankel functions

$$H_{\frac{1}{3}}^{(1)}(z) \quad , \quad H_{-\frac{1}{3}}^{(1)}(z) \quad , \quad H_{\frac{2}{3}}^{(1)}(z) \quad , \quad H_{-\frac{2}{3}}^{(1)}(z)$$

and much of the computational effort associated with their work revolved around finding the roots τ_s which were determined by the equation

$$\frac{\exp(-i\,\pi/3)\, H_{2/3}^{(1)}\left\{\frac{1}{3}(-2\tau_s)^{3/2}\right\}}{H_{1/3}^{(1)}\left\{\frac{1}{3}(-2\tau_s)^{3/2}\right\}} = -\frac{1}{\delta_e\sqrt{-2\tau_s}} \tag{2-6}$$

When Furry and his co-workers at the Radio Research Laboratory at Harvard University became interested in these problems they observed that by defining

$$\xi = \frac{2}{3}\, x^{3/2}$$

that they could simplify the theory by working with a set of modified Hankel functions of order one-third which were defined by means of the relations

$$\left.\begin{aligned} h_j(x) &= \left(\frac{2}{3}\right)^{1/3} x^{1/2}\, H_{1/3}^{(j)}\left(\frac{2}{3}x^{3/2}\right) = \xi^{1/3}\, H_{1/3}^{(j)}(\xi) \\ \frac{dh_j(x)}{dx} &= \left(\frac{2}{3}\right)^{1/3} x\, H_{-2/3}^{(j)}\left(\frac{2}{3}\, x^{3/2}\right) = \left(\frac{3}{2}\right)^{1/3} \xi^{2/3}\, H_{-2/3}^{(j)}(\xi) \end{aligned}\right\} j = 1, 2 \tag{2-7}$$

Tables of these functions (Ref. 7) were computed on the famous Automatic Sequence Controlled Calculator (or "Mark I" as it came to be affectionately called by its users) which had been presented to Harvard University by the International Business Machines Corporation in 1944. In the introduction to the tables, Furry points out that the functions are solutions of

$$\frac{d^2h_j}{dx^2} + x\, h_j = 0$$

and that they are related to the Airy functions being used by Miller and the English workers by means of the relations

$$h_1(x) = k\,[Ai(-x) - iBi(-x)]$$

$$h_2(x) = k^*[Ai(-x) + iBi(-x)]$$

where

$$k = (12)^{1/6}\exp(-i\pi/6) \quad , \quad k^* = (12)^{1/6}\exp(+i\pi/6)$$

The digression which we have made above to point out some of the differences in notation for the Airy functions is far from complete. For a more complete discussion, the reader is referred to a report by Logan (Ref. 3). However, in order to proceed with the study of the roots of $w_1'(t) - qw_1(t) = 0$, the author feels that the reader should have before him the following relations:

$$w_1(t) = -\sqrt{\pi}(12)^{-1/6}\exp(-i\tfrac{1}{6}\pi)h_1(-t) \tag{2-8a}$$

$$= \sqrt{-\tfrac{1}{3}\pi t}\,\exp(i\tfrac{2}{3}\pi)H^{(1)}_{\frac{1}{3}}[\tfrac{2}{3}(-t)^{3/2}] \tag{2-8b}$$

$$= u(t) + iv(t) \tag{2-8c}$$

$$= \sqrt{\pi}\,[Bi(t) + iAi(t)] \tag{2-8d}$$

$$= 2\sqrt{\pi}\,\exp(i\pi/6)Ai[t\exp(i\tfrac{2}{3}\pi)] \tag{2-8e}$$

From Eq. (2-8e) we readily see that the solutions a_s of

$$Ai'(-a_s) + q \exp(i\tfrac{1}{3}\pi)\, Ai(-a_s) = 0 \qquad (2\text{-}9)$$

are the same as the solutions of Eq. (2-1), namely,

$$w_1'(t_s) - q w_1(t_s) = 0$$

provided we observe that

$$t_s = a_s \exp(i\tfrac{1}{3}\pi) \qquad (2\text{-}10)$$

If we use Eq. (2-7) and Eq. (2-8b), we find that

$$q = \frac{w_1'(t_s)}{w_1(t_s)} = - \frac{\xi_s^{\frac{1}{3}} H_{-\frac{2}{3}}^{(1)}(\xi_s)}{(\frac{2}{3})^{\frac{1}{3}} H_{\frac{1}{3}}^{(1)}(\xi_s)} = \frac{\sqrt{-t_s}\exp(-i\tfrac{1}{3}\pi) H_{\frac{2}{3}}^{(1)}(\xi_s)}{H_{\frac{1}{3}}^{(1)}(\xi_s)} \qquad (2\text{-}11)$$

where

$$\xi_s = \frac{2}{3}(-t_s)^{3/2} = \frac{1}{3}(-2\tau_{s-1})^{3/2} \qquad (2\text{-}11)$$

Therefore, we observe from a comparison of Eq. (2-6) and (2-11) that the roots t_s which were studied by Fock are essentially the same as the roots τ_{s-1} which were studied by van der Pol and Bremmer* and by Bremmer (Ref. 8).

Since van der Pol and Wyngaarden (Ref. 9) were influenced by the work and the tables of Furry (Ref. 7), their Eq. (4), namely

$$h(\mu_s) = b\, h'(\mu_s) \qquad (2\text{-}12a)$$

can be shown to be

$$h_2(\mu_s) = b\, h_2'(\mu_s)\ . \qquad (2\text{-}12b)$$

Since the modified Hankel functions possess the properties

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*The relationship is $t_s = 2^{\frac{1}{3}}\tau_{s-1}$ since Fock lets $s = 1,2,3,\cdots$ whereas van der Pol and Bremmer let $s = 0,1,2,\cdots$.

$$h_1(z) = [h_2(z^*)]^* \quad , \quad h_1'(z) = [h_2'(z^*)]^*$$

(where the asterisks denote that the complex conjugate quantities are to be employed), we observe from the property

$$q = \frac{w_1'(t_s)}{w_1(t_s)} = - \frac{h_1'(-t_s)}{h_1(-t_s)} \tag{2-13}$$

that the roots t_s are related to the roots μ_s (of Ref. 9) by the simple property

$$t_s = \text{complex conjugate } (-\mu_s)$$

Further confusion has been fostered by the fact that a number of authors are employing an Airy function A(z) defined by

$$A(z) = \int_0^\infty \cos(\tau^3 - z\tau)d\tau \tag{2-14a}$$

which is related to the more conventional Airy function Ai(x) by means of the relations

$$Ai(x) = \frac{3^{\frac{1}{3}}}{\pi} A(-3^{\frac{1}{3}}x) \quad , \quad Ai'(x) = - \frac{3^{\frac{2}{3}}}{\pi} A'(-3^{\frac{1}{3}}x) \tag{2-14b}$$

From Eq. (2-9) we see that

$$q = \frac{w_1'(t_s)}{w_1(t_s)} = \frac{\exp(i\frac{2}{3}\pi)Ai'(-a_s)}{Ai(-a_s)} = \frac{3^{\frac{1}{3}}\exp(-i\frac{1}{3}\pi)A(3^{\frac{1}{3}}a_s)}{A(3^{\frac{1}{3}}a_s)} \tag{2-15}$$

where t_s and a_s are related as in Eq. (2-10).

The remarks which we have made above are sufficient to show the reader that he may expect to find the equivalent of Eq. (2-1) appearing in many disguises as he looks through the literature.

From the property

$$q(t) = \frac{d}{dt} \{\log[w_1(t)]\} = \frac{w_1'(t)}{w_1(t)} \tag{2-16}$$

we observe that the inverse problem to the finding of the roots of Eq. (2-1) can be described as "studying the behavior of the <u>logarithmic derivative</u> of the Airy function $w_1(t)$."

The occurrence of equations such as Eq. (2-1) is not limited to the radiowave propagation problems. For example, Jeffreys (Ref. 10) considered an elastic wave propagation problem which led him to study the problem of obtaining the roots ξ of the equation

$$\frac{Ai'(-\xi)}{Ai(-\xi)} = K$$

where k is a constant.

2.2 The Riccati Equation

The roots τ_s of Eq. (2-1), i.e., the roots of

$$w_1'(\tau_s) - q\, w_1(\tau_s) = 0 \qquad (2\text{-}17)$$

can often be evaluated by observing that the Airy equation

$$w_1''(\tau) = \tau\, w_1(\tau)$$

reveals that

$$\frac{dq(\tau)}{d\tau} = \frac{d}{d\tau}\,\frac{w_1'(\tau)}{w_1(\tau)} = \frac{w_1''(\tau)}{w_1(\tau)} - \left[\frac{w_1'(\tau)}{w_1(\tau)}\right]^2$$

or

$$\frac{dq}{dt} = t - q^2 \qquad \text{or} \qquad \frac{dt}{dq} = \frac{1}{t - q^2} \qquad (2\text{-}18)$$

Eq. (2-18) is a common type of non-linear first-order differential equation which is known in the literature as a <u>Riccati equation</u>. It is a simple matter to obtain the initial conditions under which Eq. (2-18) is to be integrated. Eq. (2-8e) reveals that $w_1(\tau)$ is merely a rotation of the function Ai(z). The roots of

the Airy function and its derivatives are well known. Interest in these roots goes back to as early as 1838 when Airy showed that the "rainbow integral"

$$W(m) = \int_0^\infty \cos[\tfrac{\pi}{2}(w^3 - mw)]dw$$

could be used to describe the maxima and minima of the system of spurious (or supernumerary) rainbows which are observed near the inner edge of the primary bow or the outer edge of the secondary bow. Numerical results for these quanties depended upon finding the roots of $W(m) = 0$ and $W'(m) = 0$. Airy's first numerical calculations led him to a determination of the first minimum and the first two maxima. By 1848 he had extended the results to find the location of the second zero. Then, in a classic paper which appeared in 1850, Stokes gave values (computed to 4 decimals) of the first 50 zeros and the first 10 maxima. During World War II, Miller (Ref. 5) obtained values (to 8 decimals) of the first 50 zeros of $Ai(x)$ and $Ai'(x)$. These results proved to be inadequate for some applications being made by the present author, and in 1958 he received 15-decimal values of these zeros from G. F. Miller and P. H. Haines at the National Physical Laboratory (Teddington, England). In the following year, Sherry (Ref. 11) published values for the first 50 zeros which are accurate to 25 decimals.

Let us now return to Eq. (2-17) and denote the roots for $q = 0$ by τ_s^0 and the root for $q \to \infty$ by τ_s^∞. Then we have

$$w_1(\tau_s^\infty) = 0 \quad , \quad w_1'(\tau_s^0) = 0 \tag{2-19}$$

where

$$\tau_s^\infty = \alpha_s \exp(i\tfrac{1}{3}\pi) \ , \quad \tau_s^0 = \beta_s \exp(i\tfrac{1}{3}\pi) \tag{2-20}$$

where α_s and β_s are the roots of the Airy function and its derivative

$$Ai(-\alpha_s) = 0 \quad , \quad Ai'(-\beta_s) = 0$$

Values for these constants are given in Tables 2-1 and 2-2. Values are also given for the "turning values" of the functions.

Table 2-1

ROOTS AND TURNING VALUES OF Ai(-α)

s	α_s	$Ai'(-\alpha_s)$
1	2.33810 74104 59767	+0.70121 08227 20691
2	4.08794 94441 30971	-0.80311 13696 54864
3	5.52055 98280 95551	+0.86520 40258 94152
4	6.78670 80900 71759	-0.91085 07370 49602
5	7.94413 35871 20853	+0.94733 57094 41568
6	9.02265 08533 40980	-0.97792 28085 69499
7	10.04017 43415 58086	+1.00437 01226 60312
8	11.00852 43037 33263	-1.02773 86888 20786
9	11.93601 55632 36263	+1.04872 06485 88189
10	12.82877 67528 65757	-1.06779 38591 57428
11	13.69148 90352 10718	+1.08530 28313 50700
12	14.52782 99517 75335	-1.10150 45702 77497
13	15.34075 51359 77997	+1.11659 61779 32656
14	16.13268 51569 45771	-1.13073 23104 93188
15	16.90563 39974 29943	+1.14403 66732 73553
16	17.66130 01056 97057	-1.15660 98491 16566
17	18.40113 25992 07115	+1.16853 47844 87525
18	19.12638 04742 46952	-1.17988 07298 70146
19	19.83812 98917 21500	+1.19070 61311 58776
20	20.53733 29076 77567	-1.20106 07915 19823
21	21.22482 99436 42097	+1.21098 75148 68287
22	21.90136 75955 85131	-1.22052 33738 97260
23	22.56761 29174 96503	+1.22970 07015 09681
24	23.22416 50011 21681	-1.23854 78753 29632
25	23.87156 44555 35918	+1.24708 99452 59407
26	24.51030 12365 89678	-1.25534 91404 75735
27	25.14082 11661 48964	+1.26334 52827 50799
28	25.76353 14009 82756	-1.27109 61262 18604
29	26.37880 50521 37232	+1.27861 76388 24258
30	26.98698 51116 06368	-1.28592 42371 22704
31	27.58838 78098 82445	+1.29302 89834 49956
32	28.18330 55026 32645	-1.29994 37525 11048
33	28.77200 91652 37435	+1.30667 93729 32094
34	29.35475 05587 66288	-1.31324 57481 80648
35	29.93176 41190 86556	+1.31965 19603 77514
36	30.50326 86114 18505	-1.32590 63598 38441
37	31.06946 85851 83756	+1.33201 66426 47702
38	31.63055 56580 12659	-1.33798 99181 42291
39	32.18670 96529 52051	+1.34383 27678 48983
40	32.73809 96090 00269	-1.34955 12971 47445
41	33.28488 46819 01402	+1.35515 11807 15907
42	33.82721 49495 08652	-1.36063 77026 40532
43	34.36523 21338 63659	+1.36601 57919 26784
44	34.89907 02503 45312	-1.37129 00540 34239
45	35.42885 61927 47888	+1.37646 47989 60084
46	35.95471 02618 98629	-1.38154 40663 17105
47	36.47674 66443 74809	+1.38653 16477 85955
48	36.99507 38469 94501	-1.39143 11072 66471
49	37.50979 50920 05016	+1.39624 57990 06725
50	38.02100 86772 55254	-1.40097 88839 49769
51	38.52880 83050 94249	+1.40563 33445 05322
52	39.03328 33832 72514	-1.41021 19979 25998
53	39.53451 93007 23018	+1.41471 75084 44110
54	40.03259 76807 54176	-1.41915 23983 05068
55	40.52759 66138 89718	+1.42351 90578 16189
56	41.01959 08723 32490	-1.42781 97545 15052

Table 2-2

ROOTS AND TURNING VALUES OF Ai′(−β)

s	β_s	$Ai(-\beta_s)$
1	1. 01879 29716 47471	+0. 53565 66560 15700
2	3. 24819 75821 79837	−0. 41901 54780 32564
3	4. 82009 92111 78736	+0. 38040 64686 28153
4	6. 16330 73556 39487	−0. 35790 79437 12292
5	7. 37217 72550 47770	+0. 34230 12444 11624
6	8. 48848 67340 19722	−0. 33047 62291 47967
7	9. 53544 90524 33547	+0. 32102 22881 94716
8	10. 52766 03969 57407	−0. 31318 53909 78682
9	11. 47505 66334 80245	+0. 30651 72938 82777
10	12. 38478 83718 45747	−0. 30073 08293 22645
11	13. 26221 89616 65210	+0. 29563 14810 01913
12	14. 11150 19704 62995	−0. 29108 16772 03539
13	14. 93593 71967 20517	+0. 28698 07069 99202
14	15. 73820 13736 92538	−0. 28325 27361 25021
15	16. 52050 38254 33794	+0. 27983 93053 60411
16	17. 28469 50502 16437	−0. 27669 44450 68930
17	18. 03234 46225 04393	+0. 27378 13856 46685
18	18. 76479 84376 65955	−0. 27107 02785 76971
19	19. 48322 16565 67231	+0. 26853 65782 82176
20	20. 18863 15094 63373	−0. 26615 98682 15709
21	20. 88192 27555 16738	+0. 26392 29929 60829
22	21. 56388 77231 98975	−0. 26181 14056 94794
23	22. 23523 22853 43913	+0. 25981 26701 51466
24	22. 89658 87388 74619	−0. 25791 60753 32572
25	23. 54852 62959 28802	+0. 25611 23337 79654
26	24. 19155 97095 26354	−0. 25439 33426 46825
27	24. 82615 64259 21155	+0. 25275 19925 76574
28	25. 45274 25617 77650	−0. 25118 20133 88409
29	26. 07170 79351 73912	+0. 24967 78484 21125
30	26. 68341 03283 22450	−0. 24823 45513 98365
31	27. 28817 91215 23985	+0. 24684 77011 60296
32	27. 88631 84087 68461	−0. 24551 33306 87119
33	28. 47810 96831 02278	+0. 24422 78676 45060
34	29. 06381 41626 38199	−0. 24298 80842 90143
35	29. 64367 48146 32016	+0. 24179 10550 23721
36	30. 21791 81244 68575	−0. 24063 41202 44844
37	30. 78675 56480 12503	+0. 23951 48554 15564
38	31. 35038 53790 83035	−0. 23843 10444 66267
39	31. 90899 29584 30463	+0. 23738 06563 33468
40	32. 46275 27462 38480	−0. 23636 18275 53143
41	33. 01182 87766 34287	+0. 23537 28399 36488
42	33. 55637 56097 89422	−0. 23441 21104 38024
43	34. 09653 90948 09138	+0. 23347 81753 92842
44	34. 63245 70546 35866	−0. 23256 96793 53833
45	35. 16425 99025 53408	+0. 23168 53648 03788
46	35. 69207 11985 10469	−0. 23082 40630 53231
47	36. 21600 81523 35199	+0. 22998 46861 64426
48	36. 73618 20799 46803	−0. 22916 62197 06428
49	37. 25269 88178 54148	+0. 22836 77166 46281
50	37. 76565 91005 38871	−0. 22758 82910 18357
51	38. 27515 89047 30879	+0. 22682 71133 87890
52	38. 78128 97640 80369	−0. 22608 34059 36628
53	39. 28413 90572 98596	+0. 22535 64383 68475
54	39. 76379 02724 68233	−0. 22464 55241 61432
55	40. 28032 32499 03719	+0. 22395 00171 79277
56	40. 77381 44056 64866	−0. 22326 93086 02552

There are two series which have played such a role in the calculating of the roots $\tau_s(q)$ of Eq. (2-17) that we shall refer to then as the classical expansions. For small values of q, we can express $\tau_s(q)$ in the form

$$\tau_s(q) = \tau_s^o + \sum_{n=1}^{\infty} A_n(\tau_s^o)\, q^n \tag{2-21a}$$

where

$$A_1(\tau) = \frac{1}{\tau} \tag{2-21b}$$

$$A_2(\tau) = -\frac{1}{2\tau^3} \tag{2-21c}$$

$$A_3(\tau) = \frac{1}{3\tau^2} + \frac{1}{2\tau^5} \tag{2-21d}$$

$$A_4(\tau) = -\frac{7}{12\tau^4} - \frac{5}{8\tau^7} \tag{2-21e}$$

$$A_5(\tau) = \frac{1}{5\tau^3} + \frac{21}{20\tau^6} + \frac{7}{8\tau^9} \tag{2-21f}$$

$$A_6(\tau) = -\frac{29}{45\tau^5} - \frac{77}{40\tau^8} - \frac{21}{16\tau^{11}} \tag{2-21g}$$

$$A_7(\tau) = \frac{1}{7\tau^4} + \frac{76}{45\tau^7} + \frac{143}{40\tau^{10}} + \frac{33}{16\tau^{13}} \tag{2-21h}$$

$$A_8(\tau) = -\frac{97}{140\tau^6} - \frac{163}{40\tau^9} - \frac{429}{64\tau^{12}} - \frac{429}{128\tau^{15}} \tag{2-21i}$$

$$A_9(\tau) = \frac{1}{9\tau^5} + \frac{13661}{5670\tau^8} + \frac{6769}{720\tau^{11}} + \frac{2431}{192\tau^{14}} + \frac{715}{128\tau^{17}} \tag{2-21j}$$

$$A_{10}(\tau) = -\frac{2309}{3150\tau^7} - \frac{820573}{113400\tau^{10}} - \frac{37961}{1800\tau^{13}} - \frac{46189}{1920\tau^{16}} - \frac{2431}{256\tau^{19}} \tag{2-21k}$$

$$A_{11}(\tau) = \frac{1}{11\tau^6} + \frac{1057}{330\tau^9} + \frac{263653}{13200\tau^{12}} + \frac{4641}{100\tau^{15}} + \frac{29393}{640\tau^{18}} + \frac{4199}{256\tau^{21}} \tag{2-21l}$$

$$A_{12}(\tau) = -\frac{31907}{41580\tau^8} - \frac{3089423}{267300\tau^{11}} - \frac{18592951}{356400\tau^{14}} - \frac{48347}{480\tau^{17}} - \frac{676039}{7680\tau^{20}} - \frac{29393}{1024\tau^{23}} \tag{2-21m}$$

(Text continues on p. 2-16)

Table 2-3

COEFFICIENTS FOR POLYNOMIAL REPRESENTATION FOR $A_n(\tau)$

m	A_n^m	
	$n = 1$	
1	.100000000000000000000E	1
	$n = 2$	
1	-.500000000000000000000E	+0
	$n = 3$	
1	.500000000000000000000E	+0
2	.333333333333333333333E	+0
	$n = 4$	
1	-.625000000000000000000E	+0
2	-.583333333333333333333E	+0
	$n = 5$	
1	.875000000000000000000E	+0
2	.105000000000000000000E	1
3	.200000000000000000000E	+0
	$n = 6$	
1	-.131250000000000000000E	1
2	-.192500000000000000000E	1
3	-.644444444444444444444E	+0
	$n = 7$	
1	.206250000000000000000E	1
2	.357500000000000000000E	1
3	.168888888888888888888E	1
4	.142857142857142857142E	+0
	$n = 8$	
1	-.335156250000000000000E	1
2	-.670312500000000000000E	1
3	-.407500000000000000000E	1
4	-.692857142857142857142E	+0

m	A_n^m	
	$n = 9$	
1	.558593750000000000000E	1
2	.126614583333333333333E	2
3	.940138888888888888888E	1
4	.240934744268077601410E	1
5	.111111111111111111111E	+0
	$n = 10$	
1	-.949609375000000000000E	1
2	-.240567708333333333333E	2
3	-.210894444444444444444E	2
4	-.723609347442680776009E	1
5	-.733015873015873015871E	+0
	$n = 11$	
1	.164023437500000000000E	2
2	.459265625000000000000E	2
3	.464100000000000000000E	2
4	.199737121212121212121E	2
5	.320303030303030303030E	1
6	.909090909090909090909E	-1
	$n = 12$	
1	-.287041015625000000000E	2
2	-.880259114583333333333E	2
3	-.100722916666666666666E	3
4	-.52168773849607[illegible]98E	2
5	-.115578862701084923306E	2
6	-.767364117364117364117E	+0
	$n = 13$	
1	.507841796875000000000E	2
2	.169280598958333333333E	3
3	.216316145833333333333E	3
4	.131031602132435465766E	3
5	.372195660306771417878E	2
6	.406286343286343286343E	1
7	.769230769230769230769E	-1

Table 2-3 (Cont'd)

m	B_n^m	
	n = 14	
1	-.90686035156250000000E	2
2	-.32646972656250000000E	3
3	-.46077968750000000000E	3
4	-.31964939935064935064E	3
5	-.11105727272727272727E	3
6	-.17180342990342990342E	2
7	-.79739308310736882165E	+0
	n = 15	
1	.[illegible]3496328125000000000E	3
2	.[illegible]7[illegible]3046875000000000E	3
3	.[illegible]7[illegible]1011111111111111111E	3
4	.76235712809644059641 8E	3
5	.31366598973865640531 4E	2
6	.49830089769137388184 6E	1
7	.49830089769137388184 6E	1
8	.66666666666666666666E	-1
	n = 16	
1	-.29586318969726562500 0E	3
2	-.12229011840820312499 6E	4
3	-.20524708355034722221 4E	4
4	-.17857347920236592110 8E	4
5	-.84984627105379188709 5E	3
6	-.21293557969362598991 7E	3
7	-.24235204989014512823 7E	2
8	-.82408147408147408147 4E	+0
	n = 17	
1	.53951522827148437500 0E	3
2	.23738670043945312500 0E	4
3	.43009719238281250000 0E	4
4	.41218220424107142855 2E	4
5	.22287611562500000000 0E	4
6	.66857173231180584119 7E	3
7	.10093180453440117305 4E	3
8	.59586162856751092044 7E	1
9	.58823529411764705882 3E	-1
	n=18	
1	-.98911125183105468750 0E	3
2	-.46158525085449218747 8E	4
3	-.89788421223958333333 3E	4
4	-.93983624385127314809 7E	4
5	-.56935902500643004112 2E	4
6	-.19930080621348463123 1E	4
7	-.37769770414792806464 5E	3
8	-.32847100873789326947 9E	2
9	-.84810708536198732276 8E	+0
	n = 19	
1	.18220470428466796875 0E	4
2	.89887654113769531250 0E	4
3	.18684250651041666666 6E	5
4	.21209592793330439813 3E	5
5	.14233877254565329217 1E	5
6	.57022866320945328976 4E	4
7	.13062754115831898450 6E	4
8	.15271929201962489429 1E	3
9	.69856030150442652284 9E	1
10	.52631578947368421052 6E	-1
	n = 20	
1	-.33707870292663574218 7E	4
2	-.17528092552185058592 6E	5
3	-.38772403857421875000 0E	5
4	-.47443618808593750000 0E	5
5	-.34945928671875000000 0E	5
6	-.15780716617397584032 6E	5
7	-.42516639773894406245 2E	4
8	-.63060087229987075186 3E	3
9	-.43134433346721095504 1E	2
10	-.86995915333376633605 7E	+0

Table 2-3 (Cont'd)

n = 21

m	B_n^m	
1	.626003305435180664063E	4
5	.844860306172839506099E	5
9	.221807719711782510385E	3
2	.342215140304565429660E	5
6	.424838043379530747451E	5
10	.806048826489458882758E	1
3	.802630810485839843680E	5
7	.131854862585951041879E	5
11	.476190476190476190476E	-1
4	.105318880057715360440E	6
8	.238420821587937468108E	4
		+0.0

n = 22

m	B_n^m	
1	-.116664252376556396484E	5
5	-.201567174401580286976E	6
9	-.100291468665130554674E	4
2	-.668875046958923339788E	5
6	-.111744674790466258042E	6
10	-.552100998691229277871E	2
3	-.165799024261474609359E	6
7	-.3931012466645185084099E	5
11	-.890002620202451331069E	+0
4	-.232240243842043011943E	6
8	-.841890054668934984012E	4

n = 23

m	B_n^m	
1	.218111428356170654297E	5
5	.475390678258928571378E	6
9	.411886369631848301327E	4
2	.130866857013702392566E	6
6	.288142979686010313186E	6
10	.311546186453569473920E	3
3	.341843103881835937462E	6
7	.113410197140524639597E	6
11	.918026828171452694853E	1
4	.509116055648922610531E	6
8	.281456646361638824524E	5
12	.434782608695652173913E	-1

n = 24

m	B_n^m	
1	-.408958928167819976607E	5
5	-.110991397644613217352E	7
9	-.156932087522754605138E	5
2	-.256260928318500518773E	6
6	-.730359011452684181048E	6
10	-.153256841610405073658E	
3	-.703621666968027750568E	6
7	-.318224546027872606852E	6
11	-.691820872483155839496E	2
4	-.111027728216546349007E	7
8	-.899677545873361404822E	5
12	-.908517103462587376971E	+0

n = 25

m	B_n^m	
1	.768842784955501556397E	5
5	.256827714614291433175E	7
9	.562917033135728290407E	5
13	.400000000000000000000E	-1
2	.502310619504261016790E	6
6	.182371638640773763620E	7
10	.679950468032934688407E	4
3	.144608711121365865053E	7
7	.871902051762040236717E	6
11	.425547281640320188148E	3
4	.241003322866257422826E	7
8	.276997776726792547006E	6
12	.103423234102239381071E	2

n = 26

m	B_n^m	
1	-.144897294087767601013E	6
5	-.589565132268546919298E	7
9	-.192132790058221428422E	6
13	-.925721666221164031195E	+0
2	-.985301599796819686773E	6
6	-.449413675579853634576E	7
10	-.278221472923467098508E	5
3	-.296795293913681030231E	7
7	-.234005721157937292219E	7
11	-.226479555810393823808E	4
4	-.520950635707243485804E	7
8	-.826105244679143668050E	6
12	-.851539687318596132586E	2

Table 2-3 (Cont'd)

n = 27

m	B_n^m	
1	.273694888832449913025E	6
5	.134374505289727923837E	8
9	.629008669227862490990E	6
13	.115443475348750467116E	2
2	.193411054774931271846E	7
6	.109458011043905248659E	8
10	.106640162339384200720E	6
14	.370370370370370370370E	-1
3	.608390448695458306109E	7
7	.616771069907858162333E	7
11	.108034426488708692999E	5
4	.112183141044618414315E	8
8	.239715060246664417853 1E	7
12	.567681032941667107411E	3

n = 28

m	B_n^m	
1	-.518065325289994478226E	6
5	-.304300245406774467754E	8
9	-.198742410461970741227E	7
13	-.103225324428571140090E	3
2	-.379914571879329283979E	7
6	-.263816640420520492100E	8
10	-.387252737866785193645E	6
14	-.941791185622695343281E	+0
3	-.124571514935259289188E	8
7	-.159984327319558255999E	8
11	-.472786392241173595010E	5
4	-.240750657777134261851E	8
8	-.679209804635185435452E	7
12	-.325277504802706190441E	4

n = 29

m	B_n^m	
1	.982537685894817113876E	6
5	.685094351527194882206E	8
9	.609002390690492040579E	7
13	.742069190714986568101E	3
2	.746728641280061006436E	7
6	.629896370727299426884E	8
10	.134377283861388064618E	7
14	.127842937775186016728E	2
3	.254805103662321567484E	8
7	.409120542583336357359E	8
11	.192941280292826168173E	6
15	.344827586206896551724E	-1
4	.515047991621348825983E	8
8	.188461235530006903840E	8
12	.166129090115963891564E	5

n = 30

m	B_n^m	
1	-.186682160320015251636E	7
5	-.153422227089157018162E	9
9	-.181695120445933290557E	8
13	-.455826866225021524436E	4
2	-.146856632785078664597E	8
6	-.149121739515205056488E	9
10	-.448529916041438925783E	7
14	-.123492100520371284947E	3
3	-.520699807126342296485E	8
7	-.103297458045362313138E	9
11	-.743068151696005201308E	6
15	-.956867527724712135595E	+0
4	-.109871013399435436176E	9
8	-.513323141719165900825E	8
12	-.774738326124100379836E	5

Numerical values for the coefficients which appear in Eqs. (2-21b) through (2-21m) are given in Table 2-3. We have expressed the explicit dependence of the $A_n(\tau)$ upon τ in the following manner:

$$A_n(\tau) = \frac{1}{\tau^{2n-1}} \sum_{m=1}^{M} A_n^m \, \tau^{m-1} \tag{2-21n}$$

where M is the "integral part" of $[(n + 1)/2]$. The coefficients in Table 2-3 have been obtained by means of the NPREC subroutines which have been discussed in the Preface. The reader is advised to be very careful if he employs these tables for $n > 20$ since the format employed in the computer output has resulted in a table in which m successively takes on the values 1,5,9,... in the order listed under the heading "m".

When the magnitude of τ is very large, the dominant terms in the expansion will be the terms

$$A_1(\tau) = \frac{1}{\tau}$$

$$A_3(\tau) = \frac{1}{3\tau^2} + \cdots$$

$$A_5(\tau) = \frac{1}{5\tau^3} + \cdots$$

We recognize these coefficients to be the coefficients in the well known expansion

$$\frac{\tanh^{-1}x}{x} = \frac{\coth^{-1}(1/x)}{x} = \frac{1}{2x}\left[\log(1 + x) - \log(1 - x)\right]$$

$$= 1 + \frac{1}{3}x^2 + \frac{1}{5}x^4 + \frac{1}{7}x^6 + \frac{1}{9}x^8 + \ldots, \quad x^2 < 1$$

Therefore, it appears that for very large values of τ_s^o that the series behaves like the series expansion of

$$t_s^o + (1/\sqrt{t_s^o})\tanh^{-1}(q/\sqrt{t_s^o})$$

Therefore, in order for the series in Eq. (2-21) to converge, we must require that the magnitude of $(q/\sqrt{\tau_s^o})$ be less than unity.* More precise definitions of the radius of convergence of the series in Eq. (2-21) will be possible after we discuss the roots t_s^c for which $\tau_s(q_c) = q_c^2$. We will postpone this discussion until after we have considered another classical expansion for $\tau_s(q)$

For large values of q, we can express $\tau_s(q)$ in the form

$$\tau_s(q) = \tau_s^\infty + \sum_{n=1}^{\infty} B_n(\tau_s^\infty)\, q^{-n} \qquad (2\text{-}22a)$$

where

$$B_1(\tau) = 1 \qquad (2\text{-}22b)$$

$$B_2(\tau) = 0 \qquad (2\text{-}22c)$$

$$B_3(\tau) = \frac{\tau}{3} \qquad (2\text{-}22d)$$

$$B_4(\tau) = \frac{1}{4} \qquad (2\text{-}22e)$$

$$B_5(\tau) = \frac{\tau^2}{5} \qquad (2\text{-}22f)$$

$$B_6(\tau) = \frac{7}{18}\tau \qquad (2\text{-}22g)$$

$$B_7(\tau) = \frac{1}{7}\tau^3 + \frac{5}{28} \qquad (2\text{-}22h)$$

$$B_8(\tau) = \frac{29}{60}\tau^2 \qquad (2\text{-}22i)$$

$$B_9(\tau) = \frac{1}{9}\tau^4 + \frac{41}{81}\tau \qquad (2\text{-}22j)$$

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*The author has not found the time to investigate the possible advantages of rearranging the series in Eq. (2-21) so that it is of the form

$$\tau_s(q) = \tau_s^o + (1/\sqrt{\tau_s^o})\tanh^{-1} z + \sum_{n=2}^{\infty} C_n(\tau_s^o)\, z^n$$

where $z = q/\sqrt{\tau_s^o}$.

$$B_{10}(\tau) = \frac{97}{175}\tau^3 + \frac{423}{2520} \qquad (2\text{-}22k)$$

$$B_{11}(\tau) = \frac{1}{11}\tau^5 + \frac{947}{990}\tau^2 \qquad (2\text{-}22l)$$

$$B_{12}(\tau) = \frac{2309}{3780}\tau^4 + \frac{18842}{27216}\tau \qquad (2\text{-}22m)$$

$$B_{13}(\tau) = \frac{1}{13}\tau^6 + \frac{247464}{163800}\tau^3 + \frac{11781}{65520} \qquad (2\text{-}22n)$$

Numerical values for the coefficients which appear in Eqs. (2-22b) through (2-22n) are given in Table 2-4. We have expressed the explicit dependence of the $B_n(\tau)$ upon τ in the following manner:

$$B_n(\tau) = \sum_{m=1}^{M} B_n^m \tau^{m-1} \qquad (2\text{-}22o)$$

where M is the "integral part" of [(n + 1)/2]. This definition is quite "wasteful" since the series actually progress in successive power of τ^3. However, the "book-keeping" is somewhat easier if we use the form in Eq. (2-22o) so as to have a single form which is useful for all n instead of having one definition for n = 1,4,7,10,..., another for n = 2,5,8,11,... and a third definition for n = 3,6,9,12,....

When the magnitude of τ is very large, the dominant terms in the expansion will be the terms

$$B_1(\tau) = \tau$$

$$B_3(\tau) = \frac{1}{3}\tau^3 + \cdots$$

$$B_5(\tau) = \frac{1}{5}\tau^5 + \cdots$$

$$B_7(\tau) = \frac{1}{7}\tau^7 + \cdots$$

(Text continues on p. 2-29)

Table 2-4

COEFFICIENTS FOR POLYNOMIAL REPRESENTATION FOR $B_n(\tau)$

m	B_n^m
	n = 1
1	.10000000000000000000E 1
	n = 2
1	+0.0
	n = 3
1	+0.0
2	.33333333333333333333E +0
	n = 4
1	.25000000000000000000E +0
2	+0.0
	n = 5
1	+0.0
2	+0.0
3	.19999999999999999999E +0
	n = 6
1	+0.0
2	.38888888888888888888E +0
3	+0.0
	n = 7
1	.17857142857142857142 8E +0
2	+0.0
3	+0.0
4	.14285714285714285714 2E +0
	n = 8
1	+0.0
2	+0.0
3	.48333333333333333333E +0
4	+0.0
	n = 9
1	+0.0
2	.50617283950617283950 6E +0
3	+0.0
4	+0.0
5	.11111111111111111111E +0
	n = 10
1	.16785,142857142857142E +0
2	+0.0
3	+0.0
4	.554285714285,14285714E +0
5	+0.0
	n = 11
1	+0.0
2	+0.0
3	.9565656565656565655E +0
4	+0.0
5	+0.0
6	.90909090909090909090909E -1
	n = 12
1	+0.0
2	.69231334509112286899E +0
3	+0.0
4	+0.0
5	.61084656084656084656 0E +0
6	+0.0
	n = 13
1	.17980769230769230769 2E +0
2	+0.0
3	+0.0
4	.15107692307692307692 3E 1
5	+0.0
6	+0.0
7	.76923076923076923076 9E -1

Table 2-4 (Cont'd)

n = 14

m	B_n^m
1	+0.0
2	+0.0
3	.175694415813463432510E 1
4	+0.0
5	+0.0
6	.657740672026386312099E +0
7	+0.0

n = 15

m	B_n^m
1	+0.0
2	.974186262982559278853E +0
3	+0.0
4	+0.0
5	.215486067019400352733E 1
6	+0.0
7	+0.0
8	.666666666666666666666E -1

n = 16

m	B_n^m
1	.208001373626373626373E +0
2	+0.0
3	+0.0
4	.353486679986679986679E 1
5	+0.0
6	+0.0
7	.697718947718947718946E +0
8	+0.0

n = 17

m	B_n^m
1	+0.0
2	+0.0
3	.310391373159813869430E 1
4	+0.0
5	+0.0
6	.287828715324513643840E 1
7	+0.0
8	+0.0
9	.588235294117647058823E -1

n = 18

m	B_n^m
1	+0.0
2	.139632680457783338441E 1
3	+0.0
4	+0.0
5	.618831535268572305608E 1
6	+0.0
7	+0.0
8	.732516865850199183530E +0
9	+0.0

n = 19

m	B_n^m
1	.253145914236139800049E +0
2	+0.0
3	+0.0
4	.758753905242627047136E 1
5	+0.0
6	+0.0
7	.367279988182243821340E 1
8	+0.0
9	+0.0
10	.526315789473684210526E -1

n = 20

m	B_n^m
1	+0.0
2	+0.0
3	.535450640036152018721E 1
4	+0.0
5	+0.0
6	.986945583769897495385E 1
7	+0.0
8	+0.0
9	.763296376825788590492E +0
10	+0.0

Table 2-4 (Cont'd)

n = 21

m	B_n^m
1	+0.0
2	.202745417063871194494E 1
3	+0.0
4	+0.0
5	.157440383904422705127E 2
6	+0.0
7	+0.0
8	.453178007010206556690E 1
9	+0.0
10	+0.0
11	.476190476190476190476E -1

n = 22

m	B_n^m
1	.319402531209392111647E +0
2	+0.0
3	+0.0
4	.154052789982054989492E 2
5	+0.0
6	+0.0
7	.147214497607521611381E 2
8	+0.0
9	+0.0
10	.790871957576151215327E +0
11	+0.0

n = 23

m	B_n^m
1	+0.0
2	+0.0
3	.908978191389067118850E 1
4	+0.0
5	+0.0
6	.291909600070679138053E 2
7	+0.0
8	+0.0
9	.544979968294424511692E 1
10	+0.0
11	+0.0
12	.434782608695652173913E -1

n = 24

m	B_n^m
1	+0.0
2	.297219554537243524432E 1
3	+0.0
4	+0.0
5	.369624356174421784159E 2
6	+0.0
7	+0.0
8	.208794775752038891212E 2
9	+0.0
10	+0.0
11	.815835735185580386817E +0
12	+0.0

n = 25

m	B_n^m
1	.414012988280516476003E +0
2	+0.0
3	+0.0
4	.300837610574237167456E 2
5	+0.0
6	+0.0
7	.498402656363921738934E 2
8	+0.0
9	+0.0
10	.642232530380461706099E 1
11	+0.0
12	+0.0
13	.400000000000000000000E -1

n = 26

m	B_n^m
1	+0.0
2	+0.0
3	.152534220791393930795E 2
4	+0.0
5	+0.0
6	.761166640397124134664E 2
7	+0.0
8	+0.0
9	.284716455967603149469E 2
10	+0.0
11	+0.0
12	.838631172427003732591E +0
13	+0.0

Table 2-4 (Cont'd)

n = 27

m	B_n^m
1	+0.0
2	.438946419058804238827E 1
3	+0.0
4	+0.0
5	.819389915648406185682E 2
6	+0.0
7	+0.0
8	.798931574591626546954E 2
9	+0.0
10	+0.0
11	.744551208285921380796E 1
12	+0.0
13	+0.0
14	.370370370370370370370E -1

n = 28

m	B_n^m
1	.547994727446052106553E +0
2	+0.0
3	+0.0
4	.570809946063368612048E 2
5	+0.0
6	+0.0
7	.150240129621784630233E 3
8	+0.0
9	+0.0
10	.376197673481907091305E 2
11	+0.0
12	+0.0
13	.859598690062509457541E +0
14	+0.0

n = 29

m	B_n^m
1	+0.0
2	+0.0
3	.253744437799965565890E 2
4	+0.0
5	+0.0
6	.194421833629856487095E 3
7	+0.0
8	+0.0
9	.121834083587262924343E 3

n = 29 (Cont'd)

m	B_n^m
10	+0.0
11	+0.0
12	.851605626923321082365E 1
13	+0.0
14	+0.0
15	.344827586206896551724E -1

n = 30

m	B_n^m
1	+0.0
2	.652069314549479024577E 1
3	+0.0
4	+0.0
5	.173986394824110754194E 3
6	+0.0
7	+0.0
8	.268603424056942921449E 3
9	+0.0
10	+0.0
11	.484400309450673263990E 2
12	+0.0
13	+0.0
14	.879005106581182320399E +0
15	+0.0

n = 31

m	B_n^m
1	.737560496541037550570E +0
2	+0.0
3	+0.0
4	.105931337154864142296E 3
5	+0.0
6	+0.0
7	.415917190062626618097E 3
8	+0.0
9	+0.0
10	.178424970411309017669E 3
11	+0.0
12	+0.0
13	.963108702619458955029E 1
14	+0.0
15	+0.0
16	.322580645161290322580E -1

Table 2-4 (Cont'd)

m	B_n^m, n = 32
1	+0.0
2	+0.0
3	.41924823166894404386E 2
4	+0.0
5	+0.0
6	.45795864028248810586 0E 3
7	+0.0
8	+0.0
9	.45300040334300482172 5E 3
10	+0.0
11	+0.0
12	.61043568909858093206 0E 2
13	+0.0
14	+0.0
15	.89706330724191762712 4E +0
16	+0.0

m	B_n^m, n = 33
1	+0.0
2	.97330541062746599701 7E 1
3	+0.0
4	+0.0
5	.35724643093490827057 4E 3
6	+0.0
7	+0.0
8	.82051430433305492515 8E 3
9	+0.0
10	+0.0
11	.25269916625353609904 1E 3
12	+0.0
13	+0.0
14	.10788065322958948605 1E 2
15	+0.0
16	+0.0
17	.30303030303030303030 3E -1

m	B_n^m, n = 34
1	.10063566436218606220 3E 1
2	+0.0
3	+0.0
4	.19315907481137165490 2E 3
5	+0.0
6	+0.0
7	.10786215300780753327 1E 4
8	+0.0
9	+0.0
10	.72837770410598695873 6E 3
11	+0.0
12	+0.0
13	.75536946274867306754 6E 2
14	+0.0
15	+0.0
16	.91394576031068786930 1E +0
17	+0.0

m	B_n^m, n = 35
1	+0.0
2	+0.0
3	.66893075841245259571 4E 2
4	+0.0
5	+0.0
6	.10330010103579762234 9E 4
7	+0.0
8	+0.0
9	.15164566894809756879 6E 4
10	+0.0
11	+0.0
12	.34795562755234278393 1E 3
13	+0.0
14	+0.0
15	.11984821953929999787 2E 2
16	+0.0
17	+0.0
18	.28571428571428571428 5E -1

Table 2-4 (Cont'd)

n = 36

m	B_n^m
1	+0.0
2	.145855972182003225723E 2
3	+0.0
4	+0.0
5	.714072457090325560480E 3
6	+0.0
7	+0.0
8	.232746875958386466983E 4
9	+0.0
10	+0.0
11	.112546659995619454984E 4
12	+0.0
13	+0.0
14	.920225807542393257765E 2
15	+0.0
16	+0.0
17	.929794173880234481875E +0
18	+0.0

n = 37

m	B_n^m
1	.138882245330245081978E 1
2	+0.0
3	+0.0
4	.347208602601564697373E 3
5	+0.0
6	+0.0
7	.265621102200827879020E 4
8	+0.0
9	+0.0
10	.265600600709386626111E 4
11	+0.0
12	+0.0
13	.467753264383959057265E 3
14	+0.0
15	+0.0
16	.132193093687430132033E 2
17	+0.0
18	+0.0
19	.270270270270270270270E -1

n = 38

m	B_n^m
1	+0.0
2	+0.0
3	.112703307190175578946E 3
4	+0.0
5	+0.0
6	.225003818225715207750E 4
7	+0.0
8	+0.0
9	.467809904302998191936E 4
10	+0.0
11	+0.0
12	.168141547349817696730E 4
13	+0.0
14	+0.0
15	.110599106464364087814E 3
16	+0.0
17	+0.0
18	.944726195[illegible]7637E +0
19	+0.0

n = 39

m	B_n^m
1	+0.0
2	.219305190621489160779E 2
3	+0.0
4	+0.0
5	.139619178164647016059E 4
6	+0.0
7	+0.0
8	.622326727508128649438E 4
9	+0.0
10	+0.0
11	.444676141484563893247E 4
12	+0.0
13	+0.0
14	.615905494956780323200E 3
15	+0.0
16	+0.0
17	.144897636706678639639E 2
18	+0.0
19	+0.0
20	.256410256410256410256E -1

Table 2-4 (Cont'd)

n = 40

m	B_n^m
1	.193517278781114409888E 1
2	+0.0
3	+0.0
4	.616753112414319115503E 3
5	+0.0
6	+0.0
7	.627188637079385633404E 4
8	+0.0
9	+0.0
10	.886558750601482211301E 4
11	+0.0
12	+0.0
13	.244042162008581973662E 4
14	+0.0
15	+0.0
16	.131361690592345907804E 3
17	+0.0
18	+0.0
19	.958840690052710827498E +0
20	+0.0

n = 41

m	B_n^m
1	+0.0
2	+0.0
3	.183685654290390146058E 3
4	+0.0
5	+0.0
6	.476162529966723145118E 4
7	+0.0
8	+0.0
9	.135156741022118691668E 5
10	+0.0
11	+0.0
12	.716446913995236862058E 4
13	+0.0
14	+0.0
15	.796475021376847604672E 3
16	+0.0
17	+0.0
18	.157945742333268717849E 2
19	+0.0
20	+0.0
21	.243902439024390243902E -1

n = 42

m	B_n^m
1	+0.0
2	.330683222322485498817E 2
3	+0.0
4	+0.0
5	.268014536559658499441E 4
6	+0.0
7	+0.0
8	.158556403113381624728E 5
9	+0.0
10	+0.0
11	.159883034290492282070E 5
12	+0.0
13	+0.0
14	.345436142082464983944E 4
15	+0.0
16	+0.0
17	.154402310664294965345E 3
18	+0.0
19	+0.0
20	.972221447916023050730E +0
21	+0.0

n = 43

m	B_n^m
1	.271877114305880366205E 1
2	+0.0
3	+0.0
4	.108464585419184464071E 4
5	+0.0
6	+0.0
7	.143019526365464376930E 5
8	+0.0
9	+0.0
10	.275568506686418727247E 5
11	+0.0
12	+0.0
13	.111673514762345894471E 5
14	+0.0
15	+0.0
16	.101376882875827566633E 4
17	+0.0
18	+0.0
19	.171322791153042944977E 2
20	+0.0

Table 2-4 (Cont'd)

$n = 43$ (Cont'd)

m	B_n^m
21	+0.0
22	.232558139534883720930E -1

$n = 44$

m	B_n^m
1	+0.0
2	+0.0
3	.298427687071651121298E 3
4	+0.0
5	+0.0
6	.983584472976167845907E 4
7	+0.0
8	+0.0
9	.370022830041062198861E 5
10	+0.0
11	+0.0
12	.276356406029566034571E 5
13	+0.0
14	+0.0
15	.478341816034410585767E 4
16	+0.0
17	+0.0
18	.179809998641154417543E 3
19	+0.0
20	+0.0
21	.984940096971362322405E +0
22	+0.0

$n = 45$

m	B_n^m
1	+0.0
2	.499857133083891529330E 2
3	+0.0
4	+0.0
5	.506524962914818236122E 4
6	+0.0
7	+0.0
8	.388043797141578031557E 5
9	+0.0
10	+0.0
11	.532587875016492366581E 5
12	+0.0
13	+0.0

$n = 45$ (Cont'd)

m	B_n^m
14	.169119797908843669737E 5
15	+0.0
16	+0.0
17	.127233340371341982467E 4
18	+0.0
19	+0.0
20	.185015449499883776497E 2
21	+0.0
22	+0.0
23	.222222222222222222222E -1

$n = 46$

m	B_n^m
1	.384704424470165471650E 1
2	+0.0
3	+0.0
4	.169126854219205074072E 4
5	+0.0
6	+0.0
7	.316688065441289280403E 5
8	+0.0
9	+0.0
10	.807649034635335018204E 5
11	+0.0
12	+0.0
13	.460452673601165755358E 5
14	+0.0
15	+0.0
16	.649570694340746828981E 4
17	+0.0
18	+0.0
19	.207671056924903042637E 3
20	+0.0
21	+0.0
22	.997058337716530535067E +0
23	+0.0

Table 2-4 (Cont'd)

n = 47

m	B_n^m
1	+0.0
2	+0.0
3	.483529865436153278786E 3
4	+0.0
5	+0.0
6	.199031654942886173133E 5
7	+0.0
8	+0.0
9	.968418836300227247948E 5
10	+0.0
11	+0.0
12	.983113047163785843434E 5
13	+0.0
14	+0.0
15	.249707122714824143694E 5
16	+0.0
17	+0.0
18	.157695016478891751363E 4
19	+0.0
20	+0.0
21	.199011503341809346990E 2
22	+0.0
23	+0.0
24	.212765957446808510638E -1

n = 48

m	B_n^m
1	+0.0
2	.757206073914681944667E 2
3	+0.0
4	+0.0
5	.944564460544556467437E 4
6	+0.0
7	+0.0
8	.917895897862532218079E 5
9	+0.0
10	+0.0
11	.166540147102074271595E 6
12	+0.0
13	+0.0
14	.742936306917470837712E 5
15	+0.0
16	+0.0

n = 48 (Cont'd)

m	B_n^m
17	.867289629986047549843E 4
18	+0.0
19	+0.0
20	.238069250441722021130E 3
21	+0.0
22	+0.0
23	.100862968802923102630E 1
24	+0.0

n = 49

m	B_n^m
1	.547765917375865581324E 1
2	+0.0
3	+0.0
4	.327348912364216639674E 4
5	+0.0
6	+0.0
7	.683852429721010508636E 5
8	+0.0
9	+0.0
10	.225309518820853378955E 6
11	+0.0
12	+0.0
13	.174375106901002339348E 6
14	+0.0
15	+0.0
16	.360507138971250177468E 5
17	+0.0
18	+0.0
19	.193263109553788477056E 4
20	+0.0
21	+0.0
22	.213299719840503289192E 2
23	+0.0
24	+0.0
25	.204081632653061224489E -1

Table 2-4 (Cont'd)

m	B_n^m n = 50
1	+0.0
2	+0.0
3	.761599207896086962964E 3
4	+0.0
5	+0.0
6	.395661566842224857113E 5
7	+0.0
8	+0.0
9	.243942069314354378123E 6
10	+0.0
11	+0.0
12	.326988480982487738768E 6
13	+0.0
14	+0.0
15	.116523564381858609189E 6
16	+0.0
17	+0.0
18	.114008261711473815314E 5
19	+0.0
20	+0.0
21	.271085978235346670236E 3
22	+0.0
23	+0.0
24	.101970086223180909526E 1
25	+0.0

[Note added in proof:- The user who wishes to conserve storage in an electronic computer may find it convenient to replace Eq. (2-22o) by

$$B_n(\tau) = \tau^{\mu} \sum_{j=1}^{K} C_j^n \, \tau^{3(j-1)}$$

where, if we let [...] be interpreted as "the integral part of",

$$\mu = (2n + 1) - 3[(2n + 1)/3] \quad , \quad K = [(2n + 1)/3] - [n/2]$$

For example, if n = 10, we would have $\mu = 0$, K = 2; if n = 11, we would have $\mu = 2$, K = 2; and if n = 12, we would have $\mu = 1$, K = 2.]

As in the discussion following Eq. (2-21), we recognize the coefficionts in the expansion of $x^{-1}\tanh^{-1}x$ and observe that the behavior of $\tau_s(q)$ for the cases in which the magnitude of τ_s^∞ is large is given by

$$\tau_s(q) \approx \tau_s^\infty + \frac{1}{\sqrt{\tau_s^\infty}} \tanh^{-1}(\sqrt{\tau_s^\infty}/q) \quad , \quad \left|\sqrt{\tau_s^\infty}/q\right| < 1$$

Since this result suggests that we must require that the magnitude of $\sqrt{\tau_s^\infty}/q$ be less than unity in order to employ the expansion in Eq. (2-22), we see that as the magnitude of τ_s^∞ increases, the region in the q-plane for which this expansion can be employed "shrinks" until it coalesces upon the "point at infinity." This is in sharp contrast with the expansion in Eq. (2-21) since it can be employed in ever enlarging regions in the q-plane as the magnitude of τ_s^o increases.

We observe that the roots for $q = 0$ and $q \to \infty$ lie along the line $\arg(q) = 60^\circ$, and lie in the first quadrant of the τ-plane. Much attention has been given in the literature to the study of vertically-polarized waves propagated over the convex interface between free space and a highly conducting homogeneous sphere. In this case, the impedance parameter has a phase of the order of 45°. We will refer to these as the "Watson modes" since this was the type of problem in which Watson (and later van der Pol and Bremmer) was interested nearly fifty years ago. Since World War II there has developed an interest in the propagation of vertically-polarized electromagnetic waves over the convex surface of a perfect conductor which is either

a) slightly rough
b) covered with a thin dielectric layer, or
c) corrugated.

These modes were discussed in a paper by Elliott (Ref. 12) and we shall therefore refer to these as the "Elliot modes." The very important problem of absorbing layers upon a convex surface can be

often shown to correspond to the case in which the impedance parameter q has a phase of the order of 90°. Therefore, the applications which are being encountered are such that it becomes quite important to be able to find the roots $\tau_s(q)$ for an arbitrary value of q. Therefore, it has become important to study the behavior of the logarithmic derivative $q(\tau)$ in order to understand the mapping specified by the functions $\tau_s(q)$ and $q(\tau)$. The mapping is found to possess certain critical values $\tau_s^c(q)$ for which

$$\tau_s^c = q_c^2 = [w_1'(\tau_s^c)/w_1(\tau_s^c)]^2 \qquad (2\text{-}23)$$

We often refer to these points as the "saddle points." It will be observed that these points are singularities for the differential equation for dt/dq given in Eq. (2-18)

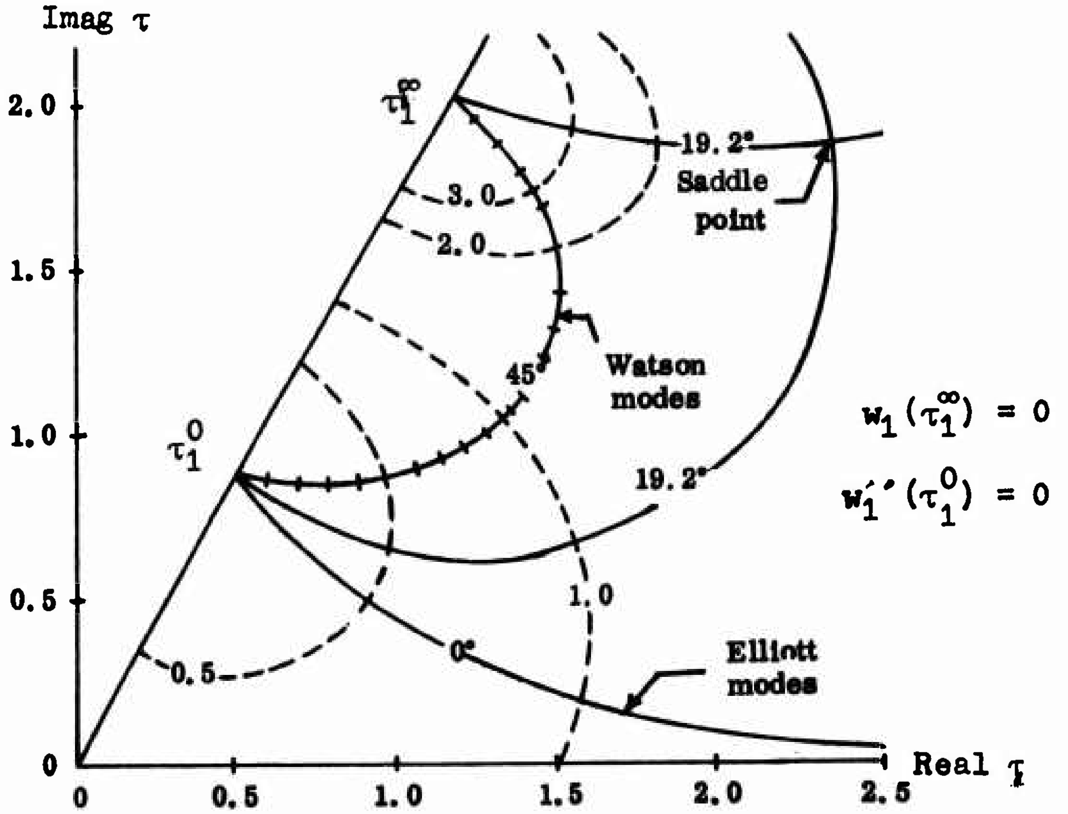

Fig. 2-1 The Logarithmic Derivative $q(\tau) = w_1'(\tau)/w_1(\tau)$

(——— phase , - - - - - modulus)

In Fig. 2-1, we have attempted to illustrate some of these properties of the logarithmic derivative function $q(\tau)$, with emphasis upon illustrating the saddle point, the Watson modes, and the Elliott modes. A more detailed illustration is given in Fig. 2-2

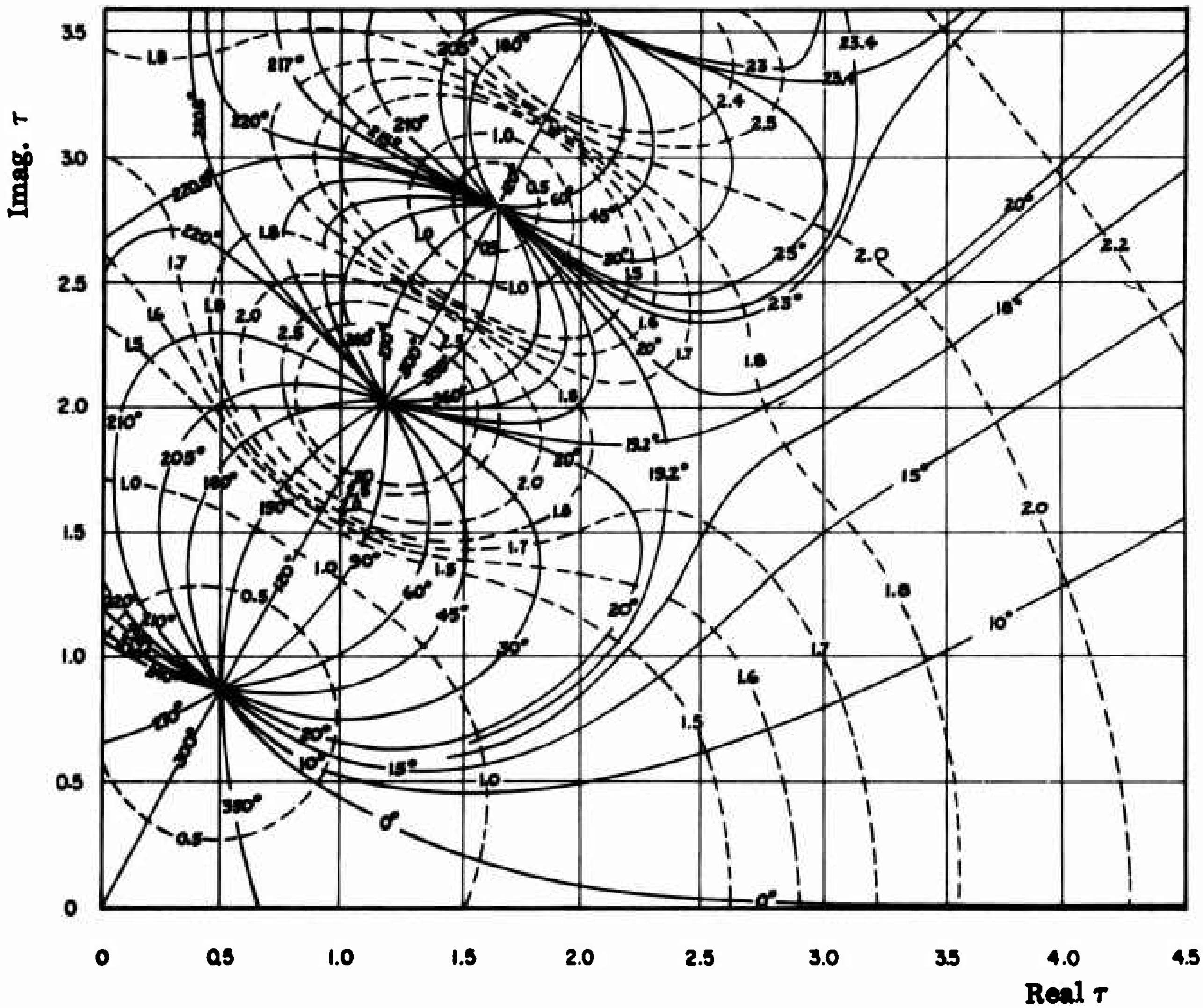

Fig. 2-2 The Logarithmic Derivative $q(\tau) = w_1'(\tau)/w_1(\tau)$

(——— phase , — — — — — modulus)

The critical points τ_s^c correspond to points at which two modes coalesce. They are very important in determining the radius of convergence (in the q-plane) of the series which we have given in Eqs. (2-21) and (2-22). The first two of these points have been located at

$$q_c \approx 1.731\exp(i19.29^{\circ}) \quad , \quad q_c \approx 2.135\exp(i23.52^{\circ})$$

The approximate location of the first 20 saddle points (or critical points) is given in Table 2-5. For $s > 3$, these critical points can be found from an asymptotic representation. From the definition given in Eq. (2-23), we see that the location of the critical points is equivalent to the problem of finding the roots of the function $f(\tau) = [w_1'(\tau)]^2 - \tau[w_1(\tau)]^2$. We can show that

Table 2-5

THE CRITICAL POINTS τ_s^c AND CORRESPONDING VALUES OF q_c

s	Real τ_s^c	Imag τ_s^c	Mod q_c	Arg q_c (deg.)
1	2.343	1.869	1.731	19.29
2	3.118	3.347	2.135	23.52
3	3.771	4.573	2.433	25.25
4	4.353	5.658	2.672	26.21
5	4.898	6.658	2.875	26.83
6	5.409	7.591	3.053	27.264
7	5.894	8.472	3.213	27.586
8	6.358	9.311	3.358	27.836
9	6.804	10.114	3.491	28.036
10	7.234	10.888	3.616	28.200
11	7.651	11.636	3.732	28.337
12	8.056	12.361	3.841	28.453
13	8.450	13.066	3.945	28.553
14	8.835	13.752	4.043	28.640
15	9.212	14.423	4.137	28.717
16	9.580	15.078	4.227	28.785
17	9.941	15.719	4.313	28.845
18	10.295	16.348	4.396	28.900
19	10.643	16.966	4.475	28.949
20	10.986	17.572	4.552	28.994

$f(\tau)$ is a solution of the differential equation

$$\frac{d^3f}{d\tau^3} - 4\tau \frac{df}{d\tau} + 2f = 0$$

This differential equation can be used to generate a Taylor series for $f(\tau)$. If we then use the fact that $f(\tau_s^c) = 0$, we see that the finding of the roots is reduced to the problem of inverting the series

$$f(\tau_s^c) = 0 = f(\tau) + f'(\tau)(\tau_s^c - \tau) + \sum_{n=2}^{\infty} \frac{f^{(n)}(\tau)}{n!}(\tau_s^c - \tau)^n$$

Since $f'(\tau) = -[w_1(\tau)]^2$ and $f''(\tau) = -2w_1(\tau)w_1'(\tau)$, it is a rather straight foward calculation to find a more accurate value of τ_s^c once we know $w_1(\tau)$ and $w_1'(\tau)$ at a neighboring point τ.

Our study of the asymptotic location of the τ_s^c for $s \to \infty$ has indicated that they can be represented in the form

$$\tau_s^c = S(s)\exp[i(\tfrac{1}{3}\pi - T(s)] \qquad (2\text{-}24a)$$

where

$$S(s) = (3/4)(X^2 + Y^2)^{\frac{1}{3}} \quad , \quad T(s) = \tfrac{2}{3}\tan^{-1}(Y/X)$$

where $X = X(s)$ and $Y = Y(s)$ possess the asymptotic behavior

$$X(s) \xrightarrow[s \to \infty]{} 2\pi s - \frac{\log(12\pi s)}{2\pi s} \qquad (2\text{-}24b)$$

$$Y(s) \xrightarrow[s \to \infty]{} \log[12\pi s - \frac{3}{\pi s}\log(12\pi s)] \qquad (2\text{-}24c)$$

If we study the behavior of the lines of constant phase in Fig. 2-2 we observe that when $30^\circ < \arg q < 210^\circ$ that all the root loci which start from τ_s^0 move along paths that end up at τ_s^∞. However, for $-150^\circ < \arg q < 30^\circ$ there is a type of root loci which starts from τ_s^0 but which "slips away" and runs off to infinity. We will designate this root as $\tau_0(q)$. From Fig. 1, we see that the root $\tau_0(q)$ corresponds to $\tau_1(q)$ for $-139.2^\circ < \arg q < 19.2^\circ$, to $\tau_2(q)$ for $-143.4^\circ < \arg q < -139.2^\circ$ and $19.2^\circ \leq \arg q < 23.4^\circ$, etc.

For the special case of the surface wave modes which we have designated as the "Elliott modes," the impedance parameter is characterized by arg $q = 0^\circ$. For this case we can compute $\tau_0(q)$ for $q >> 2$ by means of the relations

$$\text{Real } \tau_0(q) \approx q^2 + \frac{1}{2}\frac{1}{q} + \frac{1}{8}\frac{1}{q^4} + \frac{5}{32}\frac{1}{q^7} + \frac{11}{32}\frac{1}{q^{10}} + \frac{539}{512}\frac{1}{q^{13}} + \frac{8337}{2048}\frac{1}{q^{16}}$$

$$+ \frac{9659}{512}\frac{1}{q^{19}} + \frac{416349}{4096}\frac{1}{q^{22}} + O(q^{-25}) \qquad \text{(2-25a)}$$

$$\text{Imag } \tau_0(q) \approx 2q^2 \exp\left[-\frac{4}{3}q^3 - 1 - \frac{7}{12q^3} - \frac{31}{48q^6}\right.$$

$$\left. - \frac{397}{288q^9} - \frac{6427}{1536q^{12}} - \frac{248025}{15360q^{15}} - \cdots\right] \qquad \text{(2-25b)}$$

The asymptotic expansions which we have given in Eq. (2-25) for the case of q real (i.e., arg $q = 0^\circ$) can also be employed to obtain asymptotic series for $\tau_0(q)$ for $-150^\circ <$ arg $q < 30^\circ$ if we write

$$\tau_0(q) \approx [q^2 + \frac{1}{2}\frac{1}{q} + \cdots] + i2q^2 \exp[-\frac{4}{3}q^3 - 1 - \cdots] \qquad \text{(2-26)}$$

where further terms in the expansion can be obtained from Eqs. (2-25a) and (2-25b) where now the values of q are complex.

In Fig. 2-3 we have indicated the regions in which one can employ the representations for $\tau_s(q)$ which we have given in Eqs. (2-21), (2-22) and (2-26). The boundary for Eq. (2-26) has been drawn with dashed lines because this is an asymptotic representation and the expansion does not possess an exact boundary outside of which it can be employed. It is important to observe that there

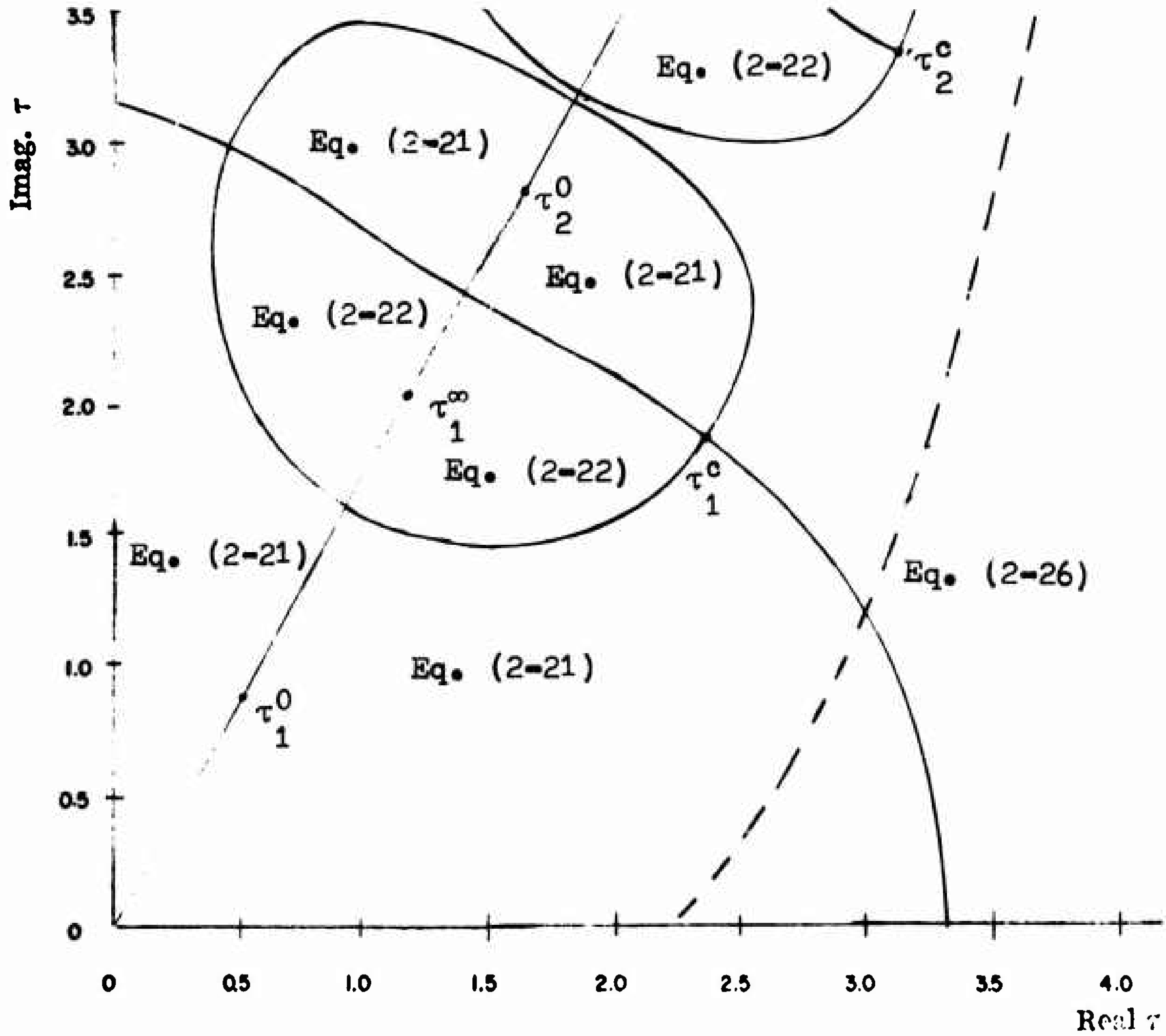

Fig. 2-3 Boundaries for Regions of Applicability of Eqs. (2-21), (2-22) and (2-26)

are root loci (computed on the basis of Eq. (2-21)) which start from $q = 0$ at τ_2^0 and which cannot be "joined" to the root loci (computed on the basis of Eq. (2-22)) which start from $q = \infty$ at τ_2^∞. Since the expansions given by Eqs. (2-21) and (2-22) become useless (from a practical point of view) before the theoretical boundaries are reached, there are considerable "gaps" in the complex q-plane which cannot be reached by employing these classical expansions.

2.3 The Root Locus for the Ground Wave Problem

Before we continue with the discussion of methods for the numerical evaluation of the roots $\tau_s(q)$, let us consider in more detail the root locus which we have referred to as the "Watson modes." Up until about a decade ago the only interest in these roots was that which arose out of the problem of predicting the diffraction effects associated with the propagation of the ground wave around the earth's surface. The rapidly advancing technology associated with the development of missiles and satellites has opened up a broad new range of problems in which the microwave engineer requires the ability to predict the diffraction effects when radiowaves are radiated from (or received by) an antenna mounted upon a convex surface. Another broad new area of interest is that associated with the radar reflection characteristics of bodies whose shape includes include portions that come within the scope of the problem of reflection from and diffraction around convex surfaces. The development of the laser has provided physicists with a source of coherent electromagnetic waves and now (Ref. 20) there exists an interest in this diffraction theory even in the range of the optical wavelengths. The recent paper by Streifer and Kodis (Ref. 21) is probably only the precusor of the flood of papers which will appear within the next decade in which the logarithmic derivative of the Airy function will play a prominent role.

From the rotational property of the Airy functions, we know that

$$q(t) = \frac{w_1'(t)}{w_1(t)} = \exp(i\tfrac{2}{3}\pi)\ \frac{v'[t\exp(i\frac{2}{3}\pi)]}{v[t\exp(i\frac{2}{3}\pi)]}$$

$$= \exp(i\tfrac{2}{3}\pi)\ \frac{Ai'[t\exp(i\frac{2}{3}\pi)]}{Ai[t\exp(i\frac{2}{3}\pi)]} \qquad (2\text{-}27)$$

Therefore, by rotating the contour plot given in Fig. 2-2 and

by relabeling the lines of constant phase, we obtain the contour plot of the logarithmic derivative $L(t) = Ai'(t)/Ai(t)$ which is given in Fig. 2-4.

Let us consider briefly a problem in which both the logarithmic derivative $q(t) = w_1'(t)/w_1(t)$ and the logarithmic derivative $L(t) = v'(t)/v(t)$ occur. In the study of diffraction by circular cylinders we have to solve the two dimensional differential equation

$$\left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + k^2\right]\Psi(\rho,\varphi) = 0 \qquad (2\text{-}28)$$

Let the electrical properties of the cylinder be characterized by k_1 while that of the outside is described by k. The cylinder has a radius a. We will use an $\exp(-i\omega t)$ time dependence. Since the wave must be outgoing at infinity and finite at $\rho = 0$, we choose

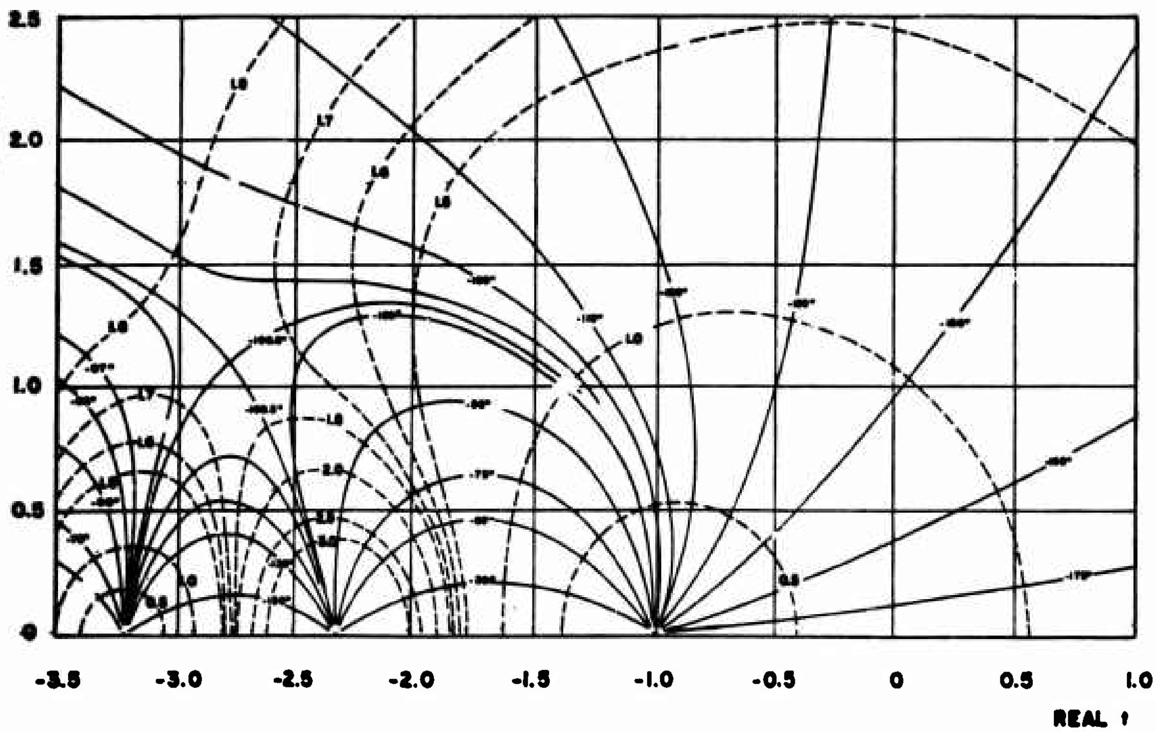

Fig. 2-4 The Logarithmic Derivative $L(t) = Ai'(t)/Ai(t)$
(———— phase , – – – – – modulus)

a solution proportional to

$$\exp(\pm i\upsilon\varphi)\, H_{\upsilon}^{(1)}(k\rho) \quad , \ \rho > a$$

$$\exp(\pm i\upsilon\varphi)\, J_{\upsilon}(k_1\rho) \quad , \ \rho < a$$

where

$$k^2 = \omega^2 \varepsilon\mu \quad , \ \rho > a \qquad (2\text{-}29a)$$

$$k_1^2 = \omega^2 \varepsilon_1\mu_1 + i\omega\mu_1\sigma_1 \quad , \ \rho < a \qquad (2\text{-}29b)$$

In electromagnetic diffraction problems which involve a circular cylinder, there are generally two cases of interest:

(a) $\Psi = E_z$, electric field parallel to the axis of the cylinder

(b) $\Psi = H_z$, magnetic field parallel to the axis of the cylinder

The continuity requirements on the electric and magnetic field at the interface $\rho = a$ lead to an eigenvalue problem for the determination of the roots υ_s defined by

$$\mu_1 k\, J_{\upsilon_s}(k_1 a) H_{\upsilon_s}^{(1)\prime}(ka) - \mu k_1 J'_{\upsilon_s}(k_1 a) H_{\upsilon_s}^{(1)}(ka) = 0 \ , \ \Psi = E_z \qquad (2\text{-}30a)$$

$$\mu k_1\, J_{\upsilon_s}(k_1 a) H_{\upsilon_s}^{(1)\prime}(ka) - \mu_1 k J'_{\upsilon_s}(k_1 a) H_{\upsilon_s}^{(1)}(ka) = 0 \ , \ \Psi = H_z \qquad (2\text{-}30b)$$

If the magnitude of k_1 is much greater than that of k, we can show that the magnitude of $k_1 a$ is much greater than the magnitude of υ_s (for small values of s if we order the υ_s and let υ_1 be the root with the smallest imaginary part, υ_2 be the root with the next smallest imaginary part, etc.) and then we can use the approximation

$$\frac{J'_{\upsilon_s}(k_1 a)}{J_{\upsilon_s}(k_1 a)} \approx -i \sqrt{(k_1 a)^2 - \upsilon_s^2}\, /(k_1 a) \qquad (2\text{-}31)$$

Therefore, we find that we have to solve the problem defined by

$$\frac{H_{\upsilon_s}^{(1)\prime}(ka)}{H_{\upsilon_s}^{(1)}(ka)} \approx \begin{cases} \cdot i(\mu k_1/\mu_1 k)\sqrt{(k_1 a)^2 - \upsilon_s^2}\,/(k_1 a), \; \Psi = E_z & (2\text{-}32a) \\ -i(\mu_1 k/\mu k_1)\sqrt{(k_1 a)^2 - \upsilon_s^2}\,/(k_1 a), \; \Psi = H_z & (2\text{-}32b) \end{cases}$$

When ka is very large, there are roots υ_s which are in the vicinity of $\upsilon_s = ka$. Therefore, we can use the Lorenz-Nicholson asymptotic approximation

$$H_\upsilon^{(1)}(ka) \approx -(i/\sqrt{\pi})(ka/2)^{-\frac{1}{3}} w_1(t) \tag{2-33}$$

where $w_1(t)$ is the Airy function and t is defined by

$$\upsilon = ka + (ka/2)^{\frac{1}{3}} t \tag{2-34}$$

Therefore, we can employ the approximation

$$\frac{H_{\upsilon_s}^{(1)\prime}(ka)}{H_{\upsilon_s}^{(1)}(ka)} \approx -(ka/2)^{-\frac{1}{3}} \frac{w_1'(t)}{w_1(t)} \tag{2-35}$$

to obtain the following problem which is to be solved for the roots t_s

$$t_s = (ka/2)^{-\frac{1}{3}}(\upsilon_s - ka) \tag{2-36}$$

defined by

$$\frac{w_1'(t_s)}{w_1(t_s)} \approx i(ka/2)^{\frac{1}{3}} \begin{cases} \dfrac{\mu k_1}{\mu_1 k} \dfrac{\sqrt{(k_1 a)^2 - \upsilon_s^2}}{(k_1 a)}, \; \Psi = E_z & (2\text{-}37a) \\[2ex] \dfrac{\mu_1 k}{\mu k_1} \dfrac{\sqrt{(k_1 a)^2 - \upsilon_s^2}}{(k_1 a)}, \; \Psi = H_z & (2\text{-}37b) \end{cases}$$

If we make the further approximation of replacing υ_s by (ka) in the radicals of Eq. (2-37) we can express the eigenvalue problem in the form

$$q(t_s) = \frac{w_1'(t_s)}{w_1(t_s)} = \begin{cases} q_v \quad , \quad \Psi = E_z & \text{(2-38a)} \\ q_h \quad , \quad \Psi = H_z & \text{(2-38b)} \end{cases}$$

where the impedance parameters are

$$q_v = i(ka/2)^{\frac{1}{3}}(k/k_1)\sqrt{1 - (k/k_1)^2} \qquad \text{(2-39a)}$$

$$q_h = i(ka/2)^{\frac{1}{3}}(k_1/k)\sqrt{1 - (k/k_1)^2} = (k_1/k)^2 q_v \qquad \text{(2-39b)}$$

Although we have derived Eq. (2-38) for the case of a circular cylinder, the analysis for the lossy dielectric sphere leads to an identical equation in the limit as $ka \to \infty$. If the diffracting obstacle is lossless, we observe that

$$\arg(q_v) = \arg(q_h) = 90^\circ$$

However, if $\sigma_1 >> \omega\varepsilon_1$, we find that

$$\arg(q_v) = 45^\circ$$

$$\arg(q_h) = 135^\circ$$

Therefore, we observe that

$$45^\circ < \arg(q_v) < 90^\circ$$

$$90^\circ < \arg(q_h) < 135^\circ$$

Most of the papers published to date have dealt with the eigenvalue problem defined by Eq. (2-38). This approximation is generally adequate for the problem of studying the propagation of the ground wave around the earth's surface. However, in problems which arise in the applications of microwaves, the condition that k_1a be much larger than ka is not satisfied in many of the situations for which data on the propagation phenomena is desired. Let us sketch a method which can be employed under much less stringent conditions than those which must be satisfied in order to have the problem reduce to Eq. (2-38).

We need an asymptotic approximation for the Bessel functions which is valid when both ka and k_1a are large, but the relation $|k_1 a| >> |ka|$ need not hold. A suitable asymptotic estimate is that which was employed in the mid-1930's by Langer and which has been employed in the more recent studies made by Olver (Ref. 22). In order to simplify the asymptotic formulae, let us make a few definitions. Let $\zeta = \zeta(z)$ be defined by

$$\frac{2}{3} \zeta^{3/2} = \log[(1 + \sqrt{1 - z^2})/z] - \sqrt{1 - z^2} \tag{2-40a}$$

and then define

$$\Phi(\zeta) = [4\zeta/(1 - z^2)]^{1/4} \tag{2-40b}$$

$$\Psi(\zeta) = 2/[z\Phi(\zeta)] \tag{2-40c}$$

The Langer-Olver asymptotic estimates are then given by

$$\sqrt{\pi}\ H_\upsilon^{(1)}(\upsilon z) \approx -i\Phi(\zeta)\upsilon^{-\frac{1}{3}} w_1(\upsilon^{\frac{2}{3}}\zeta) \tag{2-41}$$

$$\sqrt{\pi}\ H_\upsilon^{(1)\prime}(\upsilon z) \approx i\Psi(\zeta)\upsilon^{-\frac{2}{3}} w_1'(\upsilon^{\frac{2}{3}}\zeta) \tag{2-42}$$

$$\sqrt{\pi}\ J_\upsilon(\upsilon z) \approx \Phi(\zeta)\upsilon^{-\frac{1}{3}}\ v(\upsilon^{\frac{2}{3}}\zeta) \tag{2-43}$$

$$\sqrt{\pi}\ J_\upsilon'(\upsilon z) \approx -\Psi(\zeta)\upsilon^{-\frac{2}{3}} v'(\upsilon^{\frac{2}{3}}\zeta) \tag{2-44}$$

In order to use Eqs. (2-41) through (2-44) in Eq. (2-30), we define the quantities z and z_1 and ζ and ζ_1 by means of the relations

$$z = ka/\upsilon_s \quad , \quad z_1 = k_1 a/\upsilon_s$$
$$\zeta = \zeta(z) \quad , \quad \zeta_1 = \zeta(z_1)$$

and then observe that the logarithmic derivatives of the Bessel functions which occur in Eq. (2-30) can be expressed in the form

$$\frac{H_\upsilon^{(1)\prime}(ka)}{H_\upsilon^{(1)}(ka)} \approx - 2\upsilon^{-\frac{1}{3}}(\upsilon/ka)\sqrt{[1 - (ka/\upsilon)^2/(4\zeta)]}\;\frac{w_1'(\upsilon^{\frac{2}{3}}\zeta)}{w_1(\upsilon^{\frac{2}{3}}\zeta)} \tag{2-45}$$

$$\frac{J_\upsilon'(k_1a)}{J_\upsilon(k_1a)} \approx - 2\upsilon^{-\frac{1}{3}}(\upsilon/k_1a)\sqrt{[1 - (k_1a/\upsilon)^2/(4\zeta_1)]}\;\frac{v'(\upsilon^{\frac{2}{3}}\zeta_1)}{v(\upsilon^{\frac{2}{3}}\zeta_1)} \tag{2-46}$$

The subscript s on υ_s has been suppressed in writing Eqs. (2-45) and (2-46). We can then replace Eq. (2-30a) by the asymptotic approximation

$$\frac{w_1'(\upsilon_s^{\frac{2}{3}}\zeta)}{w_1(\upsilon^{\frac{2}{3}}\zeta)} = \Gamma\,\frac{v'(\upsilon_s^{\frac{2}{3}}\zeta_1)}{v(\upsilon_s^{\frac{2}{3}}\zeta_1)}\left[\frac{\zeta}{\zeta_1}\;\frac{\upsilon_s^2 - (k_1a)^2}{\upsilon_s^2 - (ka)^2}\right]^{\frac{1}{2}} \tag{2-47}$$

where

$$\Gamma = \frac{\mu k_1}{\mu_1 k}$$

We can replace Eq. (2-30b) by a similar equation in which the only difference is that Γ is replaced by Γ^{-1}.

Eq. (2-47) provides an example of the direction in which future work will undoubtedly be directed. In Eq. (2-38) the right hand side is a constant, whereas in Eq. (2-47) we not only have a right hand side which is variable, but it has the interesting property of involving the logarithmic derivative of the Airy function v(t). In some situations we may expect to find roots for which $\upsilon_s \approx k_1a$ and k_1a and ka are widely enough separated that we can use an asymptotic estimate for the left hand side of Eq. (2-47) and then find that the problem to be solve is

$$\frac{v'(\upsilon_s^{\frac{2}{3}}\zeta_1)}{v(\upsilon^{\frac{2}{3}}\zeta_1)} = \frac{Ai'(\upsilon^{\frac{2}{3}}\zeta_1)}{Ai(\upsilon^{\frac{2}{3}}\zeta_1)} = F(\upsilon_s, k_1a, ka, \Gamma) \tag{2-48}$$

where $F(\upsilon_s, k_1 a, ka, \Gamma)$ may in some problems be so slowly varying that we can consider it to be a constant and therefore find that the problem to be solved is

$$v'(t) - F\, v(t) = 0 \qquad (2\text{-}49a)$$

which is equivalent to

$$w_1'[t\exp(i\tfrac{4}{3}\pi)] - F\exp(-i\tfrac{4}{3}\pi)\; w_1[t\exp(i\tfrac{4}{3}\pi)] = 0 \qquad (2\text{-}49b)$$

We want now to turn away from this more general problem and look more closely at the classical ground wave propagation problem. However, we hope that the reader sees from the discourse above that there exists many more reasons to study the roots of

$$w_1'(\tau) - q\, w_1(\tau) = 0$$

than merely the fact that this equation plays a fundamental role in the problem of propagation of radio waves around the earth's surface.

The calculations made by van der Pol and Bremmer (Ref. 6), Norton (Ref. 13), Wyngaarden and van der Pol (Ref. 9), and Johler, et. al. (Ref. 15) have all been made for specific frequencies and electrical properties of the earth. Such calculations play an important role in specific engineering applications. However, from the point of view of making calculations which can be adapted by the potential user to a class of problems, there are advantages to be had by following the procedures which were employed by Domb* (Ref. 19) and Belkina (Ref. 17).

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*Although the reference is to a paper by Domb, the author would like to remind the reader that this work was primarily under the technical direction of M. H. L. Pryce. The calculations were made in 1942-4 by the Admiralty Computing Service and the contributors included J. C. P. Miller, L. Fox, D. H. Sadler, R. H. Corkan, P. H. Haines, R. G. Taylor, and E. M. Wilson.

In the tables of Ref. 19, it is assumed that the impedance parameter q is of the form

$$q = Q\exp(i45^{o})$$

where Q is real and positive. Calculations were made for the first five roots for

$$Q = 0.0(0.1)1.0 \qquad , \text{ and } \frac{1}{Q} = 0.0(0.1)1.0$$

In Table 2-5 we list the values that were obtained for the first root $\tau_1(q)$ by Domb and his co-workers. The "tic" marks that appear on the locus for 45^{o} (the "Watson modes") correspond to the entries in Table 2-6.

The present author feels that there is a definite advantage in the presenting of the results in terms of dimensionless parameters in the manner employed by Pryce and Domb (Ref. 25), Pryce (Ref. 24) and Domb (Ref. 19). In order to emphasize this point, we present in Table 2-7 the values for the root $\tau_1(q)$ which were obtained by Johler, Walters and Lilley for the case of propagation over typical land ($\varepsilon_1 = 15$, $\sigma_1 = 0.005$ mho/m) and over typical sea ($\varepsilon_1 = 80$, σ_1 5 mho/m) for frequencies between 0 and 2000 kc.

Table 2-6

THE ROOT τ_1 AS CALCULATED BY ADMIRALTY COMPUTING SERVICE

Q	Real τ_1	Imag τ_1	1/Q	Real τ_1	Imag τ_1
0.0	0.509	0.882	0.0	1.169	2.025
0.1	0.604	0.862	0.1	1.240	1.953
0.2	0.698	0.850	0.2	1.311	1.877
0.3	0.790	0.848	0.3	1.383	1.790
0.4	0.880	0.854	0.4	1.449	1.685
0.5	0.967	0.868	0.5	1.497	1.559
0.6	1.050	0.891	0.6	1.510	1.420
0.7	1.128	0.921	0.7	1.486	1.293
0.8	1.201	0.958	0.8	1.439	1.189
0.9	1.267	1.002	0.9	1.383	1.110
1.0	1.326	1.051	1.0	1.326	1.051

Table 2-7

TABLE OF ROOTS $\tau_1(q)$ OBTAINED BY JOHLER, WALTERS AND LILLEY

Frequency (kc)	$\varepsilon_1 = 15$, $\sigma = 0.005$ mho/m		$\varepsilon_1 = 80$, $\sigma = 5$ mho/m	
	Real τ_1	Imag τ_1	Real τ_1	Imag τ_1
0	0.5094	0.8823	0.5094	0.8823
0.1	0.5115	0.8811	0.5095	0.8823
0.2	0.5131	0.8813	0.5095	0.8823
0.5	0.5173	0.8802	0.5096	0.8822
1.0	0.5235	0.8786	0.5098	0.8822
2	0.5346	0.8759	0.5102	0.8821
3	0.5447	0.8735	0.5105	0.8820
4	0.5542	0.8714	0.5108	0.8819
5	0.5634	0.8694	0.5111	0.8818
6	0.5722	0.8676	0.5114	0.8818
7	0.5808	0.8659	0.5117	0.8817
8	0.5892	0.8643	0.5119	0.8816
9	0.5974	0.8628	0.5122	0.8816
10	0.6055	0.8614	0.5124	0.8815
20	0.6801	0.8519	0.5148	0.8809
30	0.7477	0.8483	0.5170	0.8803
50	0.8696	0.8542	0.5210	0.8793
60	0.9254	0.8625	0.5229	0.8788
70	0.9782	0.8739	0.5248	0.8783
80	1.0281	0.8879	0.5266	0.8778
90	1.0753	0.9045	0.5284	0.8774
100	1.1197	0.9233	0.5301	0.8770
200	1.4111	1.1944	0.5463	0.8731
300	1.4628	1.4552	0.5612	0.8698
400	1.4573	1.6096	0.5752	0.8669
500	1.4173	1.6992	0.5886	0.8644
600	1.3835	1.7538	0.6016	0.8620
700	1.3569	1.7906	0.6142	0.8600
800	1.3361	1.8171	0.6264	0.8581
900	1.3193	1.8373	0.6384	0.8564
1000	1.3055	1.8531	0.6502	0.8549
1100	1.2940	1.8660	0.6618	0.8536
1200	1.2843	1.8766	0.6732	0.8524
1300	1.2759	1.8856	0.6843	0.8513
1400	1.2687	1.8933	0.6954	0.8504
1500	1.2623	1.8999	0.7062	0.8497
1600	1.2566	1.9057	0.7170	0.8490
1700	1.2516	1.9108	0.7276	0.8485
1800	0.2471	1.9154	0.7380	0.8481
1900	1.2430	1.9195	0.7484	0.8479
2000	1.2393	1.9232	0.7586	0.8477

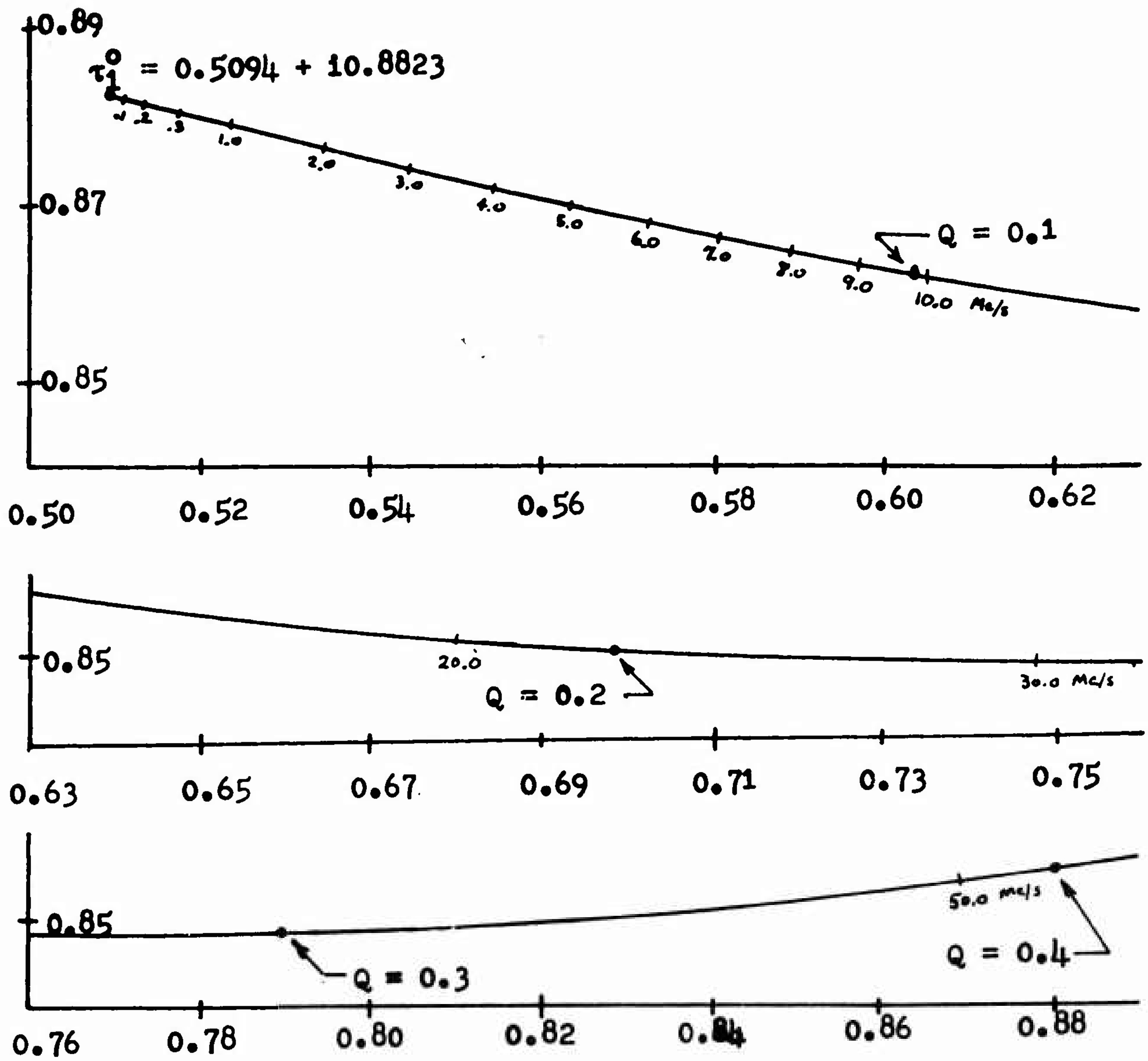

Fig. 2-5 The Root Locus of $\tau_1[Q\exp(i45^\circ)]$ for $0 < Q < 0.4$

In Fig. 2-5 we have presented (in strip form) a greatly magnified version of the root locus for $\arg(q) = 45^\circ$ which has been illustrated in Figs. 2-1 and 2-2. Over this range it is not possible to distinguish between this locus and the locus of points which are obtained from the tables of Ref. 15* for frequencies up to

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*Ref. 15 can be used to extend Table 2-6 to 10,000 kc.

50 kc over "typical land" and up to 3000 kc over "typical sea." The portion of the locus which is given in this figure is an interesting region because it contains the "minimum" in the imaginary part of $\tau_1[Q\exp(i45^\circ)]$ which occurs between $Q = 0.25$ and 0.30. This occurs for a frequency of the order of 30 kc over "typical land" and of the order of 2000 kc over "typical sea."

The data given in Ref. 15 reveals that when the frequency is 10,000 kc that the corresponding value of Q is of the order of unity in the case of "typical sea." If one plots the data of Table 2-6 for the case of "typical land" on a scale such as employed in Fig. 2-5, it will be observed that as the frequency continues to increase that the locus pulls slightly away from the locus for $\arg(q) = 45^\circ$. The reader who is interested in the relation of the impedance parameter to the frequency will find tables in Ref. 15 for quantities denoted by K_e and ψ_e. The relation to q is

$$q = \frac{1}{2^{\frac{1}{3}}K_e} \exp[i(\frac{\pi}{4} + \psi_e)]$$

In Fig. 2-6 we have plotted several sets of data which are related to the behavior of the root locus for high frequencies when $\tau_1(q)$ approaches $\tau_1^\infty = 1.169 + i2.025$ as the magnitude of q tends to infinity. The lowest set of points (labeled NBS DATA) is taken from Ref. 15 for the case of "typical land." The highest frequency considered was 10 Mc. The highest set of data in this figure is the root locus for $\arg(q) = 45^\circ$. Over the range covered by this illustration these two loci are virtually parallel. The reader will observe that the magnitude of q on the locus based on the data of Ref. 15 is of the order of 10 at a frequency of 3 Mc.

The C.C.I.R. atlas of ground wave curves which was prepared by Wyngaarden and van der Pol does not overlap with the data of Ref. 15 since the frequency range is taken to be that from 30 Mc to 300 Mc. Furthermore, the electrical parameters are not the

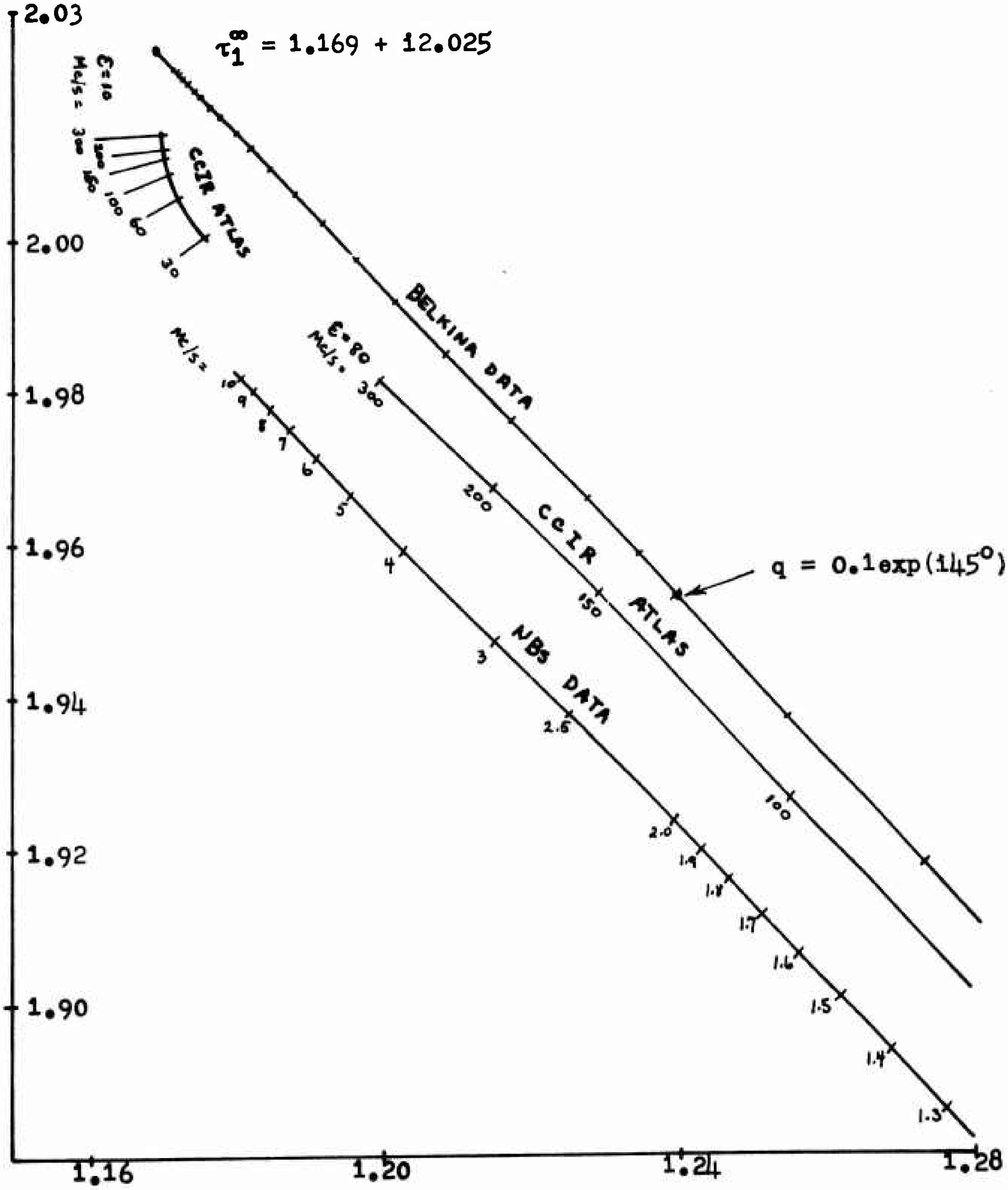

Fig. 2-6 Loci of Roots in Vicinity of $\tau_1^\infty = 1.169 + i2.025$ for Propagation Around the Earth's Surface

same in the two sets of calculations. The parameters employed were as indicated below.

Source	Johler, Walters and Lilley Ref. 15	Wyngaarden and van der Pol Ref. 9
"Typical Land"	$\varepsilon_1 = 15$ $\sigma_1 = 0.005$ mho/m	$\varepsilon_1 = 10$ $\sigma_1 = 0.001$ mho/m
"Typical Sea"	$\varepsilon_1 = 80$ $\sigma_1 = 5$ mho/m	$\varepsilon_1 = 80$ $\sigma_1 = 4$ mho/m

The curve for "typical sea" in Fig. 2-6 is virtually parallel over to the curve for arg(q) = 45°. The frequency has not been taken to be large enough for us to see how the locus of the roots for propagation over the "typical sea" approaches the limiting point at τ_1^∞. However, the curve for "typical land" in the C.C.I.R. atlas do show the manner in which that locus turns and heads into the limiting point along the locus for arg(q) = 90°. The atlas prepared by the Radio Research Laboratory (Ref. 23) covers the frequency range from 30 to 10,000 Mc , but the roots which were employed to make the calculations have not been listed in the report.

Except for the "tic" mark which is used to mark the location of the root for $q = 0.1\exp(i45^\circ)$, all the "tic" marks on the locus for arg(q) = 45° are taken from the data published by Belkina (ref. 17). The method employed by Belkina deserves further study because it is perhaps the most useful method of presenting data which has been developed to date. In Eq. (2-39) we have given expressions for the impedance parameters for vertical and horizontal polarization. They depend upon a, ω, ε_1, μ_1, and σ_1. In the ground wave problem the radius a is fixed (although it also

varies if we take into account the fact that the concept of an equivalent earth's radius is employed to take into account the rate of change of the index of refraction at the earth's surface.) but we still have the three variables ω, ε_1, and σ_1 even in those cases for which $\mu_1 = 1$. Following a suggestion make by Fock, Belkina used the approximation

$$\sqrt{1 - (k/k_1)^2} \approx \frac{1}{\sqrt{1 + (k/k_1)^2}}$$

which is valid when the magnitude of (k/k_1) is small. Belkina then observed the manner in which the frequency ω enters the expression for q_v of Eq. (2-39a) and assumed that q_v could be represented by an expression of the form

$$q = \frac{in^{5/6}}{\sqrt{1 + an}} \tag{2-50}$$

This expression has the advantage of containing only two parameters and therefore it is possible to conceive of making computations for a range of values of a and n which will serve to cover a range of values of ω, ε_1, and σ_1. This is a very clever transformation since we have two adjustable parameters at our disposal in Eq. (2-50) and the physical problem consists of two basic variables, the wavelength and the physical nature of the earth's crust over which the wave is being propagated. The fact that the properties of the earth's crust requires two real parameters, ε_1 and σ_1, for its description forces us to consider the diffraction phenomena to depend upon at least three variables (three only if we assume μ_1 to be constant and equal to the value of μ in free space). The advantage of the form for q given in Eq. (2-50) is that we have only 2 parameters to vary and hence it becomes feasible to attempt to compute some universal data which can be used in a variety of physical problems. In Table 2-8a we have reproduced Belkina's table which shows the combination of

Table 2-8a

TABLE OF VALUES OF THE PARAMETER log n SHOWING DEPENDENCE UPON PHYSICAL NATURE OF THE SOIL OR SEA AND ON THE WAVELENGTH λ

Physical Nature of Soil	Very Salty Sea Water	Slightly Salty Sea Water	Slightly Salty Sea Water	Marsh	Damp Soil and Meadow Soil	Damp Soil and Meadow Soil
ε	80	80	80	15	15	15
$\frac{\sigma_0}{\sigma}$	2.10^5	$1,10^6$	2.10^6	1.10^7	1.10^8	2.10^8
α	0,010	0,018	0,024	0,010	0,023	0,031
λ (meters)	log n					
10	0,425	0,844	1,021	1,44	2,04	2,22
20	0,124	0,543	0,720	1,143	1,744	1,924
30	−0,052	0,367	0,544	0,967	1,568	1,748
40	−0,177	0,242	0,419	0,842	1,443	1,623
50	−0,274	0,145	0,322	0,745	1,346	1,526
60	−0,353	0,066	0,243	0,666	1,267	1,447
70	−0,420	−0,001	0,176	0,599	1,200	1,380
80	−0,478	−0,059	0,118	0,541	1,142	1,322
90	−0,520	−0,110	0,067	0,490	1,091	1,271
100	−0,575	−0,156	0,021	0,444	1,045	1,225
200	−0,876	−0,457	−0,280	0,143	0,744	0,924
300	−1,052	−0,633	−0,456	−0,033	0,568	0,748
400	−1,177	−0,758	−0,581	−0,158	0,443	0,623
500	−1,274	−0,855	−0,678	−0,255	0,346	0,526
600	−1,353	−0,934	−0,757	−0,334	0,267	0,447
700	−1,420	−1,001	−0,824	−0,401	0,200	0,380
800	−1,478	−1,059	−0,882	−0,459	0,142	0,322
900	−1,529	−1,110	−0,933	−0,510	0,091	0,271
1000	−1,575	−1,156	−0,979	−0,556	0,045	0,225
1100	−1,616	−1,198	−1,020	−0,597	0,004	0,184
1200	−1,654	−1,235	−1,058	−0,635	−0,034	0,146
1300	−1,689	−1,270	−1,093	−0,670	−0,069	0,111
1400	−1,721	−1,302	−1,125	−0,702	−0,101	0,079
1500	−1,751	−1,332	−1,155	−0,732	−0,131	0,049
1600	−1,779	−1,360	−1,183	−0,760	−0,159	0,021
1700	−1,806	−1,386	−1,209	−0,786	−0,185	−0,005
1800	−1,830	−1,411	−1,234	−0,811	−0,210	−0,030
1900	−1,854	−1,435	−1,258	−0,835	−0,234	−0,054
2000	−1,876	−1,457	−1,280	−0,857	−0,256	−0,076

Note: Belkina expressed the conductivity of the earth in the form σ_0/σ where σ is the conductivity of the earth and σ_0 is the conductivity of mercury which was taken to be 10440 $(\Omega\,cm)^{-1}$.

Table 2-8b

THE FUNCTION $q(\alpha,n) = \frac{i n^{5/6}}{\sqrt{i+\alpha n}}$

lg n	α = 0,00	α = 0,01	
	Re q = Im q	Re q	Im q
−1,0	0,1037891	0,1037371	0,1038409
−0,9	0,1257434	,1256642	0,1258225
−0,8	0,1523415	0,1522206	0,1524621
−0,7	0,1845660	0,1843816	0,1847498
−0,6	0,223607	0,223325	0,223887
−0,5	0.270906	0,270476	0,271333
−0,4	0,328210	0,327555	0,328861
−0,3	0,397635	0,396635	0,398628
−0,2	0,481746	0,480219	0,483259
−0,1	0,583649	0,581317	0,585953
0	0,707107	0,703545	0,710616
0,1	0,856679	0,851236	0,862020
0,2	1,037891	1,029569	1,046016
0,3	1,257434	1,244705	1,269788
0,4	1,523415	1,503929	1,542180
0,5	1,845660	1,815804	1,874132
0,6	2,23607	2,19027	2,27921
0,7	2,70906	2,63873	2,77429
0,8	3,28210	3,17393	3,38050
0,9	3,97635	3,80968	4,12430
1,0	4,81746	4,56015	5,03891
1,1	5,83649	5,43841	6,16599
1,2	7,07107	6,45394	7,55738
1,3	8,56679	7,60847	9,27653
1,4	10,37891	8,88952	11,39862
1,5	12,57434	10,26164	14,00751
1,6	15,23415	11,65625	17,18642
1,7	18,45660	12,9648	20,9998
1,8	22,3607	14,0434	25,4659
1,9	27,0906	14,7388	30,5303
2,0	32,8210	14,9365	36,0599
2,1	39,7635	14,6074	41,8749
2,2	48,1746	13,8223	47,8098
2,3	58,3649	12,7195	53,7664
2,4	70,7107	11,4522	59,7293
2,5	85,6679	10,1478	65,7465

Table 2-8b (Cont'd)

THE FUNCTION $q(\alpha,n) = \frac{in^{\frac{5}{6}}}{\sqrt{1+i\alpha n}}$

lg n	α = 0.02		α = 0,03	
	Re q	Im q	Re q	Im q
−1,0	0,1036851	0,1038927	0,1036330	0,1039444
−0,9	0,1255848	0,1259014	0,1255018	0,1259836
−0,8	0,1520995	0,1525824	0,1519780	0,1527024
−0,7	0,1841967	0,1849331	0,1840111	0,1851159
−0,6	0,223043	0,224166	0,222760	0,224444
−0,5	0,270045	0,271758	0,269612	0,272182
−0,4	0,326896	0,329509	0,326232	0,330152
−0,3	0,395628	0,399613	0,394613	0,400591
−0,2	0,478678	0,484757	0,477123	0,486240
−0,1	0,578958	0,588229	0,576573	0,590476
0	0,699931	0,714070	0,696268	0,717469
0,1	0,845695	0,867256	0,840058	0,872384
0,2	1,021061	1,053939	1,012373	1,061652
0,3	1,231620	1,281748	1,218199	1,293298
0,4	1,483770	1,560182	1,462987	1,577381
0,5	1,784680	1,901119	1,752413	1,926529
0,6	2,14211	2,31944	2,09188	2,35658
0,7	2,56400	2,83385	2,48563	2,88730
0,8	3,05767	3,46777	2,93526	3,54298
0,9	3,62843	4,25036	3,43735	4,35271
1,0	4,27717	5,21731	3,98016	5,34946
1,1	4,99680	6,41087	4,53962	6,56711
1,2	5,76723	7,87812	5,07600	8,03403
1,3	6,54979	9,66574	5,53436	9,76282
1,4	7,28329	11,80968	5,85280	11,73901
1,5	7,88683	14,31989	5,98001	13,91601
1,6	8,27453	17,16533	5,89537	16,22415
1,7	8,3824	20,2709	5,61860	18,59365
1,8	8,1948	23,5356	5,2007	20,9779
1,9	7,7519	26,8668	4,7034	23,3633
2,0	7,1316	30,2098	4,1810	25,7645
2,1	6,4198	33,5565	3,6717	28,2121
2,2	5,6878	36,9341	3,1979	30,7423
2,3	4,9835	40,3885	2,7700	33,3909
2,4	4,3338	43,9710	2,3908	36,1908
2,5	3,7502	47,7310	2,0588	39,1718

α and n which are required to describe the variables ε_1, σ_1 and λ which enter into certain diffraction problems. In Table 2-8b we reproduce the table given by Belkina which presents the values of q which result from use of the combinations of α and n which are contained in the tables. The reader will observe that negative values of log n (i.e., $n < 1$) lead to combinations of the real and imaginary parts of q which are almost equal and therefore the argument of q is of the order of 45°. However, for large positive values of n (and $\alpha > 0$), we see that the imaginary part of q grows much faster than does the real part. Therefore, these data reveal that the family of root loci which are based upon Eq. (2-50) will parallel the locus $\arg(q) = 45^{\circ}$ for small values of n, but then will pull away from this locus and turn to approach τ_1^{∞} along a line which is parallel to the imaginary axis.

Diffraction theory needs the ingenuity and inventiveness which leads to the recognition of the importance of models such as that contained in Eq. (2-50). Since the electrical properties to be assigned to an obstacle are often not known exactly, it would be a great advantage to have the calculations which are published have their basis in a relationship similar to that of Eq. (2-50) which will permit the construction of a family of universal curves which can be relatively easily adapted to a wide variety of approximate physical models.

The importance of seeking universal curves is clearly indicated by the C.C.I.R. atlas (Ref. 9). It is not possible to plot the roots for horizontal polarization on the graph given in Fig. 2-5 because they are all closely clustered around τ_1^{∞}. This suggests that the approximations suggested in Appendix E (see Eq. (E-7) and (E-8)) could be used to obtain the data for horizontal polarization from the universal functions computed for $q = \infty$. The importance of these concepts was apparently appreciated by Pryce and Domb (Ref. 25), but later writers have apparently not been

influenced by this relatively unknown work which was carried out in England during World War II.

The papers of Norton (Ref. 13) and Pryce and Domb (Ref. 25) are seldom referred to by recent writers. However, these papers are a "must" on the reading lists of those who undertake numerical work in diffraction theory. These men were not writing a paper just to add another paper to their list of publications; they were writing to try to guide engineers towards the obtaining of numerical results for the practical problems that they might encounter.

Our discussion of the root loci for the ground wave propagation problems have led us away from our development of numerical methods for solving for the roots and associated functions. However, we are entering an era in which we can expect an increase of interest in diffraction phenomena. Even if scores of atlases are prepared which will illustrate the diffraction phenomena associated with dielectric spheres and circular cylinders, the chances are quite large that the engineer who wants an answer will have difficulty in interpolating when the exisiting data have been obtained for a set of values of the electrical parameters, the radius a, and the wavelength. Even with the ability to employ the electronic computer to make calculations, it is still worth keeping in mind that if one can find a means of computing a "universal" function that one might not need to get back on the computer to make a completely new computation when one is faced with the problem of changing the parameters of the problem.

In Appendix C we have given some coefficients which will enable the user to calculate (on a desk calculator) the roots for arg(q) in the vicinity of 45° and for arg(q) in the vicinity of 0°. These two cases appear to be the ones in which there is the most interest at present.

2.4 Numerical Values for the Classical Expansions

The expansions which are designated as Eq. (2-21) and Eq. (2-22) have been employed in the papers of many writers since the first several terms were given in the classic papers of van der Pol and Bremmer (Ref. 6). Norton (Ref. 13) gave the first several terms with numerical values for the coefficients. Howe (Ref. 14) discussed the obtaining of numerical values of A_n and B_n on an electronic computer. Johler, Walters, and Lilley (Ref. 15) have given (in a slightly different notation) the explicit dependence of the coefficients upon τ for terms up to and including $A_{10}(\tau)$ and $B_{11}(\tau)$.

If one wants specific numerical values for the $A_n(\tau)$ and $B_n(\tau)$, the use of the explicit forms for these coefficients as a function of τ is to be discouraged since the coefficients (as has been shown by Howe(Ref. 14) and by a group at Leningrad University (Ref. 16)) can be readily obtained by means of recursion formulae. For $n > 2$, the recursion formula for the A_n is

$$nA_oA_n = -[(n-1)A_1A_{n-1} + (n-2)(A_2 - 1)A_{n-2} + (n-3)A_3A_{n-3} + \cdots + 2A_{n-2}A_2 + A_{n-1}A_1]$$

$$= (n-2)A_{n-2} - \sum_{m=1}^{n-1} mA_mA_{n-m} \tag{2-51}$$

where $A_o = \beta_s \exp(i\frac{1}{4}\pi)$ and $A_oA_1 = 1$. For $n > 3$, the recursion formula for the B_n is

$$nB_n = B_1B_{n-3} + 2B_2B_{n-4} + \cdots + (n-2)B_{n-2}B_1 + (n-1)B_{n-1}B_o$$

$$= \sum_{m=1}^{n-2} mB_mB_{n-m-2} \tag{2-52}$$

where $B_o = \alpha_s \exp(i\frac{1}{3}\pi)$, $B_1 = 1$ and $B_2 = 0$.

Since the $A_n(\tau)$ and $B_n(\tau)$ are complex quantities, the calculations based upon Eqs. (2-51) and (2-52) are somewhat more involved than one might suspect. However, one can readily see that each of these quantities must be of the form of a real number times the factor $\exp(i\frac{1}{3}m\pi)$, where m is an integer. This is quite obvious since each of the explicit forms is representable as an integral power of τ multiplied times a polynomial in $z = \tau^3$ in which the coefficients are real. Since $\tau = \alpha_s \exp(i\frac{1}{3}\pi)$ or $\tau = \beta_s \exp(i\frac{1}{3}\pi)$, $\tau^3 = -\alpha_s^3$ or $\tau^3 = -\beta_s^3$, and hence the polynomial is real. If one "plays around" with the explicit forms for a while, it becomes clear that a suitable set of real coefficients can be defined by means of the relations

$$C_n(\beta) = A_n(\tau)\exp(-i\tfrac{1}{3}\pi)\exp(i\tfrac{2}{3}n\pi) \quad , \quad \tau = \beta\exp(i\tfrac{1}{3}\pi) \tag{2-53}$$

$$D_n(\alpha) = B_n(\tau)\exp(-i\tfrac{1}{3}\pi)\exp(-i\tfrac{2}{3}n\pi) \quad , \quad \tau = \alpha\exp(i\tfrac{1}{3}\pi) \tag{2-54}$$

Eqs. (2-21) and (2-22) then take the forms

$$\tau_s(q) = [\ \beta_s + \sum_{n=1}^{\infty} C_n(\beta_s) q^n \exp(-i\tfrac{2}{3}n\pi)\ ]\exp(i\tfrac{1}{3}\pi) \tag{2-55}$$

$$\tau_s(q) = [\ \alpha_s + \sum_{n=1}^{\infty} D_n(\alpha_s) q^{-n} \exp(i\tfrac{2}{3}n\pi)\ ]\exp(i\tfrac{1}{3}\pi) \tag{2-56}$$

The expressions in the brackets [•••] becomes purely real when $q = Q\exp(i\frac{2}{3}\pi)$ or $q = Q\exp(-i\frac{1}{3}\pi)$, where Q is real and positive. In this case the roots $\tau_s(q)$ lies along the ray $\arg(\tau) = \frac{1}{3}\pi$ and we observe that this is illustrated by the root loci depicted in Fig. 2-2. In Table 2-9 we present values for the coefficients C_n for use in Eq. (2-55) for s = 1(1)5. Similar tables of the coefficients D_n for use in Eq. (2-56) are given in Table 2-10.

(Text continues on p. 2-64)

Table 2-9

COEFFICIENTS IN THE EXPANSION OF $\tau_s(q)$ IN SERIES OF POWERS OF q

s = 1

n	C_n		n	C_n	
0	0.10187927916474710890 2E	1	40	-.43450039839476856133 5E	-12
1	0.98155368934565643349 7E	+0	41	-0.59936583720035197224 8E	-12
2	-0.47283779525397403410 4E	+0	42	.25776260002643435788 4E	-12
3	.13440524531392944729 6E	+0	43	.13277983638019863636 3E	-12
4	-.71602938859209580269E	-2	44	-.10744291383610046579 5E	-12
5	-.98774282120637696024 7E	-2	45	-.19142228801387823900 6E	-13
6	.20128847168449974735 6E	-2	46	.37439079134687626594 5E	-13
7	.13462118035028363782 9E	-2	47	-.17666437483341173401 8E	-14
8	-.56668183510136790690 5E	-3	48	-.11337107645703595881 4E	-13
9	-.19194548486765931239 5E	-3	49	.29013724000544148587 5E	-14
10	.15557115743545349859 6E	-3	50	.29519485284101323658 6E	-14
11	.18916689089679143143 8E	-4	51	-.15230115342664931307 0E	-14
12	-.40870251099793881790 8E	-4	52	-.61133877896127375017 7E	-15
13	.24179084285559739402 2E	-5	53	.60615193890988927859 2E	-15
14	.10054412469337743780 2E	-4	54	.66506395408094948795 1E	-16
15	-.24604427443158134182 1E	-5	55	-.20495253271518981323 8E	-15
16	-.22325772962695294382 9E	-5	56	.21678887050098105155 0E	-16
17	.10845740096401966706 7E	-5	57	.60237154078484583664 7E	-16
18	.40970966684901685698 3E	-6	58	-.19511946278463794373 8E	-16
19	-.38009141647458677677 1E	-6	59	-.15019459909324023156 2E	-16
20	-.42322222524222359577 5E	-7	60	.92845571586849734978 1E	-17
21	.11617517415717004888 6E	-6	61	.28244523089292907622 6E	-17
22	-.10491804636115500769 6E	-7	62	-.35367534944795050297 1E	-17
23	-.31448243029625606851 73E	-7	63	-.15803605765732402167 0E	-18
24	.92452961606875395493 4E	-8	64	.11573629847955301414 0E	-17
25	.73792846770156102050 2E	-8	65	-.19181766556719008652 6E	-18
26	-.41842339811879769425 8E	-8	66	-.32833642943736497918 3E	-18
27	-.13477462039333872944 0E	-8	67	.12985412991371753757 9E	-18
28	.15124611057458781508 1E	-8	68	.77486418714831781100 8E	-19
29	.98706365158131984016 6E	-10	69	-.57664810893608478424 5E	-19
30	-.47296345270391202255 7E	-9	70	-.12651177866742399956 9E	-19
31	.65725283233185074546 7E	-10	71	.21074349462059793418 5E	-19
32	.12943890327127395662 9E	-9	72	-.37644689028468273486 0E	-21
33	-.46134257672511148689 3E	-10	73	-.66616749277340296364 2E	-20
34	-.29941065860438752794 6E	-10	74	.15187711426620977814 4E	-20
35	.20144823552162073834 2E	-10	75	.18143181729565122691 5E	-20
36	.50372341365009438348 1E	-11	76	-.8676867431[illegible]6634096336E	-21
37	-.71968806609140234318 7E	-11	77	-.39931899469301251568 9E	-21
38	-.73940983916856290706 7E	-13	78	.36222565456751322006 9E	-21
39	.22291110964115985238 3E	-11	79	.51994725285181650888 5E	-22
40	-.43450039839476856133 5E	-12	80	-.12723204976377352460 8E	-21

Table 2-9 (Cont'd)

COEFFICIENTS IN THE EXPANSION OF $\tau_s(q)$ IN SERIES OF POWERS OF q

s = 2

n	C_n	
0	.324819758217983653788E	1
1	.307863045489033608842E	+0
2	-0.14589576585143720362 E	-1
3	-.302104232271669809476E	-1
4	.507636401516794284515E	-2
5	.496357483168173154451E	-2
6	-.163001684510479447964E	-2
7	-.867500517483510109251E	-3
8	.493464344311700642769E	-3
9	.134137872203975243362E	-3
10	-.141345979831441940990E	-3
11	-.113523648020419922238E	-4
12	.380967883037748295575E	-4
13	-.341814629228917655617E	-5
14	-.951586690798386831721E	-5
15	.258573114727559952403E	-5
16	.212899805242624463328E	-5
17	-.109777188099395559095E	-5
18	-.390070771176994339800E	-6
19	.380750549933740978042E	-6
20	.386692227605892910735E	-7
21	-.115967611967868557819E	-6
22	.111545492010122803628E	-7
23	.313824352431622371733E	-7
24	-.936163920447205976821E	-8
25	-.734933451691403001706E	-8
26	.420375242229676693554E	-8
27	.134011649965973437688E	-8
28	-.151552226809119708002E	-8
29	-.969258894716789854955E	-10
30	.473391658285887383021E	-9
31	-.661166927469486406184E	-10
32	-.129485445986394645358E	-9
33	.462162994489588250939E	-10
34	.299422979722732188682E	-10
35	-.201612935664335765043E	-10
36	-.503586127899763183811E	-11
37	.720004564890996011502E	-11
38	.733523576541123496435E	-13
39	-.222969009686513222418E	-11
40	.434676696846899579207E	-12

s = 3

n	C_n	
0	.482009921117873563940E	1
1	.207464609375841886668E	+0
2	-0.4464800645975191658 E	-2
3	-.141550160444320033007E	-1
4	.107033024610476208534E	-2
5	.170281861344961540837E	-2
6	-.241121909196354754901E	-3
7	-.237240232898045160655E	-3
8	.523878105409942686049E	-4
9	.347205864561481425397E	-4
10	-.110847293206668620191E	-4
11	-.509262971019762066958E	-5
12	.229344432003435231376E	-5
13	.719927990245891298779E	-6
14	-.464675836922533221622E	-6
15	-.926805103766439269921E	-7
16	.921857331996273102090E	-7
17	.939977449180465655624E	-8
18	-.178832584017718984382E	-7
19	-.226866192977492983902E	-9
20	.338340427419252037447E	-8
21	-.253987849844268536819E	-9
22	-.621575694010238398604E	-9
23	.104374508205004731252E	-9
24	.110103603200657041250E	-9
25	-.302326570193217174594E	-10
26	-.185818109578670601724E	-10
27	.761675450137240960198E	-11
28	.292220522577934860157E	-11
29	-.177027558147829935726E	-11
30	-.407888061678396884549E	-12
31	.388643862699195822224E	-12
32	.436247386021380888751E	-13
33	-.814427055782162970277E	-13
34	-.822449874867149474170E	-15
35	.163529341266589164244E	-13
36	-.142868310022285407985E	-14
37	-.314346127489149969692E	-14
38	.595927018064842592704E	-15
39	.575205522222226238022E	-15
40	-.177196226505388955337E	-15

Table 2-9 (Cont'd)

COEFFICIENTS IN THE EXPANSION OF $\tau_s(q)$ IN SERIES OF POWERS OF q

s = 4			s = 5		
n	C_n		n	C_n	
0	.616330735563948654764E	1	0	.737217725504777017709E	1
1	.162250548657936102033E	+0	1	.135645137847885376500E	+0
2	-0.2135642360567450681 E	-2	2	-0.1247908371246331331 E	-0
3	-.871885888108483636903E	-2	3	-.611024012145296672606E	-2
4	.402410619842825876227E	-3	4	.196956728154579785104E	-3
5	.835169061579160738271E	-3	5	.492636395584029387124E	-3
6	-.715411717297918463230E	-4	6	-.293739478791104806902E	-4
7	-.940484577604947548660E	-4	7	-.469441355702789480174E	-4
8	.123250806694419674755E	-4	8	.425278425395890034599E	-5
9	.113556493600618525638E	-4	9	.482898644434376956019E	-5
10	-.207942946138092999231E	-5	10	-.604210743853502578962E	-6
11	-.141552083148389594045E	-5	11	-.517255923549343302118E	-6
12	.345288938080386428785E	-6	12	.846807951845107587519E	-7
13	.178310153576483229811E	-6	13	.566222248307377168268E	-7
14	-.565743468169250793020E	-7	14	-.117408786069155518748E	-7
15	-.223398023785964983281E	-7	15	-.626014512469271172530E	-8
16	.915864613690264057954E	-8	16	.161306389701828703852E	-8
17	.274003833322792581555E	-8	17	.692921199471174813863E	-9
18	-.146580941264089365525E	-8	18	-.219817072813413395125E	-9
19	-.322156766946156574482E	-9	19	-.761879603802540392693E	-10
20	.231954836896550163320E	-9	20	.297285485754083347592E	-10
21	.349876185077911130826E	-10	21	.825167140366261595929E	-11
22	-.362814929059010130047E	-10	22	-.399126970144840181272E	-11
23	-.320816010636888886667E	-11	23	-.870853477307795498949E	-12
24	.560585749105978553231E	-11	24	.531994566730915210003E	-12
25	.166174637256453008339E	-12	25	.880790136775914933886E	-13
26	-.854721725044155760428E	-12	26	-.703920256953288535190E	-13
27	.238564649205608451232E	-13	27	-.827931750282463915706E	-14
28	.128403787802000396711E	-12	28	.924391820918759622175E	-14
29	-.111124625526267436876E	-13	29	.672761927061615099381E	-15
30	-.189657055735403621270E	-13	30	-.120430423707409114542E	-14
31	.282262923572534173224E	-14	31	-.358559142730492700989E	-16
32	.274576360166112777714E	-14	32	.155569225959342660292E	-15
33	-.598879977921170830455E	-15	33	-.194805975168616173696E	-17
34	-.387878416428699593968E	-15	34	-.199110512956005452574E	-16
35	.116112243130160990110E	-15	35	.109889316153168559930E	-17
36	.530932527256580213881E	-16	36	.252242072240334733227E	-17
37	-.212835013734951366795E	-16	37	-.250691683238504948922E	-18
38	-.696143796633045599963E	-17	38	-.315880900649625920202E	-18
39	.374870866724477213762E	-17	39	.462655764337535659503E	-19
40	.856136183499375499007E	-18	40	.390343629727189369135E	-19

Table 2-10

COEFFICIENTS IN THE EXPANSION OF $\tau_s(q)$ IN SERIES OF POWERS OF $\frac{1}{q}$

s = 1

n	D_n		n	D_n	
0	0.233810741045976703849E	1	40	-.700691631307825915940E	7
1	-0.100000000000000000000E	1	41	-0.172926385120568210974E	8
2	0.000000000000000000000E	0	42	.302402694909812084630E	8
3	.779369136819922346163E	+0	43	.296000071565089620851E	8
4	-.250000000000000000000E	+0	44	-.102975125398930052371E	9
5	-.109334925256937550787E	1	45	-.191061717492799559388E	8
6	.909263992956576070523E	+0	46	.300923950953512690432E	9
7	.164740570689507454422E	1	47	-.133232366387904713760E	9
8	-.264226069370932414401E	1	48	-.764222912713176726357E	9
9	-.213710405546052183249E	1	49	.850994329133701780251E	9
10	.691693414275288923156E	1	50	.162675880891976768802E	10
11	.112297788720997886992E	1	51	-.341788385744768603123E	10
12	-.166366387432300987771E	2	52	-.247043062983368563710E	10
13	.656326185545065316075E	1	53	.112194546138271726615E	11
14	.363549113840041841541E	2	54	-.815317134722237362844E	6
15	-.366549831925517812206E	2	55	-.318350972974930095218E	11
16	-.690160345205570526425E	2	56	.198978249303328147564E	11
17	.131615045605843940331E	3	57	.780657206146936957945E	11
18	.981386219147608638941E	2	58	-.105461120000807422780E	12
19	-.393408408913159305757E	3	59	-.156329720659350104282E	12
20	-.213671996252882455892E	2	60	.398523735722135716625E	12
21	.103281515436985573212E	4	61	.196308040671876342016E	12
22	-.557005440585398217545E	3	62	-.126091466443172167816E	13
23	-.2381019464111483180501E	4	63	.229775799379091896537E	12
24	.289471314674802657161E	4	64	.346149996770127167665E	13
25	.458512006481859752533E	4	65	-.283001749930191764780E	13
26	-.104802124890582268055E	5	66	-.813564326005073585241E	13
27	-.596169025865546235994E	4	67	.132197256989378198077E	14
28	.317985515060398198593E	5	68	.150225118847528990443E	14
29	-.318078127520901200706E	4	69	-.473745720389461439785E	14
30	-.842338171333703320929E	5	70	-.133433575875402257474E	14
31	.599334412880064201119E	5	71	.144494564473025958993E	15
32	.193151282158636859046E	6	72	-.525601236919508639310E	14
33	-.281949289996349905166E	6	73	-.382518041958423898398E	15
34	-.357114653412512609012E	6	74	.392093656248619176020E	15
35	.994137337829276568794E	6	75	.854485756965898248908E	15
36	.373873844573088761079E	6	76	-.166790003306127340751E	16
37	-.297606709730805582319E	7	77	-.141119396700401082819E	16
38	.797089484675489713151E	6	78	.569998253336990313415E	16
39	.776451218363028956443E	7	79	.479966949271702811900E	15
40	-.700691631307825915940E	7	80	-.167565476382404105172E	17

Table 2-10 (Cont'd)

COEFFICIENTS IN THE EXPANSION OF $\tau_s(q)$ IN SERIES OF POWERS OF $\frac{1}{q}$

S = 2			S = 3		
n	D_n		n	D_n	
0	0.408794944413097061664E	1	0	0.552055982809555105913E	1
1	-0.100000000000000000000E	1	1	-0.100000000000000000000E	1
2	0.000000000000000000000E	0	2	0.000000000000000000000E	0
3	.136264981471032353888E	1	3	.184018660936518368637E	1
4	-.250000000000000000000E	+0	4	-.250000000000000000000E	+0
5	-.334226613155414233100E	1	5	-.609531616311647605221E	1
6	.158975811716204412869E	1	6	.214688437759271430077E	1
7	.958072498187466118153E	1	7	.238568268211730906384E	2
8	-.807714315125584396657E	1	8	-.147303473941981504595E	2
9	-.289606323947006886556E	2	9	-.100408084579340719114E	3
10	.376982129296736853842E	2	10	.930894880661515916773E	2
11	.877995882095470272155E	2	11	.436994176766931869203E	3
12	-.167760105020341424762E	3	12	-.563545753786133821636E	3
13	-.255967606493247866609E	3	13	-.192348223313878665179E	4
14	.721539426891974386250E	3	14	.331909679840334836491E	4
15	.674081123208715342341E	3	15	.842204563946376258303E	4
16	-.301494237401875427363E	4	16	-.191560266673467278722E	5
17	-.135361694564129419441E	4	17	-.360834044520073565891E	5
18	.122526485280284000809E	5	18	.108731918893099484244E	6
19	.157476716681860290409E	3	19	.147974673976141442151E	6
20	-.483523552613142565644E	5	20	-.608060015755490440192E	6
21	.200064782665822597609E	5	21	-.558442645316957552551E	6
22	.184496002719801956776E	6	22	.335250812599213363659E	7
23	-.160211145918213774909E	6	23	.175863507603203542119E	7
24	-.675271406334836458618E	6	24	-.182217600841141022970E	8
25	.945821006563287773898E	6	25	-.287070720194862564449E	7
26	.233658608898229907082E	7	26	.975638613200809204214E	8
27	-.490494388913865440049E	7	27	-.195130038916713413993E	8
28	-.742563511396239639021E	7	28	-.513873835732817131141E	9
29	.235417329246284616321E	8	29	.289882554726960439069E	9
30	.202066569627730891038E	8	30	.265661541890561436602E	10
31	-.106819168797585985831E	9	31	-.253045511042209585852E	10
32	-.360831999548602782712E	8	32	-.134352379599288361076E	11
33	.462423864951584340725E	9	33	.186528961643430774135E	11
34	-.581352106544805958278E	8	34	.661264657333605684386E	11
35	-.191499451904169358447E	10	35	-.126085268866129378934E	12
36	.111523166809649654370E	10	36	-.314169700105549980136E	12
37	.756873375397621319870E	10	37	.806910306330725607249E	12
38	-.816870519515023714101E	10	38	.142076288139476560057E	13
39	-.283192995670739877330E	11	39	-.496486728517105679654E	13
40	.476366930503784312220E	11	40	-.595276431077473231451E	13

Table 2-10 (Cont'd)

COEFFICIENTS IN THE EXPANSION OF $\tau_s(q)$ IN SERIES OF POWERS OF $\frac{1}{q}$

s = 4			s = 5		
n	D_n		n	D_n	
0	0.678670809007175899878E	1	0	0.79441335871208531231 4E	1
1	-0.1000000000000000000000E	1	1	-0.1000000000000000000000E	1
2	0.0000000000000000000000E	0	2	0.0000000000000000000000E	0
3	.226223603002391966626E	1	3	.264804452904028437438E	1
4	-.250000000000000000000E	+0	4	-.250000000000000000000E	+0
5	-.921188133996909257102E	1	5	-.126218516900043266556E	2
6	.263927536836123961064E	1	6	.308938528388033177011E	1
7	.444773925819637973453E	2	7	.714426256730153383981E	2
8	-.222620465715919737132E	2	8	-.305028082508437894178E	2
9	-.232283524422012097350E	3	9	-.438509840023860337373E	3
10	.173097283218019533699E	3	10	.277722387611299512983E	3
11	.126483119480225848301E	4	11	.281597043881791580871E	4
12	-.129111934807469749795 4E	4	12	-.242736672093095400322E	4
13	-.704435665627940513355E	4	13	-.185773887047251189273E	5
14	.938908856439637652459E	4	14	.206998541918064754035E	5
15	.396453872687442843453E	5	15	.124542709565643700008E	6
16	-.670718686514875577695E	5	16	-.173599803803909220190E	6
17	-.223444958939151863042E	6	17	-.842217355393014798904E	6
18	.472652417387517293896E	6	18	.143802444978587472313E	7
19	.125109248560697874781E	7	19	.571296131911481063048E	7
20	-.329346321240754150219E	7	20	-.117958786076780056427E	8
21	-.689937578474802978072E	7	21	-.386839168525765089033E	8
22	.227230904694428226376E	8	22	.959684340540354019398E	8
23	.370601560258901527364F	8	23	.260242873753620300697E	9
24	-.155351878390258495614E	9	24	-.775166338622123638896E	9
25	-.190612090195564545633E	9	25	-.173028616833558957695E	10
26	.105282025903881159807E	10	26	.622017948453909056987F	10
27	.909347963725712870424E	9	27	.112940731678008807032E	11
28	-.707299041497392062528E	10	28	-.496039157176117248623E	11
29	-.372921331264621445658F	10	29	-.716856606306146334992E	11
30	.470950839860001634444E	11	30	.393203952651806751464E	12
31	.972990353516545271368E	10	31	.435647688768783477689E	12
32	-.310656496081978471794E	12	32	-.309834813243092928861E	13
33	.334100163974231074149E	11	33	-.246178414794792126796E	13
34	.202869118979671587764E	13	34	.242673194642102824360E	14
35	-.860188381526750810370E	12	35	.120705188769981359165E	14
36	-.131023879812984766356E	14	36	-.188891710891867397151E	15
37	.985400897742123353667E	13	37	-.396899763955661396635E	14
38	.835769118622300534071E	14	38	.146074712467919800538E	16
39	-.921547327665523841005E	14	39	-.106641737201243257643E	15
40	-.525531653958306525130E	15	40	-.112183000963845279888E	17

By using Eqs. (2-55) and (2-56) in Eqs. (2-21) and (2-22), respectively, it can be seen that the C_n and D_n satisfy the recursion formulae

$$nC_oC_n = -(n-2)C_{n-2} + \sum_{m=1}^{n-1} mC_mC_{n-m} \tag{2-57}$$

where $C_o = \beta_s$ and $C_oC_1 = 1$. In Program 2-1 we show an algorithm which will permit one to compute these coefficients. In order to relate the FORTRAN variables to the mathematical variables, we let N denote the largest value of n for which the C_n are desired, and we let the C_n be denoted in the FORTRAN program by C(I), where n = I. Finally, we denote the value of β_s by the symbol BET. The subroutine to accomplish the setting up of the array C(1), C(2), C(3),...,C(N) has been written to yield double precision results. However, the author wishes to direct the attention of the reader to the manner in which the constants 0 and 1 have been inserted in a DATA statement. This feature is important because the use of 0.0D0 and 1.0D0 has been avoided in the program's executable statements and therefore the only changes that must be made to

Program 2-1

```
      SUBRØUTINE CØEFSQ(BET,N,C)
      DØUBLE PRECISIØN BET, C(N),ZERØ,ØNE,DFLØAT
      DATA ZERØ,ØNE/0.0D0,1.0D0/
      DFLØAT(KK) = DBLE(FLØAT(KK))
      C(1) = ØNE/BET
      C(2) = -C(1)*C(1)/(ØNE + ØNE)
      DØ 1 I = 3,N
      MF = I - 1
      C(I) = DFLØAT(I-2)*C(I-2)
      DØ 2 M = 1,MF
      C(I) = C(I) + DFLØAT(M)*C(M)*C(I-M)
    2 CØNTINUE
      C(I) = - C(I)/DFLOAT(I)
    1 CØNTINUE
      RETURN
      END
```

convert the program to single precision is to delete the DØUBLE PRECISIØN type statements, the arithmetic statement function DFLOAT, and insert "DATA ZERØ,ØNE/0.0,1.0/" in place of the DATA statement that appears in the program.*

In order to check on the numerical stability of the recursion formula in Eq. (2-57), the calculation of the $C_n(\tau_1^o)$ in Table 2-9 was repeated in single precision arithmetic. The results are listed in Table 2-11. The process seems to be "self-correcting" since there are some entries which are good to only four figures in the middle of the table, but the result for n = 80 displays agreement to six figures (after round off is used to write C_{80} = -0.127232E-21).

In order to obtain the recursion formula for the coefficients D_n of Eq. (2-56), we need merely insert Eq. (2-54) in Eq. (2-52) and find that

$$nD_n = -\sum_{m=1}^{n-2} mD_m D_{n-m-2} \qquad (2\text{-}58)$$

where $D_o = \alpha_s$, $D_1 = -1$ and $D_2 = 0$. In Program 2-2 we present an algorithm for the obtaining of the array $D_1, D_2, D_3, \cdots, D_N$. However, before we consider this program, we will make a digression in order to emphasis the importance in practical problems of the case

*It is a good programming practice to avoid the use of constants such as 0.0, 1.0, and 3.1415927 in the "body" of any program in which there exists the possibility that one might need to go to DØUBLE PRECISIØN since one would then have to replace many of the executable statements with new statements containing 0.0D0, 1.0D0, and 3.14159265358979320D0, respectively. The use of devices such as ZERØ,ØNE,PI in the executable statements and the inserting of the constants by means of a DATA statement permits one to readily convert a program written in double precision to use it in single precision, or vice versa.

of q very large. We will then introduce another array which is important in practical problems, and use Program 2-2 to set up both of these arrays.

In Fig. 2-5 we have presented several curves which show that for vertically polarized waves the cases in which q tends to infinity plays a vital role include virtually all cases of propagation

Table 2-11

THE COEFFICIENTS $C_n(\tau_1^0)$ AS COMPUTED IN SINGLE PRECISION

n	C_n	n	C_n	n	C_n
0	0.10187930E 01	34	-0.29925379E-10	67	0.12985392E-18
1	0.98155367E 00	35	0.20137136E-10	68	0.77486458E-19
2	-0.47283778E 00	36	0.50409587E-11	69	-0.57664740E-19
3	0.13440523E-00	37	-0.71986632E-11	70	-0.12651211E-19
4	-0.71602707E-02	38	-0.73094466E-13	71	0.21074336E-19
5	-0.98774498E-02	39	0.22287107E-11	72	-0.37643501E-21
6	0.20129035E-02	40	-0.43431269E-12	73	-0.66167721E-20
7	0.13461956E-02	41	-0.59945250E-12	74	0.15187661E-20
8	-0.56666862E-03	42	0.25780213E-12	75	0.18143185E-20
9	-0.19195578E-03	43	0.13276179E-12	76	-0.86768527E-21
10	0.15557887E-03	44	-0.10743466E-12	77	-0.39931940E-21
11	0.18911074E-04	45	-0.19145934E-13	78	0.36222523E-21
12	-0.40866278E-04	46	0.37440696E-13	79	0.51994960E-22
13	0.24151776E-05	47	-0.17673479E-14	80	-0.12723196E-21
14	0.10056239E-04	48	-0.11336792E-13	81	0.10060423E-22
15	-0.24616357E-05	49	0.29012308E-14	82	0.38772412E-22
16	-0.22318140E-05	50	0.29520079E-14	83	-0.11394486E-22
17	0.10840956E-05	51	-0.15230350E-14	84	-0.10073228E-22
18	0.41000339E-06	52	-0.61132877E-15	85	0.57878818E-23
19	-0.38026834E-06	53	0.60614715E-15	86	0.20240678E-23
20	-0.42217450E-07	54	0.66508472E-16	87	-0.22905735E-23
21	0.11611411E-06	55	-0.20495320E-15	88	-0.16954143E-24
22	-0.10456789E-07	56	0.21679091E-16	89	0.77408635E-24
23	-0.31502187E-07	57	0.60237029E-16	90	-0.10850725E-24
24	0.92562826E-08	58	-0.19411866E-16	91	-0.22683739E-24
25	0.73732547E-08	59	-0.15019482E-16	92	0.82855039E-25
26	-0.41809671E-08	60	0.92845527E-17	93	0.55753642E-25
27	-0.13494919E-08	61	0.28244545E-17	94	-0.38520651E-25
28	0.15133817E-08	62	-0.35367505E-17	95	-0.98833368E-26
29	0.98226314E-10	63	-0.15803798E-18	96	0.14534777E-25
30	-0.47271572E-09	64	0.11573625E-17	97	0.12684577E-27
31	0.65598900E-10	65	-0.19181695E-18	98	-0.47271444E-26
32	0.12950258E-09	66	-0.32833645E-18	99	0.95721225E-27
33	-0.46165995E-10			100	0.13275000E-26

over land when the frequency exceeds several megacycles and all cases for propagation over the sea when the frequency exceeds several hundred megacycles. For the case of horizontally polarized waves, the case $q = \infty$ plays an important role throughout the wavelength range from the infra-red to the longest waves for all the values of the conductivity and dielectric constant which occur in nature. These circumstances certainly justify our giving this case special attention. It is noteworthy that Pryce and Domb devoted their entire paper (Ref. 25) to a consideration of the limiting case in which $q = \infty$.

In many cases, the magnitude of q will be so large that one need only used the approximation of Eq. (E-7), namely,

$$\exp(i\xi\tau_s) \approx \exp(i\xi\tau_s^\infty)\exp(i\xi/q) \tag{2-59}$$

to approximate the functions $W(x,y,y_o,q)$ and $F(x,y,q)$ defined in Eqs. (E-25) and (E-26) in the following manner:

$$W(x,y,y_o,q) \approx \exp(ix/q)W(x,y,y_o,\infty) \tag{2-60}$$

$$F(x,y,q) \approx \exp(ix/q)F(x,y,\infty) \tag{2-61}$$

When Eq. (2-59) is not sufficiently accurate, we can often evaluate the functions for large q by means of the expansions given below.

Since this case of $q \to \infty$ is so important, let us evaluate the coefficients $F_n(\alpha)$ in the expansion

$$\frac{d\tau_s(q)}{d\tau_s^\infty} = 1 + \sum_{n=3}^{\infty} F_n(\alpha_s)q^{-n}\exp(i\tfrac{1}{3}n\pi) \tag{2-62}$$

The reader will have to "look ahead" to the discussion in Section 5 of the Olver relation and the "normalization constant" to grasp some of the reasons why this expansion is so important. However,

we will assert (without a demonstration of the method of proof) that the importance lies in the fact that the following relationship

$$\frac{1}{(1 - \tau_s/q^2)[w_1'(\tau_s)]^2} = \frac{1}{[w_1'(\tau_s^\infty)]^2} \frac{d\tau_s(q)}{d\tau_s^\infty}$$

$$= \frac{1}{4\pi[Ai'(-\alpha_s)]^2} \frac{d\tau_s(q)}{d\alpha_s} \qquad (2\text{-}63)$$

permits us to express the diffraction function $V_{11}(\xi,q)$ in the form

$$V_{11}(\xi,q) = -2\sqrt{-1}\pi \sum_{s=1}^{\infty} \frac{\exp(i\xi\tau_s)}{(\tau_s - q^2)[w_1(\tau_s)]^2}$$

$$= 2\sqrt{-1}\pi \sum_{s=1}^{\infty} \frac{\exp(i\xi\tau_s)}{[w_1'(\tau_s^\infty)]^2} \frac{d\tau_s(q)}{d\tau_s^\infty} \qquad (2\text{-}64)$$

Therefore, with the aid of the coefficients D_n which are needed in Eq. (2-56) to compute $\tau_s = \tau_s(q)$ and the coefficients F_n which are needed in Eq. (2-62) to compute $d\tau_s(q)/d\tau_s^\infty$, we can evaluate the diffraction function $V_{11}(\xi,q)$ for the case in which q is very large. Since $V_{11}(\xi,q)$ is the diffraction function which is required when both source and receiver are at a very great height (in terms of wavelength) we observe that this function will be very frequently encountered in practice since the case of large values of q occurs in the high frequency range and it is therefore going to generally be the case that the antennas for transmission and reception are at a "great height." We should also make the observation that the diffraction function $V_o(\xi,q)$ can be readily evaluated for the case of $q \to \infty$ by expressing the residue series representation in the form

$$V_o(\xi,q) = -i\sqrt{i\pi\xi}\; q^{-2} \sum \frac{\exp(i\xi\tau_s)}{1 - \tau_s/q^2}$$

$$= -i\,\frac{\sqrt{i\pi\xi}}{q^2} \sum \exp(i\xi\tau_s)\,\frac{d\tau_s(q)}{d(\frac{1}{q})} \qquad (2\text{-}65)$$

Therefore, only the coefficients D_n are required in order to compute $V_o(\xi,q)$ for very large values of q.

By suitably combining the ideas employed to arrive at Eqs. (2-64) and (2-65), we can also evaluate the diffraction function $V_1(\xi,q)$. Since

$$V_1(\xi,q) = 2\sqrt{i\pi} \sum \frac{\exp(i\xi\tau_s)}{(\tau_s - q^2)w_1(\tau_s)} \qquad (2\text{-}66)$$

we need only observe that

$$\frac{1}{[(1 - \tau_s/q^2)w_1'(\tau_s)]^2} = \frac{1}{[w_1'(\tau_s^\infty)]^2}\left[\frac{d\tau_s(q)}{d\tau_s^\infty}\;\frac{d\tau_s(q)}{d(\frac{1}{q})}\right] \qquad (2\text{-}67)$$

to see that we can express $V_1(\xi,q)$ in the form

$$V_1(\xi,q) = -\frac{2\sqrt{i\pi}}{q^2} \sum \frac{\exp(i\xi\tau_s)}{w_1'(\tau_s^\infty)}\left[\frac{d\tau_s(q)}{d\tau_s^\infty}\;\frac{d\tau_s(q)}{d(\frac{1}{q})}\right]^{\frac{1}{2}} \qquad (2\text{-}68)$$

These results reveal that the knowledge of the coefficients D_n and F_n permit one to evaluate the three types of diffraction functions. Let us therefore return the to the development of the background theory for Program 2-2 in which these coefficients are to be computed.

Since $F_n(\alpha) = dD_n(\alpha)/d\alpha$, Eq. (2-58) leads to

$$F_n(\alpha) = \frac{dD_n(\alpha)}{d\alpha} = - \sum_{m=1}^{n-2} m[F_m D_{n-m-2} + D_m F_{n-m-2}]$$

$$= - \frac{(n - 1)}{n} \sum_{m=1}^{n-2} F_m D_{n-m-2} \qquad (2\text{-}69)$$

where $F_o = 1$, $F_1 = 0$ and $F_2 = 0$. Program 2-2 has been used to set up an array for each of these coefficients. In writing this program we have done what is so often required in translating a

Program 2-2

```
      SUBRØUTINE LARGEQ(ALP,N,D,F)
      DØUBLE PRECISIØN  ALP,D(N),F(N),DFLØAT,ZERØ,ØNE,AN,AN1
      DATA ZERØ,ØNE/0.0D0,1.0D0/
      DFLØAT(KK) = DBLE(FLØAT(KK))
      D(1) = ALP
      D(2) = -ØNE
      D(3) = ZERØ
      F(1) = ØNE
      F(2) = ZERØ
      F(3) = ZERØ
      DØ 1 I = 4,N
      AN = DFLØAT(I-1)
      K   = I-3
      D(I) = ZERØ
      F(I) = ZERØ
      DØ 2 J = 1,K
      L    = K-J+1
      AN1  = DFLØAT(J)
      D(I) = D(I)  + AN1*D(J+1)*D(L)
      F(I) = F(I)  + AN1*(D(J+1)*F(L) + D(L)*F(J+1))
    2 CØNTINUE
      D(I) = - D(I)/AN
      F(I) = - F(I)/AN
    1 CØNTINUE
      RETURN
      END
```

mathematical expression into FORTRAN instructions - we have dimensioned the arrays D(N) and F(N), but since we need a place to put D_o and F_o, we have used the following convention

$$
\begin{array}{ll}
D_o = D(1) & F_o = F(1) \\
D_1 = D(2) & F_1 = F(2) \\
D_2 = D(3) & F_2 = F(3) \\
\vdots & \vdots \\
D_{N-1} = D(N) & F_{N-1} = F(N)
\end{array}
$$

The quantity denoted by ALP in the subroutine's argument list is the Airy function root α_s which is required in order to determine the coefficients for the root $\tau_s(q)$. In Table 2-12 we give the computer output which results from running the subroutine contained in Program 2-2. The case given in the table is that for s = 1. These tables were actually computed with the subroutine given in Program 2-3 rather than the one which is designated as Program 2-2. We take this occasion to present the instructions contained in Program 2-3 in order to permit the reader to have an example of a program which employs the NPREC subroutines which have been discussed in the Preface. There is a one-to-one correspondence between the instructions in Program 2-2 and those in Program 2-3, but the use of the NPREC subroutines requires that one put in a number of additional instructions which enable one to employ the computer software which permits the working with 21 decimal digits. The reader who is unfamiliar with NPREC will notice that in order to store the numbers one needs three conventional 36-bit "words," it is necessary to set us an array such as D(3,51) in order to obtain 51 values of D. Also, since this is a FORTRAN II program, the use of adjustatble dimensions in the arrays is not possible and therefore the integer 51 appears explicitly because we set out to obtain the coefficients for n = 0, 1, 2,...,50.

Table 2-12

THE COEFFICIENTS $D_n(a_1)$ AND $F_n(a_1)$

n	$D_n(a_1)$		$F_n(a_1)$	
0	.233810741045976700000E	1	.100000000000000000000E	1
1	-.100000000000000000000E	1	+0.0	
2	+0.0		+0.0	
3	.779369136819922333333E	+0	.333333333333333333333E	+0
4	-.250000000000000000000E	+0	-0.0	
5	-.109334925256937547187E	1	-.935242964183906799999E	+0
6	.909263992956576055554E	+0	.388888888888888888888E	+0
7	.164740570689507445404E	1	[illegible]	1
8	-.264226069370932405702E	1	-.226017049677777476666E	1
9	-.213710405546052163332E	1	-.517464491527850323975E	1
10	.691693414275288888169E	1	.909041807136252178040E	1
11	.112297788720997851922E	1	.911112745519882451207E	1
12	-.166366387432300976017E	2	-.305386585496655844333E	2
13	.656326185545065344842E	1	-.747305883153744299052E	1
14	.363549113840041806875E	2	.900680865684224769793E	2
15	-.366549831925517799523E	2	-.329562820431761948952E	2
16	-.690160345205570436150E	2	-.234546326422511212139E	3
17	.131615045605843931255E	3	[illegible]	3
18	.981386219147608437748E	2	.522730919468299964612E	3
19	-.393408408913159267568E	3	-.992323025066159018309E	3
20	-.213671996252882116077E	2	[illegible]	3
21	.103281515436985560184E	4	.338534769300161144809E	4
22	-.557005440585398234408E	3	.437853476401823925701E	3
23	-.238101946411483142016E	4	[illegible]	4
24	.289471314674802638095E	4	.495493922692861108129E	4
25	.458512006481859653460E	4	.257420606271768706578E	5
26	-.104802124890582256371E	5	-.303645112186488398690E	5
27	-.596169025865546022592E	4	-.554454619032728532684E	5
28	.317985515060398151420E	5	.122585863334508983208E	6
29	-.318078127520901523539E	4	.838693002088812401608E	5
30	-.842338171333703163827E	5	[illegible]	6
31	.599334412880064196210E	5	.128148424303529659982E	5
32	.193151282158636813687E	6	.117864891622043676654E	7
33	-.281949289996349875198E	6	[illegible]	6
34	-.357114653412512495799E	6	-.294166033377288326956E	7
35	.994137337829276410111E	6	.412402065801852456725E	7
36	.373873844573088530982E	6	[illegible]	7
37	-.297606709730805521484E	7	-.158099112391683853018E	8
38	.797089484675490000935E	6	-.747523525814095398903E	7
39	.776451218363028760465E	7	.509293688347906588641E	8
40	-.700691631307825876177E	7	-.103438493038375554854E	8

Program 2-3

```
      SUBRØUTINE LARGEQ(ALP,D,F)
      DIMENSIØN ALP(3),D(3,51),C(3,51),AN(3),AN1(3)
      CØMMØN AN,AN1
      CALL SP2NP(0.0,D(1,3))
      CALL SP2NP(-1.0 ,D(1,2))
      CALL NPREC
      D(1,1) = ALP
      CALL NPØUT
      CALL SP2NP(0.0,F(1,3))
      CALL SP2NP(0.0,F(2,3))
      CALL SP2NP(1.0,F(1,1))
      DØ 1 I = 4,51
      AN = FLØATF(I-1)
      CALL SP2NP(AN,AN)
      K  = I - 3
      CALL SP2NP(0.0,D(1,I))
      CALL SP2NP(0.0,F(1,I))
      DØ 2 J = 1,K
      L = K - J + 1
      AN1 = FLØATF(J)
      CALL SP2NP(AN1,AN1)
      CALL NPREC
      D(1,I) = D(1,I) + AN1*D(1,J+1)*D(1,L)
      F(1,I) = F(1,I) + AN1*(D(1,J+1)*F(1,L) + D(1,L)*F(1,J+1))
      CALL NPØUT
      CALL NP2SP(AN1,AN1)
    2 CØNTINUE
      CALL NPREC
      D(1,I) = - D(1,I)/AN
      F(1,I) = - F(1,I)/AN
      CALL NPØUT
      CALL NP2SP(AN,AN)
    1 CØNTINUE
      RETURN
      END
```

Before leaving the subject of these classical expansions, we will present the numerical values for the coefficients for further terms in the explicit representations for $A_n'(\tau)$ and $B_n'(\tau)$. The data which we have obtained in presented in Block 2-1. This is a FORTRAN IV BLØCK DATA subprogram and the data are placed in a labeled COMMØN which we have designated as BREMM. The data consists of the arrays BREM(18,9) and BR(20,3). The computations which led to the obtaining of these data were undertaken because

```
      BLOCK DATA
      COMMON/BREMM/BREM(18,9),BR(20,3)
      DOUBLE PRECISION BREM,BR
      DATA BREM(1,1), BREM(2,1), BREM(3,1), BREM(3,2), BREM(4,1),       BREM1  D
     A   BREM(4,2), (BREM(5,I),I=1,3), (BREM(6,I),I=1,3), (BREM(7,I),   BREM1  D
     B   I=1,4), (BREM(8,I),I=1,4),(BREM(9,I),I=1,5), (BREM(10,I),      BREM1  D
     C   I=1,5)      /                                                  BREM1  D
     D -0.10000000000000000000D+01, +0.15000000000000000000D+01,        BREM1  1
     E -0.25000000000000000000D+01, -0.66666666666666666667D+00,        BREM1  2
     F +0.43750000000000000000D+01, +0.23333333333333333333D+01,        BREM1  3
     G -0.78750000000000000000D+01, -0.63000000000000000000D+01,        BREM1  4
     H -0.60000000000000000000D+00, +0.14437500000000000000D+02,        BREM1  5
     I +0.16399999999999999999D+02, +0.32222222222222222222D+01,        BREM1  6
     J -0.26812500000000000000D+02, -0.35750000000000000000D+02,        BREM1  7
     K -0.11822222222222222222D+02, -0.571428571428571428571D+00,       BREM1  8
     L +0.50273437500000000000D+02, +0.80437500000000000000D+02,        BREM1  9
     M +0.36675000000000000000D+02, +0.415714285714285714285D+01,       BREM1 10
     N -0.94960937500000000000D+02, -0.17726041666666666667D+03,        BREM1 11
     O -0.10341527777777777778D+03, -0.19274779541446208112 8D+02,      BREM1 12
     P -0.55555555555555555555D+00, +0.18042578125000000000D+03,        BREM1 13
     Q +0.38490833333333333333D+03, +0.27416277777777777777D+03,        BREM1 14
     R +0.72360934744268077601 0D+02, +0.51311111111111111111D+01/      BREM1 15
      DATA (BREM(11,I),I=1,6),(BREM(12,I),I=1,6),(BREM(13,I),I=1,7),    BREM1  D
     A   (BREM(14,I),I=1,7), (BREM(15,I),I=1,8)     /                   BREM1  D
     B -0.34444921875000000000D+03, -0.82667812500000000000D+03,        BREM1 16
     C -0.69615000000000000000D+03, -0.23968454545454545454 5D+03,      BREM1 17
     D -0.28827272727272727272 7D+02, -0.54545454545454545454 5D+00,    BREM1 18
     E +0.66019433593750000000D+03, +0.17605182291666666666D+04,        BREM1 19
     F +0.17122895833333333333D+04, +0.73036283389450056115 7D+03,      BREM1 20
     G +0.12713674897119341563 6D+03, +0.61389129389129389129 4D+01,    BREM1 21
     H -0.12696044921875000000D+04, -0.37241731770833333333D+04,        BREM1 22
     I -0.41100067708333333333D+04, -0.20965056341189674522 6D+04,      BREM1 23
     J -0.48385435839880284324 1D+03, -0.40628634328634328634 3D+02,    BREM1 24
     K -0.53846153846153846153 8D+00, +0.24485229492187500000D+04,      BREM1 25
     L +0.78352734375000000000D+04, +0.96763734375000000000D+04,        BREM1 26
     M +0.57536891883116883116 9D+04, +0.16658590909090909090 9D+04,    BREM1 27
     N +0.20616411588411588411 6D+03, +0.71765377479663193948 6D+01,    BREM1 28
     O -0.47338110351562500000D+04, -0.16410544921875000000D+05,        BREM1 29
     P -0.22427325555555555555 5D+05, -0.15247142561928811928 8D+05,    BREM1 30
     Q -0.53323218255571588903 5D+04, -0.88712486360278952870 1D+03,    BREM1 31
     R -0.54813098746051127003 1D+02, -0.53333333333333333333D+00/      BREM1 32
```

```
 DATA (BREM(16,I),I=1,8), (BREM(17,I),I=1,9),(BREM(18,I),I=1,9)/    BREM1  D
A +0.91717588806152343750OD+04, +0.34241233154296875000OD+05,         BREM1 33
B +0.51311770887586805555D+05, +0.39286165424520502643 8D+05,         BREM1 34
C +0.16147079150022045854 8D+05, +0.34069692750980158386 8D+04,       BREM1 35
D +0.31505766485718866670 8D+03, +0.82408147408147408147 1D+01,       BREM1 36
E -0.17804002532958984375 0D+05, -0.71216010131835937500 0D+05,       BRE71 37
F -0.11612624194335937500 0D+06, -0.98923729017857142852 5D+05,       BREM1 38
G -0.46803984281250000000 0D+05, -0.12034291181612505141 5D+05,       BREM1 39
H -0.15139770680160175958 2D+02, -0.71503395428101310453 7D+02,       BREM1 40
I -0.52941176470588235294 1D+00, +0.34618893814086914062 5D+05,       BREM1 41
J +0.14770728027343750000 0D+06, +0.26038642154947916666 7D+06,       BREM1 42
K +0.24435742340133101850 5D+06, +0.13095257575147890945 8D+06,       BREM1 43
L +0.39860161242696926246 4D+05, +0.64208609705147770989 6D+04,       BREM1 44
M +0.45985941223305057727 0D+03, +0.93291779389818605504 6D+01/       BREM1 45
 DATA BR(3,1),BR(5,1),BR(6,1),BR(7,1),BR(8,1),BR(9,1),BR(9,2),        BREM2  D
A   BR(10,1),BR(11,1),BR(11,2),BR(12,1),BR(12,2),BR(13,1),BR(13,2),BREM2  D
B   BR(14,1),BR(14,2),BR(15,1),BR(15,2),BR(15,3),BR(16,1),BR(16,2) BREM2  D
C   ,(BR(17,I),I=1,3),(BR(18,I),I=1,3),(BR(19,I),I=1,3),(BR(20,I), BREM2  D
D I=1,3)/0.33333333333333333333D0,0.4D0,0.38888888888888888888D0,BREM2  1
E +0.42857142857142857142 9D+00, +0.96666666666666666666 7D+00,       BREM2  2
F +0.50617283950617283950 6D+00, +0.44444444444444444444 4D+00,       BREM2  3
G +0.16628571428571428571 4D+01, +0.19131313131313131313 1D+01,       BREM2  4
H +0.45454545454545454545 4D+00, +0.69231334509112286889 9D+00,       BREM2  5
I +0.24433862433862433862 4D+01, +0.45323076923076923076 9D+01,       BREM2  6
J +0.46153846153846153846 1D+00, +0.35138883162692686502 1D+01,       BREM2  7
K +0.32887033601319315605 0D+01, +0.97418626298255927885 3D+00,       BREM2  8
L +0.86194426807760141093 4D+01, +0.46666666666666666666 7D+00,       BREM2  9
M +0.10604600399600399600 4D+02, +0.41863136863136863136 8D+01,       BREM2 10
N +0.62078274631962773886 0D+01, +0.14391435766225682192 0D+02,       BREM2 11
O +0.47058823529411764705 8D+00, +0.13963268045778333844 1D+01,       BREM2 12
P +0.24753261410742892224 3D+02, +0.51276180609513942847 1D+01,       BREM2 13
Q +0.22762617157278811414 1D+02, +0.22036799290934629280 4D+02,       BREM2 14
R +0.47368421052631578947 3D+00, +0.10709012800723040374 4D+02,       BREM2 15
S +0.49347279188494874769 3D+02, +0.61063710146063087239 5D+01/       BREM2 16
 END
```

the author was interested in obtaining an asymptotic expansion for the function $V_2(0,q)$ which describes the "foward scattered field" which has attracted so much attention in the literature. This function can be defined by the relationship

$$V_2(0,q) = \lim_{\xi\to 0} [V_{11}(\xi,q) - \frac{\sqrt{I}}{2\sqrt{\pi}\ \xi}] \qquad (2\text{-}70)$$

An expansion in powers of q can be found "operationally" from the divergent series representation

$$V_2(0,q) = -2\sqrt{-I\pi}\sum_{s=1}^{\infty} \frac{1}{\tau_s[w_1(\tau_s^o)]^2} \frac{d\tau_s(q)}{d\tau_s^o} \qquad (2\text{-}71)$$

and an expansion in inverse powers of q comes from the divergent series representation

$$V_2(0,q) = 2\sqrt{I\pi}\sum_{s=1}^{\infty} \frac{1}{[w_1{}'(\tau_s^{\infty})]} \frac{d\tau_s^{\infty}(q)}{d\tau_s^{\infty}} \qquad (2\text{-}72)$$

Therefore, the quantities which were sought in the computer program and which have been prepared for re-entry into the computer with the aid of Block 2-1 were the derivatives $A_n(\tau_s^o)$ and $B_n(\tau_s^{\infty})$ which appear in the expressions

$$\frac{d\tau_s(q)}{d\tau_s^o} = 1 + A_1'(\tau_s^o)q + A_2'(\tau_s^o)q^2 + \ldots \qquad (2\text{-}73)$$

$$\frac{d\tau_s(q)}{d\tau_s^{\infty}} = 1 + B_1'(\tau_s^{\infty})q^{-1} + B_2'(\tau_s^{\infty})q^{-2} + \ldots \qquad (2\text{-}74)$$

where the explicit forms for the first few coefficients are

$$A_1'(\tau) = -\frac{1}{\tau^2} \quad , \quad B_1'(\tau) = B_2'(\tau) = 0$$

$$A_2'(\tau) = \frac{3}{2\tau^4} \quad , \quad B_3'(\tau) = \frac{1}{3} \ , \ B_4'(\tau) = 0$$

$$A_3'(\tau) = -\frac{5}{2\tau^6} - \frac{2}{3\tau^3} \quad , \quad B_5'(\tau) = \frac{2}{5}\tau$$

$$A_4'(\tau) = \frac{35}{8\tau^8} + \frac{7}{3\tau^5} \quad , \quad B_6'(\tau) = \frac{7}{18}$$

The array BREM(18,9) in Block 2-1 contains the numerical values of the coefficients E_n^m in the expansion

$$A_n'(\tau) = \frac{1}{\tau^{2n}} \left[\sum_{m=1}^{K} E_n^m \tau^{3m} \right] \quad , \ K = [(n-1)/2] \qquad (2\text{-}75)$$

where the [...] denote that K is the integral part of the expression enclosed in brackets. The data go as far as n = 18. The relation between the notation in Eq. (2-75) and that in Block 2-1 is E_N^M = BREM(N,M).

In Table 2-13 we give values for $C_n(\beta) = A_n(\tau)\exp[i\frac{1}{3}(2n-1)]$ and $E_n(\beta) = A_n'(\tau)\exp(i\frac{2}{3}n\pi)$ for the case in which $\beta = \beta_1$ and $\tau = \tau_1^o = \beta_1 \exp(i\frac{1}{3}\pi)$.

The numbers which appear in the array BR(20,3) contain the non-zero coefficients which appear in the expansion

$$B_n'(\tau) = \sum_{m=1}^{K} F_n^m \tau^{m-1} \quad , \quad K = [(n-1)/2] \qquad (2\text{-}76)$$

In Table 2-14 we give values of the F_n^m for $n \leq 38$. The array BR(20,3) can be used to set up an array F(20,9) which is related to the coefficients F_n^m by means of the relation F_N^M = F(N,M). This array can be set up by the following set of FORTRAN statements:

(Text continues on p. 2-84)

Table 2-13

THE COEFFICIENTS $C_n(\beta_1)$ AND $E_n(\beta_1)$

n	$C_n(\beta_1)$	$E_n(\beta_1)$
0	.101879297164747108902E 1	.100000000000000000000E 1
1	.981553689345656433497E +0	-.963447645068069415971E +0
2	-.472637795253974034105E +0	.139234704718081299304E 1
3	.134405245313929447297E +0	-.160530541253442531730E 1
4	-.716029388859209580331E -2	.164363810136019448228E 1
5	-.987742821206376960197E -2	-.156395342100306465911E 1
6	.201288471684499747311E -2	.140433001024760476667E 1
7	.134621180350283637869E -2	-.119639040814776139250E 1
8	-.566681835101367907230E -3	.972004600707734501909E +0
9	-.191945484867659312143E -3	-.757717557549668545245E +0
10	.155571157435453498406E -3	.569740055713202685596E +0
11	.189166890896791432830E -4	-.414678146099079026854E +0
12	-.408702510997938818891E -4	.292864546871329579316E +0
13	.241790842855597400745E -5	-.201148220818662611706E +0
14	.100544124693377437353E -4	.134661407109539702616E +0
15	-.246044274431581338876E -5	-.880453314035690400946E -1
16	-.223257729626952945715E -5	.563061842171953364019E -1
17	.108457400964019668246E -5	-.352627932244345050180E -1
18	.409709666849016849750E -6	.216523299144131938941E -1
19	-.380091416474586772410E -6	-.130508419632632016660E -1
20	-.423222225242223621635E -7	.772975324121657726841E -2
21	.116175174157170050392E -6	-.450227643940897757868E -2
22	-.104918046361155016318E -7	.258073416035174899469E -2
23	-.314824302962560680308E -7	-.145688604169574106685E -2
24	.924529616068753927830E -8	.810601317242295926351E -3
25	.737928467701561035388E -8	-.444789539107214678671E -3
26	-.418423398118797702311E -8	.240605606621087977668E -3
27	-.134774620393338725142E -8	-.126689699647300672113E -3
28	.151246110574567812812E -8	.679246940481439168851E -4
29	.987063651581319958735E-10	-.354287393074440126080E -4
30	-.472963452703912028671E -9	.182675229276037788850E -4
31	.657252832331850776568E-10	-.931339789873563380912E -5
32	.129438903271273955062E -9	.469676146873087925519E -5
33	-.461342576725111479054E-10	-.234407892285999801765E -5
34	-.299410658604387531830E-10	.115824554865677469284E -5
35	.201448235521620740240E-10	-.566671662757563405209E -6
36	.503723413650094374329E-11	.274538920314720872368E -6
37	-.719688066091402338794E-11	-.131766639232384328269E -6
38	-.739409839168563117529E-13	.626896037470605635857E -7
39	.222911109641159853378E-11	-.295697786310450413045E -7
40	-.434500398394768565945E-12	.138239029554649131945E -7

Table 2-14

COEFFICIENTS FOR POLYNOMIAL REPRESENTATION FOR $F_n(\tau)$

m	F_n^m
	n = 3
1	.333333333333333333333E +0
	n = 5
1	+0.0
2	.400000000000000000000E +0
	n = 6
1	.388888888888888888888E +0
2	+0.0
	n = 7
1	+0.0
2	+0.0
3	.428571428571428571428E +0
	n = 8
1	+0.0
2	.966666666666666666666E +0
3	+0.0
	n = 9
1	.506172839506172839506E +0
2	+0.0
3	+0.0
4	.444444444444444444444E +0
	n = 10
1	+0.0
2	+0.0
3	.166285714285714285714E 1
4	+0.0
	n = 11
1	+0.0
2	.191313131313131313131E 1
3	+0.0
4	+0.0
5	.454545454545454545454E +0

m	F_n^m
	n = 12
1	.692313345091122868899E +0
2	+0.0
3	+0.0
4	.244338624338624338624E 1
5	+0.0
	n = 13
1	+0.0
2	+0.0
3	.453230769230769230769E 1
4	+0.0
5	+0.0
6	.461538461538461538461E +0
	n = 14
1	+0.0
2	.351388831626926865021E 1
3	+0.0
4	+0.0
5	.328870336013193156050E 1
6	+0.0
	n = 15
1	.974186262982559278853E +0
2	+0.0
3	+0.0
4	.861944266077601410934E 1
5	+0.0
6	+0.0
7	.466666666666666666666E +0
	n = 16
1	+0.0
2	+0.0
3	.106046003996003996003E 2
4	+0.0
5	+0.0
6	.418631368631368631368E 1
7	+0.0

Table 2-14 (Cont'd)

n = 17

m	F_n^m
1	+0.0
2	.62078274631962773886OE 1
3	+0.0
4	+0.0
5	.14391435766225682192OE 2
6	+0.0
7	+0.0
8	.47058823529411764705BE +0

n = 18

m	F_n^m
1	.13963268045778333844 1E 1
2	+0.0
3	+0.0
4	.24753261410742892224 3E 2
5	+0.0
6	+0.0
7	.51276160609513942847 1E 1
8	+0.0

n = 19

m	F_n^m
1	+0.0
2	+0.0
3	.22762617157278811414 1E 2
4	+0.0
5	+0.0
6	.22036799290934629280 4E 2
7	+0.0
8	+0.0
9	.47368421052631578947 3E +0

n = 20

m	F_n^m
1	+0.0
2	.10709012800723040374 4E 2
3	+0.0
4	+0.0
5	.49347279188494874769 3E 2
6	+0.0
7	+0.0
8	.61063710146063087239 5E 1
9	+0.0

n = 21

m	F_n^m
1	.20274541706387119449 4E 1
2	+0.0
3	+0.0
4	.62976153561769082051 1E 2
5	+0.0
6	+0.0
7	.31722460490714458968 3E 2
8	+0.0
9	+0.0
10	.47619047619047619047 6E +0

n = 22

m	F_n^m
1	+0.0
2	+0.0
3	.46215836994616496847 8E 2
4	+0.0
5	+0.0
6	.88328698564512966828 7E 2
7	+0.0
8	+0.0
9	.71178476181853609379 5E 1
10	+0.0

n = 23

m	F_n^m
1	+0.0
2	.18179563827781342377 0E 2
3	+0.0
4	+0.0
5	.14595480003533956902 6E 3
6	+0.0
7	+0.0
8	.43598397463553960935 3E 2
9	+0.0
10	+0.0
11	.47826086956521739130 4E +0

Table 2-14 (Cont'd)

n = 24

m	F_n^m
1	.297219554537243524432E 1
2	+0.0
3	+0.0
4	.147849742469768713663E 3
5	+0.0
6	+0.0
7	.146156343026427223846E 3
8	+0.0
9	+0.0
10	.815835735185580386818E 1
11	+0.0

n = 25

m	F_n^m
1	+0.0
2	+0.0
3	.902512831722711502371E 2
4	+0.0
5	+0.0
6	.299041715018353043361E 3
7	+0.0
8	+0.0
9	.578009277342415535489E 2
10	+0.0
11	+0.0
12	.479999999999999999999E +0

n = 26

m	F_n^m
1	+0.0
2	.305068441582787861590E 2
3	+0.0
4	+0.0
5	.390583320198562067332E 3
6	+0.0
7	+0.0
8	.227773164774082519575E 3
9	+0.0
10	+0.0
11	.922494289667704105851E 1
12	+0.0

n = 27

m	F_n^m
1	.438946419058804238827E 1
2	+0.0
3	+0.0
4	.327755966259362474273E 3
5	+0.0
6	+0.0
7	.559252102214138582868E 3
8	+0.0
9	+0.0
10	.744551208285921380796E 2
11	+0.0
12	+0.0
13	.481481481481481481481E +0

n = 28

m	F_n^m
1	+0.0
2	+0.0
3	.171242983819010583614E 3
4	+0.0
5	+0.0
6	.901440777730707781402E 3
7	+0.0
8	+0.0
9	.338577906133716382174E 3
10	+0.0
11	+0.0
12	.103151642807501134905E 2
13	+0.0

n = 29

m	F_n^m
1	+0.0
2	.507488875599931131760E 2
3	+0.0
4	+0.0
5	.972109168149282435475E 3
6	+0.0
7	+0.0
8	.974672668698103394744E 3
9	+0.0

Table 2-14 (Cont'd)

n = 29 (Cont'd)

m	F_n^m
10	+0.0
11	.936766189615653190600E 2
12	+0.0
13	+0.0
14	.482758620689655172413E +0

n = 30

m	F_n^m
1	.652069314549479024577E 1
2	+0.0
3	+0.0
4	.695945579296443016777E 3
5	+0.0
6	+0.0
7	.188022396839860045014E 4
8	+0.0
9	+0.0
10	.484400309450673263990E 3
11	+0.0
12	+0.0
13	.114270663855553701652E 2
14	+0.0

n = 31

m	F_n^m
1	+0.0
2	+0.0
3	.317794011464592426888E 3
4	+0.0
5	+0.0
6	.249550314037575970858E 4
7	+0.0
8	+0.0
9	.160582473370178115902E 4
10	+0.0
11	+0.0
12	.115573044314335074603E 3
13	+0.0
14	+0.0
15	.483870967741935483870E +0

n = 32

m	F_n^m
1	+0.0
2	.838496463337888808773E 2
3	+0.0
4	+0.0
5	.228979320141244052940E 4
6	+0.0
7	+0.0
8	.362400322674403857380E 4
9	+0.0
10	+0.0
11	.671479258008439025267E 3
12	+0.0
13	+0.0
14	.125588863013868467797E 2
15	+0.0

n = 33

m	F_n^m
1	.973305410627465997017E 1
2	+0.0
3	+0.0
4	.142898572373963308229E 4
5	+0.0
6	+0.0
7	.574360013033138447610E 4
8	+0.0
9	+0.0
10	.252699166253536099042E 4
11	+0.0
12	+0.0
13	.140245109198466331867E 3
14	+0.0
15	+0.0
16	.484848484848484848484E +0

n = 34

m	F_n^m
1	+0.0
2	+0.0
3	.579477224434114964707E 3
4	+0.0
5	+0.0
6	.647172918046845199628E 4
7	+0.0

Table 2-14 (Cont'd)

n = 34 (Cont'd)

m	F_n^m
8	+0.0
9	.655539933695388262864E 4
10	+0.0
11	+0.0
12	.906443355298408425055E 3
13	+0.0
14	+0.0
15	.137091567046603180395E 2
16	+0.0

n = 35

m	F_n^m
1	+0.0
2	.137786191682490519142E 3
3	+0.0
4	+0.0
5	.516500505178988111748E 4
6	+0.0
7	+0.0
8	.121316535158478055037E 5
9	+0.0
10	+0.0
11	.382751190307577062323E 4
12	+0.0
13	+0.0
14	.167787507355019997021E 3
15	+0.0
16	+0.0
17	.485714285714285714284E +0

n = 36

m	F_n^m
1	.145855972182003225723E 2
2	+0.0
3	+0.0
4	.285628962836130224192E 4
5	+0.0
6	+0.0
7	.162922613170870526868E 5
8	+0.0
9	+0.0
10	.112546659995619454984E 5
11	+0.0
12	+0.0
13	.119629354980511123509E 4

n = 36 (Cont'd)

m	F_n^m
14	+0.0
15	+0.0
16	.148767067820837517100E 2
17	+0.0

n = 37

m	F_n^m
1	+0.0
2	+0.0
3	.104162580780469409211E 4
4	+0.0
5	+0.0
6	.159372661320496727412E 5
7	+0.0
8	+0.0
9	.239040540638447963500E 5
10	+0.0
11	+0.0
12	.561303917260750868718E 4
13	+0.0
14	+0.0
15	.198289640531145198050E 3
16	+0.0
17	+0.0
18	.486486486486486486485E +0

n = 38

m	F_n^m
1	+0.0
2	.225406614380351157893E 3
3	+0.0
4	+0.0
5	.112501909112857603875E 5
6	+0.0
7	+0.0
8	.374247923442398553549E 5
9	+0.0
10	+0.0
11	.164955702084799466403E 5
12	+0.0
13	+0.0
14	.154838749050109722940E 4
15	+0.0
16	+0.0
17	.160603453183037816598E 2
18	+0.0

```
      CØMMØN /BREMM/BREM(18,9),BR(20,3)
      DØUBLE PRECISIØN F(20,9),BREM,BR
      DØ 1 I = 1,20
      DØ 2 J = 1,9
      F(I,J) = 0.0D0
    1 CØNTINUE
    2 CØNTINUE
      F(3,1) = BR(3,1)
      F(5,2) = BR(5,1)
      DØ 3 K = 2,6
      I0     = 3*K
      I1     = I0 + 1
      I2     = I0 + 2
      F(I0,1)= BR(I0,1)
      F(I1,3)= BR(I1,1)
      F(I2,2)= BR(I2,1)
    3 CØNTINUE
      F(9,3) = BR(9,2)
      F(11,5)= BR(11,2)
      DØ 4 K = 4,6
      I0     = 3*K
      I1     = I0 + 1
      I2     = I0 + 2
      DØ 5 J = 2,3
      J0     = 3*J - 2
      J1     = J0 + 2
      J2     = J0 + 1
      F(I0,J0) = BR(I0,J)
      F(I1,J1) = BR(I1,J)
      F(I2,J2) = BR(I2,J)
    5 CØNTINUE
    4 CONTINUE
```

As an illustration of the usefulness of the results given in the relations which have been labeled Eq. (2-75) and Eq. (2-76), let us consider Eq. (2-72) in further detail. We can show that the function $V_2(\xi,\infty)$ is an entire function of ξ which can be expanded in a Taylor series of the form

$$V_2(\xi,\infty) = 2\sqrt{i\pi} \sum_{s=1}^{\infty} \frac{\exp(i\xi\tau_s^{\infty})}{[w_1'(\tau_s^{\infty})]^2} = \sum_{n=0}^{\infty} a_n \frac{\xi^n}{n!} \tag{2-77}$$

Let us now observe that

$$\frac{d^r V_2(\xi,\infty)}{d\xi^r} = 2\sqrt{i\pi} \sum_{s=1}^{\infty} \frac{(i\tau_s^\infty)^r \exp(i\xi\tau_s^\infty)}{[w_1'(\tau_s^\infty)]^2} = \sum_{n=r}^{\infty} a_n \frac{\xi^{n-r}}{(n-r)!} = \sum_{n=0}^{\infty} a_{n+r} \frac{\xi^n}{n!}$$

Therefore, we can express a function of the form

$$F(\xi) = 2\sqrt{i\pi} \sum_{s=1}^{\infty} [c_o + c_1\tau_s^\infty + c_2(\tau_s^\infty)^2 + \dots + c_m(\tau_s^\infty)^m] \frac{\exp(i\xi\tau_s^\infty)}{[w_1'(\tau_s^\infty)]^2}$$

in terms of the function $V_2(\xi,\infty)$ according to the rule

$$F(\xi) = \left[c_o + c_1\frac{d}{d(i\xi)} + c_2\frac{d^2}{d(i\xi)^2} + \dots + c_m\frac{d^m}{d(i\xi)^m}\right] V_2(\xi,\infty)$$

If we now use this rule in the representation which results from using Eq. (2-76) in Eq. (2-74), we find that Eq. (2-72) leads to

$$V_2(0,q) = V_2(0,\infty) + \sum_{n=1}^{\infty} A_n q^{-n} \tag{2-78}$$

where

$$A_n = \sum_{m=1}^{K} (-i)^{m-1} F_n^m a_{m-1} \tag{2-79}$$

The coefficients F_n^m are those which were defined in Eq. (2-76) and the coefficients a_j are those defined by Eq. (2-77).

The form of the result in Eq. (2-79) provides an excellent example of a situation which occurs frequently in numerical analysis. Since we know a recursion formula which will permit us to compute the quantities $B_n'(\tau) = F_n(\alpha)\exp(i\frac{2}{3}n\pi)$ without having to take recourse to the representation in Eq.(2-76) it is tempting to dismiss this representation in the form of a polynomial in τ as being "useless." However, we see in Eq. (2-79) that the coefficients in the polynomial representation can play an important role in analytical work.

2.5 Programs for Step-by-Step Integration

In this discussion we want to present some methods by which one can compute the root $t(q)$ of

$$w_1'(t) - q\, w_1(t) = 0 \qquad (2\text{-}80)$$

by integrating the differential equation

$$\frac{dt}{dq} = \frac{1}{t - q^2} \qquad (2\text{-}81)$$

with the initial conditions

$$t_o = t(q_o)$$

We can use the differential equation to compute

$$\left[\frac{d^n t}{dq^n}\right]_{q=q_o}$$

and use these expressions in the Taylor series to obtain for $t(q)$ a representation of the form

$$t(q) = t_o + \sum_{n=1}^{\infty} A_n (q - q_o)^n \qquad (2\text{-}82)$$

The explicit form of the first few A_n are

$$A_1 = \frac{1}{t_o - q^2}$$

$$A_2 = -\frac{1}{2}\left[\frac{1}{\left(t_o - q_o^2\right)^3} - \frac{2q_o}{\left(t_o - q_o^2\right)^2}\right]$$

$$A_3 = -\frac{1}{2}\frac{1}{\left(t_o - q_o^2\right)^5} - \frac{5}{3}\frac{q_o}{\left(t_o - q_o^2\right)^4} + \frac{4}{3}\frac{q_o^2}{\left(t_o - q_o^2\right)^3} + \frac{1}{3}\frac{1}{\left(t_o - q_o^2\right)^2}$$

$$A_4 = -\frac{5}{8}\frac{1}{\left(t_o - q_o^2\right)^7} + \frac{35}{12}\frac{q_o}{\left(t_o - q_o^2\right)^6} - \frac{13}{3}\frac{q_o^2}{\left(t_o - q_o^2\right)^5} + 2\frac{q_o^3}{\left(t - q_o^2\right)^4}$$

$$- \frac{7}{12}\frac{1}{\left(t_o - q_o^2\right)^4} + \frac{q_o}{\left(t_o - q_o^2\right)^3}$$

$$A_5 = \frac{7}{8}\frac{1}{\left(t_o - q_o^2\right)^9} - \frac{105}{20}\frac{q_o}{\left(t_o - q_o^2\right)^8} + \frac{34\, q_o^2}{3\left(t_o - q_o^2\right)^7} + \left(\frac{21}{20} - \frac{154}{15}q_o^3\right)\frac{1}{\left(t_o - q_o^2\right)^6}$$

$$\left(\frac{16}{5}q_o^4 - \frac{49}{15}\right)\frac{q_o}{\left(t_o - q_o^2\right)^5} + \frac{12}{5}\frac{q_o^2}{\left(t_o - q_o^2\right)^4} + \frac{1}{5\left(t_o - q_o^2\right)^3}$$

$$A_6 = -\frac{21}{16}\frac{1}{\left(t_o - q_o^2\right)^{11}} + \frac{157}{24}\frac{q_o}{\left(t_o - q_o^2\right)^{10}} - \frac{364}{18}\frac{q_o^2}{\left(t_o - q_o^2\right)^9}$$

$$+ \left(\frac{2134}{90}q_o^3 - \frac{231}{20}\right)\frac{1}{\left(t_o - q_o^2\right)^8} + \left(\frac{709}{90}q_o - \frac{1128}{90}q_o^4\right)\frac{1}{t_o - q_o^2}^7$$

$$+ \left(\frac{16}{3}q_o^5 - \frac{1006}{90}\right)\frac{1}{\left(t_o - q_o^2\right)^6} + \left(-\frac{58}{90} + \frac{16}{3}q_o^3\right)\frac{1}{\left(t_o - q_o^2\right)^5} + \frac{q_o}{\left(t_o - q_o^2\right)^4}$$

Since t_o and q_o are complex quantities, the programming of these coefficients would be quite tedious. However, we can develop a much simpler method for the computation of the A_n if we insert Eq. (2-82) in Eq. (2-81) and obtain a recursion formula for the coefficients.

We observe that

$$\frac{dt}{dq} = A_1 + \sum_{n=1}^{\infty} (n + 1)A_{n+1}(q - q_o)^n$$

and

$$t - q^2 = t - [q_o + (q - q_o)]^2$$

$$= t_o - q_o^2 + (A_1 - 2q^o)(q - q_o) + (A_2 - 1)(q - q_o)^2$$

$$+ \sum_{n=3}^{\infty} A_n(q - q_o)^n$$

The differential equation

$$(t - q^2)\frac{dt}{dq} = 1$$

then leads to the relationships

$$(t_o - q_o^2)(1A_1) = 1$$

$$(t_o - q_o^2)(2A_2) + (A_1 - 2q_o)(1A_1) = 0$$

$$(t_o - q_o^2)(3A_3) + (A_1 - 2q_o)(2A_2) + (A_2 - 1)(1A_1) = 0$$

$$(t_o - q_o^2)(4A_4) + (A_1 - 2q_o)(3A_3) + (A_2 - 1)(2A_2) + A_3(1A_1) = 0$$

The general term in the series can be evaluated from the relationship

$$(t_o - q_o^2)(nA_n) = - [(A_1 - 2q_o)(n-1)A_{n-1} + (A_2 - 1)(n-2)A_{n-2}]$$
$$- \sum_{m=3}^{n-1} (n - m)A_mA_{(n-m)} \qquad (2\text{-}83)$$

It would be rather straight foward to construct a computer program to evaluate the A_n. However, since the terms which appear in Eq. (2-82) are of the form

$$B_n = A_n(q - q_o)^n \tag{2-84}$$

it would advantageous to have the program evaluate B_n directly and to stop the summation when a sufficient number of terms have been summed. This is much better computer practice than to first construct the A_n and then go back and sum the polynomial which results when Eq. (2-82) is truncated.

Although it is wasteful of storage space in the computer, a simplication is obtained in programming this set of relationships if we define an auxiliary set of coefficients C_n by means of the following statements

$$U = (q - q_o)/(t_o - q_o^2) \tag{2-85}$$

$$C_1 = B_1 - 2q_oU \tag{2-86a}$$

$$C_2 = B_2 - U^2 \tag{2-86b}$$

$$C_n = B_n \quad , \quad n \geqslant 3 \tag{2-86c}$$

The recursion relationships then take the form

$$(t_o - q_o^2)(1B_1) = 0 \tag{2-89a}$$

$$(t_o - q_o^2)(2B_2) = - C_1B_1 \tag{2-89b}$$

$$(t_o - q_o^2)(3B_3) = - (2C_1B_2 + C_2B_1) \tag{2-89c}$$

or, in general

$$(t_o - q_o^2)(nB_n) = - \sum_{j=1}^{n-1} (n-j)C_jB_{n-j} \tag{2-89d}$$

Program 2-4

```
      SUBRØUTINE CMRTSM(Q0,T0,Q,T,EPSI,CØNVRG)
      LØGICAL CØNVRG
      CØMPLEX Q0,T0,Q,T,B(50),C(50),U,F,TEST
C-------LØGICAL VARIABLE CØNVRG = .FALSE. IF SUM ØF LAST THREE
C-------TERMS FAILS TØ BE LESS THAN EPSI*CABS(T).
      CØNVRG = .TRUE.
      U      = Q - Q0
      F      = 1.0/(T0 - Q0*Q0)
      B(1)   = U*F
      C(1)   = B(1) - 2.0*Q0*U
      T      = T0 + B(1)
      TEST   = B(1)
      B(2)   = -(F/FLØAT(2))*C(1)*B(1)
      C(2)   = B(2) - U*U
      T      = T + B(2)
      TEST   = TEST + B(2)
      B(3)   = -(FLØAT(3))*(2.0*C(1)*B(2) + C(2)*B(1))
      C(3)   = B(3)
      T      = T + B(3)
      TEST   = TEST + B(3)
      DØ 1 I = 4,50
      B(I)   = 0.0
      INDEX  = I - 1
      DØ 2 J = 1,INDEX
      B(I)   = B(I) + FLØAT(I-J)*C(J)*B(I-J)
    2 CØNTINUE
      B(I)   = - (F/FLØAT(I))*B(I)
      C(I)   = B(I)
      T      = T + B(I)
      TEST   = TEST + B(I) - B(I-3)
      TMAG   = EPSI*CABS(T)
      TESTMG = CABS(TEST)
      IF(TESTMG.LT.TMAG) GØ TØ 10
    1 CØNTINUE
      CØNVRG = .FALSE.
   10 RETURN
      END
```

In Program 2-4 we present a FORTRAN subroutine which will evaluate Eq. (2-82) by employing the recursion relation given in Eq. (2-89). The relation between the mathematical variables and the FORTRAN variables are

t_o = T0 , t = T

q_o = Q0 , q = Q

The user will have to supply the value of EPSI which is used to control the exit from the DØ-loop. The author generally employed

EPSI = 1.0E-8

However, for some applications in which less accuracy is required a smaller value of EPSI will cut down on the amount of work which the computer is asked to perform. The LØGICAL variable CØNVRG is employed to permit the user to know whether the series converged when fewer than 50 terms were summed. The criteron for convergence to the accuracy specified by EPSI is based upon whether the last three terms combine to have an absolute value which is less than EPSI times the absolute value of T.

When q_o and q are very large, the representation in Eq. (2-82) becomes quite useless. When this is the case, one can replace Eq. (2-82) with the alternative form

$$t(q) = t_o + \sum_{n=1}^{\infty} B_n (q^{-1} - q_o^{-1})^n \tag{2-90}$$

It will be convenient to define

$$x_o = \frac{1}{q_o} \quad , \quad x = \frac{1}{q}$$

and observe that Eq. (2-81) takes the form

$$(1 - x^2 t)\frac{dt}{dx} = 1 \tag{2-91}$$

Let us express Eq. (2-90) in the form

$$t(x) = t_o + \sum_{n=1}^{\infty} B_n (x - x_o)^n \tag{2-92}$$

If we insert Eq. (2-92) in Eq. (2-91) we can show that the recursion formula for the B_n is

$$(1-x_o^2 t_o)(n+1)B_{n+1} = \sum_{r=1}^{n} rB_r(B_{n-r-1} + 2x_o B_{n-r} + x_o^2 B_{n-r-1}) \quad (2\text{-}93)$$

The initial values are

$$B_{-1} = 0 \quad , \quad B_o = t_o \quad , \quad B_1 = \frac{1}{1 - x_o^2 t_o}$$

$$B_2 = \frac{2t_o x_o + x_o^2 - 2t_o^2 x_o^3}{2(1 - x_o^2 t_o)^3}$$

$$B_3 = \frac{2t_o + 4x_o + 2t_o^2 x_o^2 + 2t_o x_o^3 + (3 - 10t_o^3)x_o^4 - 6t_o^2 x_o^5 + 6t_o^4 x_o^6}{t_o(1 - x_o^2 t_o)^5}$$

For $q_o \to \infty$, we have $x_o = 0$. For this limiting case, the recursion formula in Eq. (2-93) is identical with Eq. (2-52).

Since the terms which occur in Eq. (2-92) are of the form

$$D_n = B_n(x - x_o)^n$$

it would be advantageous to program a recursion formula for the D_n and compute only as many of the D_n as are required to obtain the desired accuracy for $t(x)$. Let us define

$$U = x - x_o$$

$$A_1 = x_o^2 D_1 + 2x_o U t_o$$

$$A_2 = x_o^2 D_2 + 2x_o U D_1 + U^2 t_o^2$$

$$A_3 = x_o^2 D_3 + 2x_o U D_2 + U^2 D_1$$

and, in general

$$A_n = x_o^2 D_n + 2x_o U D_{n-1} + U^2 D_{n-2} \quad , \quad n > 3$$

We can then express Eq. (2-93) in the form

$$(1 - x_o^2 t_o)(n+1)D_{n+1} = \sum_{r=1}^{n} rD_r A_{n-r+1} \qquad (2\text{-}94)$$

The introduction of the A_n has greatly simplified the appearance of the equation that defines the recursion process by which the D_n are to be generated. Although the introduction of the A_n means that we must set up arrays in the computer storage for both the D_n and the A_n, the price paid for the using of computer storage space for the A_n is more than compensated for in the saving of the operations which would be repeated in each step of the recursion if we were to program the form in Eq. (2-93).

In Program 2-5 we present a FORTRAN subroutine which permits us to evaluate the sum in Eq. (2-92). The FORTRAN variables are related to the mathematical variables in the following manner:

$$x_o = XO \quad , \qquad x = X$$
$$t_o = TO \quad , \qquad t = T$$

The quantity EPSI plays the same role as the corresponding criterion in Program 2-4. The value

EPSI = 1.0E-8

can be employed in most applications of the program.

In Programs 2-6 and 2-7 we present double precision versions of Programs 2-4 and 2-5, respectively.

Program 2-6 provides the FORTRAN statements for

SUBRØUTINE RØØTSM(QOR,QOI,TOR,TOI,QR,QI,TR,TI,EPSI,CØNVRG)

The FORTRAN variables in the argument list are related to the

(Text continues on p. 2-99)

Program 2-5

```
      SUBRØUTINE CMRTLG(XO,TO,X,T,EPSI,CØNVRG)
      LØGICAL CONVRG
      CØMPLEX XO,TO,X,T,U,U2,C2XOU,XO2,F,D(50),A(50),TEST
C-------LØGICAL VARIABLE CØNVRG = .FALSE. IF SUM ØF LAST THREE
C-------TERMS FAILS TØ BE LESS THAN EPSI*CABS(T).
      CØNVRG = .TRUE.
      U      = X - XO
      U2     = U*U
      C2XOU  = 2.0*XO*U
      XO2    = XO*XO
      F      = 1.0/(1.0 - XO2*TO)
      T      = TO
      D(1)   = U*F
      T      = T + D(1)
      TEST   = D(1)
      A(1)   = C2XOU*TO + XO2*D(1)
      D(2)   = A(1)*D(1)*F/FLØAT(2)
      T      = T + D(2)
      TEST   = TEST + D(2)
      A(2)   = TO*U2 + C2XOU*D(1) + XO2*D(2)
      D(3)   = (2.0*A(1)*D(2) + A(2)*D(1))*F/FLØAT(3)
      T      = T + D(3)
      TEST   = TEST + D(3)
      A(3)   = U2*D(1) + C2XOU*D(2) + XO2*D(3)
      DØ 1 I = 4,50
      INDEX  = I - 1
      D(I)   = 0.0
      DØ 2 J = 1,INDEX
      D(I)   = D(I) + FLØAT(I-J)*A(J)*D(I-J)
    2 CØNTINUE
      D(I)   = D(I)*F/FLØAT(I)
      A(I)   = U2*D(I-2) + C2XOU*D(I-1) +XO2*D(I)
      T      = T + D(I)
      TEST   = TEST + D(I) - D(I-3)
      TMAG   = EPSI*CABS(T)
      TESTMG = CABS(TEST)
      IF(TESTMG.LT.TMAG) GØ TØ 10
    1 CØNTINUE
      CØNVRG = .FALSE.
   10 RETURN
      END
```

```
      SUBROUTINE ROOTSM(Q0R,Q0I,T0R,T0I,QR,QI,TR,TI,EPSI,CONVRG)
      LOGICAL CONVRG
      DOUBLE PRECISION  Q0R,Q0I,T0R,T0I,QR,QI,TR,TI,EPSI
      DOUBLE PRECISION  DFLOAT,DSQRT,BR(50),BI(50),CR(50),CI(50),UR,UI,
     A         FR,FI,TESTR,TESTI,TEST,TESTMG
     B
      DFLOAT(K) = DBLE(FLOAT(K))
      CONVRG = .TRUE.
      UR       = QR - Q0R
      UI       = QI - Q0I
      TR       = T0R - Q0R*Q0R + Q0I*Q0I
      TI       = T0I - 2.0D0*Q0R*Q0I
      FI       = TR*TR + TI*TI
      FR       = TR/FI
      FI       =-TI/FI
C-----(FR,FI) IS THE RECIPROCAL OF (T0 - Q0*Q0)
      BR(1)    = UR*FR - UI*FI
      BI(1)    = UR*FI + UI*FR
      CR(1)    = BR(1) - 2.0D0*(Q0R*UR - Q0I*UI)
      CI(1)    = BI(1) - 2.0D0*(Q0R*UI + Q0I*UR)
      TR       = T0R + BR(1)
      TI       = T0I + BI(1)
      TESTR    = BR(1)
      TESTI    = BI(1)
      CR(2)    = CR(1)*BR(1) - CI(1)*BI(1)
      CI(2)    = CR(1)*BI(1) + CI(1)*BR(1)
      BR(2)    =-(FR*CR(2) - FI*CI(2))/DFLOAT(2)
      BI(2)    =-(FR*CI(2) + FI*CR(2))/DFLOAT(2)
      CR(2)    = BR(2) - UR*UR + UI*UI
      CI(2)    = BI(2) - 2.0D0*UR*UI
      TR       = TR + BR(2)
      TI       = TI + BI(2)
      TESTR    = TESTR + BR(2)
      TESTI    = TESTI + BI(2)
```

Program 2-6 (Cont'd)

```
      CR(3)  = (DFLOAT(2))*(CR(1)*BR(2) - CI(1)*BI(2)) + CR(2)*BR(1)
     A         - CI(2)*BI(1)
      CI(3)  = (DFLOAT(2))*(CR(1)*BI(2) + CI(1)*BR(2)) + CR(2)*BI(1)
     A         + CI(2)*BR(1)
      BR(3)  =-(FR*CR(3) - FI*CI(3))/DFLOAT(3)
      BI(3)  =-(FR*CI(3) + FI*CR(3))/DFLOAT(3)
      CR(3)  = BR(3)
      CI(3)  = BI(3)
      TR     = TR + BR(3)
      TI     = TI + BI(3)
      TESTR  = TESTR + BR(3)
      TESTI  = TESTI + BI(3)
      DO 1 I = 4,50
      CR(I)  = 0.0D0
      CI(I)  = 0.0D0
      INDEX  = I - 1
      DO 2 J = 1,INDEX
      CR(I)  = CR(I) + DFLOAT(I-J)*(CR(J)*BR(I-J) - CI(J)*BI(I-J))
      CI(I)  = CI(I) + DFLOAT(I-J)*(CR(J)*BI(I-J) + CI(J)*BR(I-J))
    2 CONTINUE
      BR(I)  =-(FR*CR(I) - FI*CI(I))/DFLOAT(I)
      BI(I)  =-(FR*CI(I) + FI*CR(I))/DFLOAT(I)
      CR(I)  = BR(I)
      CI(I)  = BI(I)
      TR     = TR + BR(I)
      TI     = TI + BI(I)
      TESTR  = TESTR + BR(I) - BR(I-3)
      TESTI  = TESTI + BI(I) - BI(I-3)
      TEST   = DSQRT(TESTR**2 + TESTI**2)
      TESTMG = DSQRT(TR**2 + TI**2)*EPSI
      IF(TEST.LT.TESTMG) GO TO 10
    1 CONTINUE
      CONVRG = .FALSE.
   10 RETURN
      END
```

```
      SUBROUTINE ROOTLG(XOR,XOI,TOR,TOI,XR,XI,TR,TI,EPSI,CONVRG)
      LOGICAL CONVRG
      DOUBLE PRECISION  XOR,XOI,TOR,TOI,XR,XI,TR,TI,EPSI
      DOUBLE PRECISION  DR(50),DI(50),AR(50),AI(50),TMAG,TESTR,TESTI,
     A                  DFLOAT,UR,UI,U2R,U2I,X02R,X02I,FR,FI,U2X0R,U2X0I
     B                 ,DSQRT,TEST,TEPSI
      DFLOAT(K) = DBLE(FLOAT(K))
      CONVRG = .TRUE.
      UR     = XR - XOR
      UI     = XI - XOI
      U2R    = UR*UR - UI*UI
      U2I    = 2.0D0*UR*UI
      U2XOR  = 2.0D0*(UR*XOR - UI*XOI)
      U2XOI  = 2.0D0*(UR*XOI + UI*XOR)
      XO2R   = XOR*XOR - XOI*XOI
      XO2I   = 2.0D0*XOR*XOI
      TESTR  = 1.0D0 - XO2R*TOR + XO2I*TOI
      TESTI  =       - XO2R*TOI - XO2I*TOR
      TMAG   = TESTR*TESTR + TESTI*TESTI
      FR     = TESTR/TMAG
      FI     =-TESTI/TMAG
      TR     = TOR
      TI     = TOI
      DR(1)  = UR*FR - UI*FI
      DI(1)  = UR*FI + UI*FR
      TR     = TR + DR(1)
      TI     = TI + DI(1)
      TESTR  = DR(1)
      TESTI  = DI(1)
      AR(1)  = U2XOR*TOR - U2XOI*TOI + XO2R*DR(1) - XO2I*DI(1)
      AI(1)  = U2XOR*TOI + U2XOI*TOR + XO2R*DI(1) + XO2I*DR(1)
      AR(2)  =(AR(1)*DR(1) - AI(1)*DI(1))/DFLOAT(2)
      AI(2)  =(AR(1)*DI(1) + AI(1)*DR(1))/DFLOAT(2)
      DR(2)  = AR(2)*FR - AI(2)*FI
      DI(2)  = AR(2)*FI + AI(2)*FR
      TR     = TR + DR(2)
      TI     = TI + DI(2)
      TESTR  = TESTR + DR(2)
```

```
      TESTI   = TESTI + DI(2)
      AR(2)   = U2R*T0R - U2I*T0I + U2X0R*DR(1) - U2X0I*DI(1)
     A                            + X02R*DR(2) - X02I*DI(2)
      AI(2)   = U2R*T0I + U2I*T0R + U2X0R*DI(1) + U2X0I*DR(1)
     A                            + X02R*DI(2) + X02I*DR(2)
      AR(3)   = (DFLOAT(2)*(AR(1)*DR(2) - AI(1)*DI(2)) + AR(2)*DR(1)
     A          - AI(2)*DI(1))/DFLOAT(3)
      AI(3)   = (DFLOAT(2)*(AR(1)*DI(2) + AI(1)*DR(2)) + AR(2)*DI(1)
     A          + AI(2)*DR(1))/DFLOAT(3)
      DR(3)   = AR(3)*FR - AI(3)*FI
      DI(3)   = AR(3)*FI + AI(3)*FR
      TR      = TR + DR(3)
      TI      = TI + DI(3)
      TESTR   = TESTR + DR(3)
      TESTI   = TESTI + DI(3)
      DO 1 I = 4,50
      INDEX   = I-1
      AR(I)   = 0.0D0
      AI(I)   = 0.0D0
      DO 2 J = 1,INDEX
      AR(I)   = AR(I) + DFLOAT(I-J)*(AR(J)*DR(I-J) - AI(J)*DI(I-J))
      AI(I)   = AI(I) + DFLOAT(I-J)*(AR(J)*DI(I-J) + AI(J)*DR(I-J))
    2 CONTINUE
      DR(I)   = (AR(I)*FR - AI(I)*FI)/DFLOAT(I)
      DI(I)   = (AR(I)*FI + AI(I)*FR)/DFLOAT(I)
      TR      = TR + DR(I)
      TI      = TI + DI(I)
      TESTR   = TESTR + DR(I) - DR(I-3)
      TESTI   = TESTI + DI(I) - DI(I-3)
      TEPSI   = DSQRT(TR**2 + TI**2)*DABS(EPSI)
      TEST    = DSQRT(TESTR**2 + TESTI**2)
      IF(TEST.LT.TEPSI) GO TO 10
      AR(I)   = X02R*DR(I) - X02I*DI(I) + U2X0R*DR(I-1) - U2X0I*DI(I-1)
     A         +U2R*DR(I-2) - U2I*DI(I-2)
      AI(I)   = X02R*DI(I) + X02I*DR(I) + U2X0R*DI(I-1) + U2X0I*DR(I-1)
     A         +U2R*DI(I-2) + U2I*DR(I-2)
    1 CONTINUE
      CONVRG = .FALSE.
   10 RETURN
      END
```

mathematical variables in the following manner:

$$q_o = QOR + iQOI \quad , \quad q = QR + iQI$$

$$t_o = TOR + iTOI \quad , \quad t = TR + iTI$$

The value to be assigned to EPSI will depend upon the accuracy which the user wishes to obtain. CØNVRG is a logical variable which tells whether or not the series converged to the desired accuracy by the time it has summed 50 terms. The user will have to supply the non-executable statements

```
LØGICAL CØNVRG
DØUBLE PRECISIØN QOR,QOI,TOR,TOI,QR,QI,TR,TI,EPSI
```

in the calling program.

The statements in Program 2-7 constitute the subroutine entitled

```
SUBRØUTINE RØØTLG(XOR,XOI,TOR,TOI,XR,XI,TR,TI,EPSI,CØNVRG)
```

The calling program will have to contain the statements

```
LØGICAL CØNVRG
DØUBLE PRECISIØN XOR,XOI,TOR,TOI,XR,XI,TR,TI,EPSI
```

The FORTRAN variables are related to the mathematical variables in the following manner:

$$x_o = XOR + iXOI \quad , \quad x = XR + iXI$$

$$t_o = TOR + iTOI \quad , \quad t = TR + iTI$$

Programs 2-6 and 2-7 were employed to generate the tables which are presented in Appendix D. In Program 2-8 we present a subroutine which we employed to generate tables such as the one in Table D-1a. The values in Table D-1a were obtained with a CALL statement of the form

```
CALL PRØPSM(1.0187929716447471D0,45.0D0,0.05D0,41,TR,TI)
```

The reader will recognize the first argument to be the root β_1

defined by $Ai'(-\beta_1) = 0$.

It should be noted that Programs (2-4 through (2-7) can be used for step-by-step integration of the Riccati equation $(t-q^2)\frac{dt}{dq} = 1$ and that the calculations made in this manner (by employing a program such as in Program 2-8) do not have the boundaries of regions of applicability such as illustrated in Fig. 2-3.

Program 2-8

```
      SUBROUTINE PROPSM(T,ARG,QSTEP,N,TR,TI)
      LOGICAL CONVRG
      DOUBLE PRECISION T,ARG,QSTEP,TR(N),TI(N)
      DOUBLE PRECISION PIO180,COS60,SIN60,DFLOAT,QOR,QOI,EPSI,
     A   RAD,TOR,TOI,QR,QI,TREAL,TIMAG,DCOS,DSIN
      DATA PIO180,COS60,SIN60/0.17453292519943   D-1, 0.5D0,
     A                          0.86602540378443865D0 /
      DFLOAT(K) = DBLE(FLOAT(K))
      TR(1)  = T*COS60
      TI(1)  = T*SIN60
      QOR    = 0.0D0
      QOI    = 0.0D0
      EPSI   = 1.0D-17
      RAD    = PIO180*ARG
      DO 1 I = 2,N
      INDEX  = I
      TOR    = TR(I-1)
      TOI    = TI(I-1)
      QI     = QSTEP*DFLOAT(I-1)
      QR     = QI*DCOS(RAD)
      QI     = QI*DSIN(RAD)
      CALL ROOTSM(QOR,QOI,TOR,TOI,QR,QI,TREAL,TIMAG,EPSI,CONVRG)
      TR(I)  = TREAL
      TI(I)  = TIMAG
      QOR    = QR
      QOI    = QI
      IF(.NOT.CONVRG) GO TO 2
    1 CONTINUE
      RETURN
    2 WRITE(6,150) INDEX,T,ARG,QSTEP
      DO 9 I = INDEX,N
      TR(I)  = 0.0D0
      TI(I)  = 0.0D0
    9 CONTINUE
      RETURN
  150 FORMAT(46H0  SUBROUTINE PROPSM COULD NOT HANDLE INDEX = ,I4/
     A 11H   WITH T = ,D22.16,8H   ARG = ,D22.16,10H   QSTEP = ,D22.16)
      END
```

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Section 3

EXPANSIONS IN THE LIGHTED REGION OF A CIRCULAR CYLINDER

Consider the problem which arises when one wants to compute the field in the lighted region when the plane wave $\exp(-ikx - i\omega t)$ illuminates a perfectly conducting cylinder of radius $\underline{a}$. If the radius of the cylinder is much greater than the wavelength, and if the field is observed for points outside the shadow region, the field can be described by the optical approximations

$$u(\rho,\varphi) - \exp(-ikx - i\omega t) \pm \sqrt{\frac{a \cos \alpha}{2D + a \cos \alpha}} \exp\left[ik(D - a \cos \alpha) - i\omega t\right] \tag{3-1}$$

where the + sign is used when the magnetic vector is parallel to the axis of the cylinder and $U(\rho,\varphi)$ represents the magnetic field (the so-called Neuman problem since the derivative of the wave function must vanish at the surface) and the - sign is used when the electric vector is parallel to the axis of the cylinder and $U(\rho,\varphi)$ represents the electric field (the so-called Dirichlet problem since the wave function must vanish at the surface). The geometry is illustrated in Fig. 3-1. The angle α is the angle of

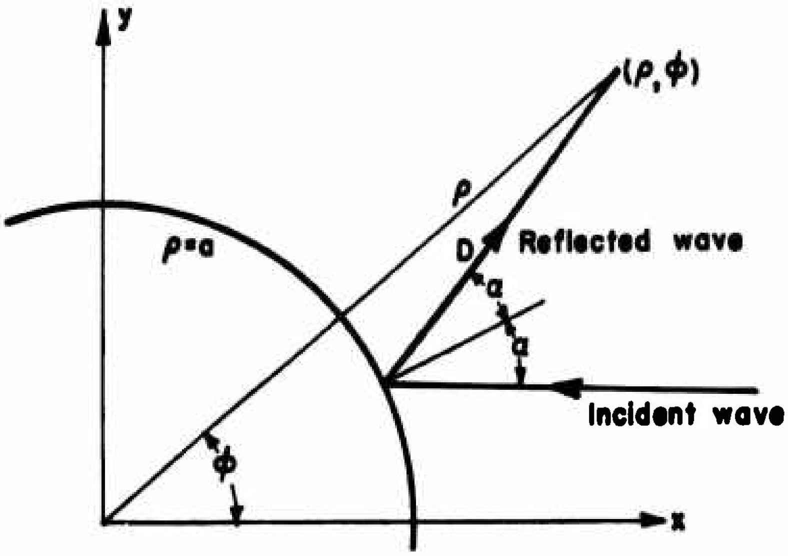

Fig. 3-1 Geometry of the Reflection Problem

incidence of the wave at the point of reflection, and the distance D is the distance of the observation point (ρ,φ) from the point of reflection.

In the following discussion we will present several extensions of the optical approximation which are based upon the diffraction function

$$V_1(\xi,\zeta,q) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(i\xi t) \left[v(t-\zeta) - \frac{v'(t) - q\,v(t)}{w_1'(t) - q\,w_1(t)}\, w_1(t-\zeta) \right] dt \quad (3\text{-}2)$$

which has been employed by Fock (Ref. 1), Belkina and Vainshtein (Ref. 2), Azriliant and Belkina (Ref. 3), Fedorov (Ref. 4), Wait (Ref. 5), Goodrich (Ref. 6), Logan and Yee (Ref. 7), and many other authors. Fock showed that for $\sqrt{\zeta} - \xi > 0$ that $V_1(\xi,\zeta,q)$ can be expressed in the form

$$V_1(\xi,\zeta,q) = \exp\ i\left(\xi\zeta - \frac{\xi^3}{3}\right) + P(\xi,\zeta,q) \quad (3\text{-}3)$$

$$P(\xi,\zeta,q) = -\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(i\xi t)\, \frac{v'(t) - q\,v(t)}{w_1'(t) - q\,w_1(t)}\, w_1(t-\zeta)\,dt \quad (3\text{-}4)$$

and that $P(\xi,\zeta,q)$ can be shown to have the asymptotic property

$$P(\xi,\zeta,q) \xrightarrow[p>>1]{} -\frac{q - ip}{q + ip}\sqrt{\frac{p}{\sigma}}\ \exp\left[i\,\frac{1}{12}\,(\sigma^3 + 3\sigma^2 p - 3\sigma p^2 + 3p^3)\right] \quad (3\text{-}5)$$

where

$$p = \frac{1}{3}\left[\sqrt{\xi^2 + 3\zeta} - 2\xi\right], \qquad \sigma = \sqrt{\xi^2 + 3\zeta} \quad (3\text{-}6)$$

Let us express $P(\xi,\zeta,q)$ in the form of a product of a slowly varying complex amplitude and the phase factor given appearing in Eq. (3-5).

$$P(\xi,\zeta,q) = A(p,\sigma,q)\exp\left[i\frac{1}{12}(\sigma^3 + 3\sigma^2 p - 3\sigma p^2 + 3p^3)\right] \tag{3-7}$$

Since Fock has shown that $P(\xi,\zeta,q)$ is a solution of

$$\frac{\partial^2 P}{\partial\zeta^2} + i\frac{\partial P}{\partial\xi} + \zeta P = 0 \tag{3-8}$$

we can show that $A(p,\sigma,q)$ is a solution of

$$\frac{1}{4\sigma^2}\frac{\partial^2 A}{\partial p^2} + \frac{9}{4\sigma^2}\frac{\partial^2 A}{\partial\sigma^2} + \frac{3}{2\sigma^2}\frac{\partial^2 A}{\partial\sigma\partial p} - \frac{3}{4\sigma^3}\frac{\partial A}{\partial p} + \left(2i - \frac{9}{4\sigma^3}\right)\frac{\partial A}{\partial\sigma} + \frac{i}{\sigma}A = 0 \tag{3-9}$$

In order to see how $A(p,\sigma,q)$ can be used to describe the reflected wave from the circular cylinder, we can start from a form of the Laplace operator which was used by Keller, Lewis, and Seckler (Ref. 8) to study a slightly different aspect of this problem. The form which we need is

$$\nabla^2 = \left(1 + \frac{u}{s^2}\right)\frac{\partial^2}{\partial s^2} + \left(\frac{1}{s} - \frac{w}{s^2} - \frac{u}{s^3}\right)\frac{\partial}{\partial s} - \frac{2v}{s^2}\frac{\partial^2}{\partial s\partial\beta} + \frac{1}{s^2}\frac{\partial^2}{\partial\beta^2} + \frac{v}{s^3}\frac{\partial}{\partial\beta} \tag{3-10}$$

where

$$s = D + \frac{a}{2}\cos\alpha \qquad \beta = \pi - 2\alpha$$

and

$$u = \frac{9}{16}a^2\cos^2\beta/2 \qquad v = -\frac{3}{4}a\cos\beta/2 \qquad w = \frac{3}{8}a\sin\beta/2$$

If we make a change of scale from α to σ and from $\sin(\beta/2)$ to p

$$p = \left(\frac{ka}{2}\right)^{1/3}\sin\beta/2 \qquad \sigma = 2\left(\frac{ka}{2}\right)^{1/3}\frac{s}{a} \tag{3-11}$$

the wave equation takes the form

$$\left(\nabla^2 + k^2\right)\Phi = \left[\left(\frac{ka}{2}\right)^{4/3}L_o + \left(\frac{ka}{2}\right)^{2/3}L_1 + \left(\frac{ka}{2}\right)^{0}L_2\right]\Phi \tag{3-12}$$

where L_o is the differential operator

$$L_0 = \frac{9}{\sigma^2}\frac{\partial^2}{\partial\sigma^2} + \frac{1}{\sigma^2}\frac{\partial^2}{\partial p^2} + \frac{6}{\sigma^2}\frac{\partial^2}{\partial\sigma\partial p} - \frac{3}{\sigma^3}\frac{\partial}{\partial p} + \left(8i - \frac{9}{\sigma^3}\right)\frac{\partial}{\partial\sigma} + \frac{4i}{\sigma} \qquad (3\text{-}13)$$

and L_1 and L_2 are similar differential operators which involve only σ and ρ. If we assume that $\Phi(\rho,\varphi)$ can be expressed in the form of an asymptotic series

$$\Phi(\rho,\sigma) = \left(\sum_{n=0}^{\infty} \Phi_n(p,\sigma)\ (ka/2)^{-\frac{2}{3}n}\right) \exp[ik(D - a\cos\alpha)] \qquad (3\text{-}14)$$

we then find that

$$L_0\Phi_0 = 0$$

so that Φ_0 satisfies the same partial differential equation as does $A(p,\sigma,q)$. An examination of the boundary conditions reveals that we can identify Φ_0 with the slowly-varying function $A(p,\sigma,q)$.

$$\Phi_0(p,\sigma) = A(p,\sigma,q) \qquad (3\text{-}15)$$

Therefore, we can express $U(\rho,\varphi)$ in the form

$$U(\rho,\varphi) = \exp(-ikx - i\omega t) + A(p,\sigma,q)\exp\left[ik(D - a\cos\alpha) - i\omega t\right] \qquad (3\text{-}16)$$

provided the higher-order terms in $(ka/2)^{-\frac{2}{3}}$ are neglectable. Since

$$A(p,\sigma,q) \xrightarrow[p>>1]{} -\frac{q-ip}{q+ip}\sqrt{\frac{p}{\sigma}} = -\frac{q-ip}{q+ip}\sqrt{\frac{a\cos\alpha}{2D + a\cos\alpha}} \qquad (3\text{-}17)$$

we can express the approximation for the case when the magnetic vector is parallel to the axis of the cylinder in the form

$$H_z(\rho,\varphi) = \exp(-ikx - i\omega t) + A(p,\sigma,0)\exp\left[ik(D - a\cos\alpha) - i\omega t\right] \qquad (3\text{-}18)$$

and the case when the electric vector is parallel to the axis in the form

$$E_z(\rho,\varphi) = \exp(-ikx - i\omega t) + A(p,\sigma,\infty)\exp\left[ik(D - a\cos\alpha) - i\omega t\right] \qquad (3\text{-}19)$$

If we assume that $A(p,\sigma,q)$ possesses an asymptotic expansion of the form

$$A(p,\sigma,q) = \sqrt{\frac{p}{\sigma}} \sum_{n=0}^{\infty} \frac{A_n(p,q)}{\sigma^n} \tag{3-20}$$

then the differential equation Eq. (3-9) leads to the recursion formulae

$$\begin{aligned} A_{n+1}(p,q) = -\frac{1}{2(n+1)} \Bigg\{ & \left[\frac{1}{4}\frac{d^2A_n}{dp^2} + \frac{1}{4p}\frac{dA_n}{dp} - \frac{1}{16p^2}A_n\right] \\ & - \left[\frac{3}{2}n\frac{dA_{n-1}}{dp} + \frac{3n}{4p}A_{n-1}\right] \\ & + \left[\frac{45}{16} + \frac{9}{4}(n-2)(n+1)\;A_{n-2}\right]\Bigg\} \end{aligned} \tag{3-21}$$

for which the initial conditions are

$$A_0(p,q) = \sqrt{2/p}\;\; V_{11}(-2p,q)\; \exp(-i\tfrac{2}{3}p^3),\; A_m = 0,\; m < 0 \tag{3-22}$$

Logan (Ref. 9) has shown that $V_{11}(\xi,0)$ and $V_{11}(\xi,\infty)$ can be expanded in asymptotic series of the form

$$V_{11}(\xi,0) \xrightarrow[\xi\to-\infty]{} -\frac{\sqrt{-\xi}}{2}\exp\left(-i\frac{\xi^3}{12}\right)\left\{1+i\frac{2}{\xi^3}-\frac{28}{\xi^6}-i\frac{896}{\xi^9}+\frac{43120}{\xi^{12}}+i\frac{2754752}{\xi^{15}}+\dots\right\} \tag{3-23}$$

$$V_{11}(\xi,\infty) \xrightarrow[\xi\to-\infty]{} \frac{\sqrt{-\xi}}{2}\exp\left(-i\frac{\xi^3}{12}\right)\left\{1-i\frac{2}{\xi^3}+\frac{20}{\xi^6}+i\frac{560}{\xi^9}-\frac{25520}{\xi^{12}}-i\frac{16[illegible]1600}{\xi^{15}}+\dots\right\} \tag{3-24}$$

In Tables 3-1 and 3-2 we list some of the leading terms in these asymptotic expansions. The author has tables which contain further terms in these series, but until appropriate summation techniques for summing these types of expansions can be found their usefulness is limited.

Table 3-1

LEADING TERMS IN THE ASYMPTOTIC EXPANSION OF $A_n(p,0)$

$$A_0(p,0) = -1 + \frac{1}{4}\,\frac{1}{p^3} + \frac{7}{16}\,\frac{1}{p^6} - i\,\frac{7}{4}\,\frac{1}{p^9} + \ldots$$

$$A_1(p,0) = -i\,\frac{1}{32}\,\frac{1}{p^2} + \frac{35}{128}\,\frac{1}{p^5} - i\,\frac{1001}{512}\,\frac{1}{p^8} + \ldots$$

$$A_2(p,0) = -i\,\frac{3}{16}\,\frac{1}{p} + \frac{465}{2048}\,\frac{1}{p^4} - i\,\frac{10857}{8192}\,\frac{1}{p^7} + \ldots$$

$$A_3(p,0) = +i\,\frac{15}{32}\,\frac{1}{p^0} + \frac{45}{512}\,\frac{1}{p^3} - i\,\frac{63525}{65536}\,\frac{1}{p^6} + \ldots$$

$$A_4(p,0) = -\frac{105}{1024}\,\frac{1}{p^2} - i\,\frac{28665}{32768}\,\frac{1}{p^5} + \ldots$$

$$A_5(p,0) = -\frac{315}{512}\,\frac{1}{p} - i\,\frac{45675}{65536}\,\frac{1}{p^4} + \ldots$$

$$A_6(p,0) = +\frac{3465}{2048}\,\frac{1}{p^0} - i\,\frac{3465}{16384}\,\frac{1}{p^3} + \ldots$$

$$A_7(p,0) = +i\,\frac{45045}{65536}\,\frac{1}{p^2} + \ldots$$

$$A_8(p,0) = +i\,\frac{135135}{32768}\,\frac{1}{p} + \ldots$$

$$A_9(p,0) = -i\,\frac{765765}{65536}\,\frac{1}{p^0} + \ldots$$

Table 3-2

LEADING TERMS IN THE ASYMPTOTIC EXPANSION OF $A_n(p,\infty)$

$$A_0(p,\infty) = 1 + i\,\frac{1}{4}\,\frac{1}{p^3} + \frac{5}{16}\,\frac{1}{p^6} - i\,\frac{35}{32}\,\frac{1}{p^9} + \ldots$$

$$A_1(p,\infty) = + i\,\frac{1}{32}\,\frac{1}{p^2} + \frac{35}{128}\,\frac{1}{p^5} - i\,\frac{715}{512}\,\frac{1}{p^8} + \ldots$$

$$A_2(p,\infty) = + i\,\frac{3}{16}\,\frac{1}{p} + \frac{495}{2048}\,\frac{1}{p^4} - i\,\frac{8745}{8192}\,\frac{1}{p^7} + \ldots$$

$$A_3(p,\infty) = - i\,\frac{15}{32}\,\frac{1}{p^0} + \frac{75}{512}\,\frac{1}{p^3} - i\,\frac{60315}{65536}\,\frac{1}{p^6} + \ldots$$

$$A_4(p,\infty) = + \frac{105}{1024}\,\frac{1}{p^2} - i\,\frac{30135}{32768}\,\frac{1}{p^5} + \ldots$$

$$A_5(p,\infty) = + \frac{315}{512}\,\frac{1}{p} - i\,\frac{55125}{65536}\,\frac{1}{p^4} + \ldots$$

$$A_6(p,\infty) = - \frac{3465}{2048}\,\frac{1}{p^0} - i\,\frac{10395}{16384}\,\frac{1}{p^3} + \ldots$$

$$A_7(p,\infty) = - i\,\frac{45045}{65536}\,\frac{1}{p^2} + \ldots$$

$$A_8(p,\infty) = - i\,\frac{135135}{32768}\,\frac{1}{p} + \ldots$$

$$A_9(p,\infty) = + i\,\frac{765765}{65536}\,\frac{1}{p^0} + \ldots$$

Our present state of knowledge of the type of series which are displayed in Tables 3-1 and 3-2 is so limited that we do not know how to "sum" the series when $p \to 0$. This type of expansion can only be used when p is very large and positive, which corresponds to the physical situation of being well above the horizon in the lighted region, and the radius of curvature of the cylinder being very large in comparison with the wavelength. The author has had some limited success with obtaining numerical results from these series by employing a special form of the Euler transformation, namely,

$$S = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$
$$= (1 - x)^{-1} \left(a_0 + \sum_{k=1}^{\infty} [x/(1 - x)]^k \Delta^k a_0 \right) \tag{3-25}$$

where

$$\Delta a_0 = a_1 - a_0$$
$$\Delta^2 a_0 = \Delta a_1 - \Delta a_0$$
$$\dots\dots\dots$$

Since the series are characterized by the appearance of factors such as signs +, i, -1, -i, etc, the author has found it most useful to let $\exp(i\pi/2)$ be a factor of the variable x which appears in Eq. (3-25). Although the author has been able to use methods of this type to extrapolate values from the asymptotic series to much smaller value than can be obtained by a straightfoward use of the asymptotic series, the method does not appear useful enough at the present state of its development to warrant a further discussion at this time.

In order to obtain practical results for smaller values of p, it is much simpler to reconsider Eq. (3-22) and to realize that the function $V_{11}(-2p,q)$ offers a means of obtaining approximate values of $A(p,\sigma,q)$ provided σ is very large. Since $V_{11}(-2p,q)$ can be

numerically integrated by employing an integral representation such as that in Eq. (1-50), a more attractive approach to the computation of $A(p,\sigma,q)$ would appear to be to use Eq. (3-21) to find the $A_n(p,q)$ by means of numerical integration. Work along these lines has been begun by the author but the progress to date has been too limited to be able to report at this time upon the usefulness of this method. Furthermore, since Eq. (3-20) is an asmptotic series in inverse powers of σ, the usefulness of this representation (for small values of p) may be so limited that these studies may be more "academic" than "practical."

However, there are some very practical aspects of Eq. (3-16) which should be discussed at this point. When Eq. (3-22) is inserted into Eq. (3-20) and Eq. (3-16), there remains the limitation that p must be positive. It is relatively easy to eliminate this restriction and obtain a representation which holds down to $p = 0$ (i.e., on the horizon) and even to points slightly below the horizon. In order to achieve this result, we need to return to Eq. (3-7) in which the function $A(p,\sigma,q)$ was defined in terms of the function $P(\xi,\zeta,q)$. From Eq. (3-6), we find that the inverse relations are

$$2\xi = \sigma - 3p \quad , \qquad 4\zeta = \sigma^2 + 2\sigma p - 3p^2 \tag{3-26}$$

We can then show that for large positive values of σ,

$$\frac{2}{3}\zeta^{3/2} \longrightarrow \frac{1}{12}(\sigma^3 + 3\sigma^2 p - 3\sigma p^2 + 3p^3) - \frac{2}{3}p^3 \tag{3-27}$$

We now want to use an asymptotic property of Eq. (3-2) in order to show that in the vicinity of the "horizon" (defined as $\xi - \sqrt{\zeta} = 0$) that one can obtain an extension of Eq. (3-16) which shows that the field in this region is very closely related to the classical Fresnel integral representation for the field behind an opaque screen. The asymptotic formulae which contain this result are

described by the relations

$$V_1(\xi,\zeta,q) - \exp[i(\xi\zeta - \tfrac{1}{3}\xi^3)] \xrightarrow[\substack{\zeta\to\infty\\ \xi<0}]{} F(\zeta)[-\mu K(-\tau) + V_2(x,q)] \quad (3\text{-}28)$$

$$V_1(\xi,\zeta,q) \xrightarrow[\substack{\zeta\to\infty\\ \xi>0}]{} F(\zeta)[\ \mu K(\tau) + V_2(x,q)] \quad (3\text{-}29)$$

where the parameters μ, x, and τ are defined by

$$\mu = \zeta^{1/4} \quad , \quad x = \xi - \sqrt{\zeta} \quad , \quad \tau = \mu x$$

and $F(\zeta)$ denotes the factor

$$F(\zeta) = [\zeta^{-1/4}] \exp(i\tfrac{2}{3}\zeta^{3/2})$$

and $K(\tau)$ denotes a special form of the familiar Fresnel integrals

$$K(\tau) = \exp[-i(\tau^2 + \pi/4)] \frac{1}{\sqrt{\pi}} \int_{\tau}^{\infty} \exp(is^2)ds \quad (3\text{-}30)$$

which has the properties

$$K(0) = \frac{1}{2}$$

$$K(\tau) \xrightarrow[\tau\to\infty]{} \frac{\exp(i\pi/4)}{2\sqrt{\pi}\ \tau}$$

The function $V_2(x,q)$ has already been defined by Eq. (1-51) in which it is related to a function $V_{11}(x,q)$ which was defined in Eq. (1-50). From these former results we see that we can express $V_2(x,q)$ in the form

$$V_2(x,q) = -\frac{\exp(i\ \pi/4)}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(ixt) \frac{v'(t) - qv(t)}{w_1'(t) - qw_1(t)} dt - \frac{\exp(i\ \pi/4)}{2\sqrt{\pi}\,x} \quad (3\text{-}31)$$

However, this is not a convenient form for numerical evaluation since the portion taken along the negative t-axis gives rise to a singular function of x which is exactly cancelled off by the factor involving the reciprocal of x. Since $V_2(x,q)$ is actually

an entire function of x, it can actually be developed in a Taylor series about $x = 0$ if one starts from a proper representation. One of the most convenient representations for use with numerical integration techniques is obtained by making use of the property of the Airy functions which assert that

$$w_1(t\exp(i\tfrac{2}{3}\pi)) = \exp(i\tfrac{1}{3}\pi)w_2(t)$$

$$v(t\exp(i\tfrac{2}{3}\pi)) = \frac{1}{2}\exp(-i\pi/6)w_1(t)$$

where, for the sake of the present discussion, we can assume that t is real and positive. These properties permit us to deform the section of the contour in Eq. (3-31) that runs from $-\infty$ to 0 to a contour that runs from $\infty\exp(i\tfrac{2}{3}\pi)$ to 0 and arrive at the representation

$$V_2(x,q) = -\frac{\exp(i\pi/4)}{\sqrt{\pi}}\left[\int_0^{\infty}\exp(ixt)\frac{v'(t) - qv(t)}{w_1'(t) - qw_1(t)}\,dt + \exp(-i\tfrac{1}{3}\pi)\int_0^{\infty}\exp\left[-xt\frac{(\sqrt{3}+i)}{2}\right]\frac{v'(t) - q^*v(t)}{w_2'(t) - q^*w_2(t)}\,dt\right] \tag{3-32}$$

where

$$q^* = q\exp(i\tfrac{2}{3}\pi)$$

and $w_1(t)$ and $w_2(t)$ are related to the real-valued functions (i.e., real-valued when t is real) $u(t)$ and $v(t)$ by means of the relations

$$w_1(t) = u(t) + i\,v(t)$$

$$w_2(t) = u(t) - i\,v(t)$$

Let us now collect some of the properties of Eq. (3-16) which can be deduced from the results which we have been discussing.

We observe that for points "high" above the horizon (i.e., for $p > 0$ and $\sigma \to \infty$) that Eq. (3-22) leads to

$$U(\rho,\varphi) - \exp(-ikx - i\omega t) \approx$$

$$\sqrt{2/\sigma}\ V_{11}(-2p,q)\exp(-i\tfrac{2}{3}p^3)\exp[ik(D - a\cos\alpha) - i\omega t] \qquad (3\text{-}33)$$

If we then let $p \to \infty$, we arrive at the asymptotic approximation

$$U(\rho,\varphi) - \exp(-ikx - i\omega t) - \frac{q-ip}{q+ip}\sqrt{\frac{p}{\sigma}}\exp\left[ik(D-a\cos\alpha) - i\omega t\right] \qquad (3\text{-}34)$$

which is the extension to the impedance boundary condition

$$\frac{\partial U}{\partial \rho} = ikZU, \qquad \text{for } \rho = a$$

$$q = -i(ka/2)^{\frac{1}{3}}Z$$

of the special problems which were expressed in Eq. (3-1) since the introduction of the definitions of p and σ change the appearance of (3-34) to the following form

$$U(\rho,\varphi) - \exp(-ikx - i\omega t) - \frac{q-ip}{q+ip}\sqrt{\frac{a\cos\alpha}{2D + a\cos\alpha}}\exp\left[ik(D-a\cos\alpha) - i\omega t\right] \qquad (3\text{-}35)$$

However, the analytical "tools" which we have developed by the introduction of the functions $V_{11}(x,q)$ and $V_2(x,q)$ permit us to use either Eq. (3-28) or Eq. (3-29) to arrive at a result which holds on the horizon, i.e., for $p = 0$, for the case when $\sigma \to \infty$. This result is of the form

$$U(\rho,\varphi) \xrightarrow[p\to 0]{} \left|\frac{1}{2} + \left(\frac{ka}{2}\right)^{1/3}\sqrt{\frac{2}{kD}}\ V_2(0,q)\right| \exp(-ikx - i\omega t) \qquad (3\text{-}36)$$

The extensions to the optical results which were made in the paper by Keller, Lewis, and Seckler (Ref. 8) belong to the class of results of which the expansions in Tables 3-1 and 3-2 are examples. However, these representations (which are effectively in inverse

integral powers of (ka) become useless for $p \to 0$. These new results have the considerable advantage of being useful for $p \to 0$ as well as $p \to \infty$.

The results which we have obtained provide us with an interesting means of obtaining the fields at the surface of the cylinder. If we consider the meanings of p and σ, we see that the fields on the surface of the cylinder are obtained by letting $\sigma \to p$. From Eq. (3-16) we see that

$$\begin{aligned} U(a,\varphi) &= [1 + A(p,p,q)] \exp(-ika\cos\varphi - i\omega t) \\ &= V_1(-p,q) \exp[-i\tfrac{1}{3}p^3 - ika\cos\varphi - i\omega t] \end{aligned} \tag{3-37}$$

where $V_1(z,q)$ (which was already introduced in Eq. (1-31) is defined by means of either a Fourier integral

$$V_1(z,q) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(izt) \frac{1}{w_1'(t) - q w_1(t)} dt \tag{3-38}$$

or a residue series

$$V_1(z,q) = i 2\sqrt{\pi} \sum_{s=1}^{\infty} \frac{\exp(i z t_s)}{(t_s - q^2) w_1(t_s)} = -i 2\sqrt{\pi} \frac{1}{q} \sum_{s=1}^{\infty} \frac{\exp(i z t_s)}{\left(1 - \frac{t_s}{q^2}\right) w_1'(t_s)} \tag{3-39}$$

where the roots t_s are solutions of the transcendental equation

$$w_1'(t_s) - q w_1(t_s) = 0 .$$

Let us employ these results for the case of the electromagnetic scattering problems which we considered in Eqs. (3-18) and (3-19). When the magnetic field of the incident plane wave is parallel to the axis of the cylinder, we obtain the magnetic field at the surface of the cylinder by considering the case $\sigma \to p$. For this case we obtain

$$H_z(a,\varphi) \sim \left[1 + A(p,p,0)\right] \exp(-ika\cos\varphi - i\omega t)$$
$$= G(-p)\exp\left[-ika\cos\varphi - i\omega t\right] \qquad (3\text{-}40)$$

where G(-p) is the current distribution function

$$G(-p) = \exp\left(-i\frac{p^3}{3}\right)\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{\exp(-ipt)}{w_1'(t)}\,dt \qquad (3\text{-}41)$$

which has the asymptotic property that as $p \to \infty$

$$G(-p) \approx 2\left\{1 - i\frac{1}{4p^3} - \frac{1}{p^6} - i\frac{469}{64}\frac{1}{p^9} + \frac{5005}{64}\frac{1}{p^{12}} + i\frac{1122121}{1024}\frac{1}{p^{15}} + \ldots\right\} \qquad (3\text{-}42)$$

Since the tangential component of the electric field vanishes on the surface of the cylinder, we observe that $V_1(-p, 0, \infty) = 0$ and hence Eq. (3-3) will not directly give us the field distribution upon the cylinder. However, if we use the Maxwell equation

$$\sqrt{\frac{\mu}{\epsilon}}\,H_\varphi = i\frac{\partial E_z}{\partial(k\rho)}$$

we find that Eq. (3-19) leads to

$$\left[\sqrt{\frac{\mu}{\epsilon}}\,H_\varphi\right]_{\rho=a} = i\left(\frac{2}{ka}\right)^{1/3}\left[-i2p + \frac{1}{p}\frac{\partial A}{\partial\sigma}\right]_{\sigma=p}\exp(-ika\cos\varphi - i\omega t)$$

$$= i\left(\frac{2}{ka}\right)^{1/3} F(-p)\exp(-ika\cos\varphi - i\omega t) \qquad (3\text{-}43)$$

where F(-p) is the current distribution function

$$F(-p) = \exp\left(-i\frac{p^3}{3}\right)\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{\exp(-ipt)}{w_1(t)}\,dt$$

which has the asymptotic property that as $p \to \infty$

$$F(-p) = -i2p + \frac{1}{2p^2} - \sum_{n=1}^{\infty}\frac{n\,A_n(p,\infty)}{p^{n+1}}$$
$$\approx -i2p\left\{1 + \frac{i}{4p^3} + \frac{1}{2p^6} - i\frac{175}{64}\frac{1}{p^9} - \frac{395}{16}\frac{1}{p^{12}} + i\frac{318175}{1024}\frac{1}{p^{15}} + \ldots\right\} \qquad (3\text{-}44)$$

The expansions in inverse integral powers of p which have been given in Tables 3-1 and 3-2, as well as the expansions for G(-p) and F(-p) which have been given in Eqs. (3-42) and (3-44), respectively, become useless when p tends to zero. The physical interpretation of p in terms of the cosine of the angle of incidence reveals that all the expansions of this type become useless for grazing incidence. However, we must bear in mind that even in some cases where the expansions in inverse powers of p appear to be useful, the terms of order $(ka/2)^{-\frac{2}{3}}$ which we have neglected in Eq. (3-14) may become important. We have shown that by means of the functions such as $V_2(x,q)$ and $V_1(x,q)$ that the restriction to non-grazing angles of incidence can be readily relaxed by evaluating these integrals by numerical means. However, it is very difficult to relax the restrictions to large values of ka and much research is need in this direction. It is interesting to observe that the above results reveal that for p tending to zero, i.e., for grazing incidence, it is no longer satisfactory to decompose the fields into the unperturbed field and the reflected field. Since the obstacle tends to block off the incident field, a Fresnel integral representation such as given in Eqs. (3-28) and (3-29) becomes necessary. The author's experience with these types of expansions had led him to conjecture that one cannot expect to be able to analytically continue the expansions in inverse powers of (ka) from the lighted region into the shadow region. However, in Section 2 the author has discussed an example in which it is shown how one can start with the residue series which is valid in the shadow region and analytically extend these series into the lighted region provided one introduces the diffraction functions as analytical "tools" to be employed to cross the horizon. The situation that exists in the vicinity of the horizon appears to be be a type of "Stokes phenomenon." Two types of series are required in the lighted region (namely, the incident and the reflected waves), whereas

in the shadow region one works with the total field. It is the experience of the present author that the "connection formulae" can be found when one wishes to pass from the shadow region into the lighted region, but that the connection formulae for the reverse direction present a much higher order of difficulty. Since we are actually working with a partial differential equation (the wave equation) the phenomenon to which we refer is actually a generalization of the "Stokes phenomenon" since the latter has arisen as a topic in the discussion of the asymptotic solution of ordinary differential equations.

In the course of studying these expansions in inverse integral power of (ka), the author has extended the computations of Keller, Lewis, and Seckler (Ref. 8) so as to obtain further terms in the asymptotic expansions. In fact, some of the further terms in the series employed by Keller and his associates are identical with the terms which have been displayed in Tables 3-1 and 3-2. The additional terms which have been evaluated are, at the moment, more or less of purely academic interest. However, if means of "summing" these asymptotic series can be discovered, then the additional terms may prove to be of considerable interest. Therefore, the author is taking the opportunity presented by the occasion of preparing this report to include these tables. The tables were generated on the IBM 7090 by employing a FORTRAN II program which was augmented by the use of the NPREC subroutines which the author has discussed in the Preface. In order to make it easier for the reader to utilize these results, we will introduce the notation which appears in Ref. 8. In Fig. 3-2 we reproduce the illustration (Fig. 10) of Ref. 8 which provides the geometric meanings for the symbols which are employed.

A plane wave exp(ikx) is incident from the left on a circular cylinder whose equation is

$$x = - b_o \cos \varphi = - b_o \sin \beta/2 \ , \ y = b_o \sin \varphi = b_o \cos \beta/2 \ .$$

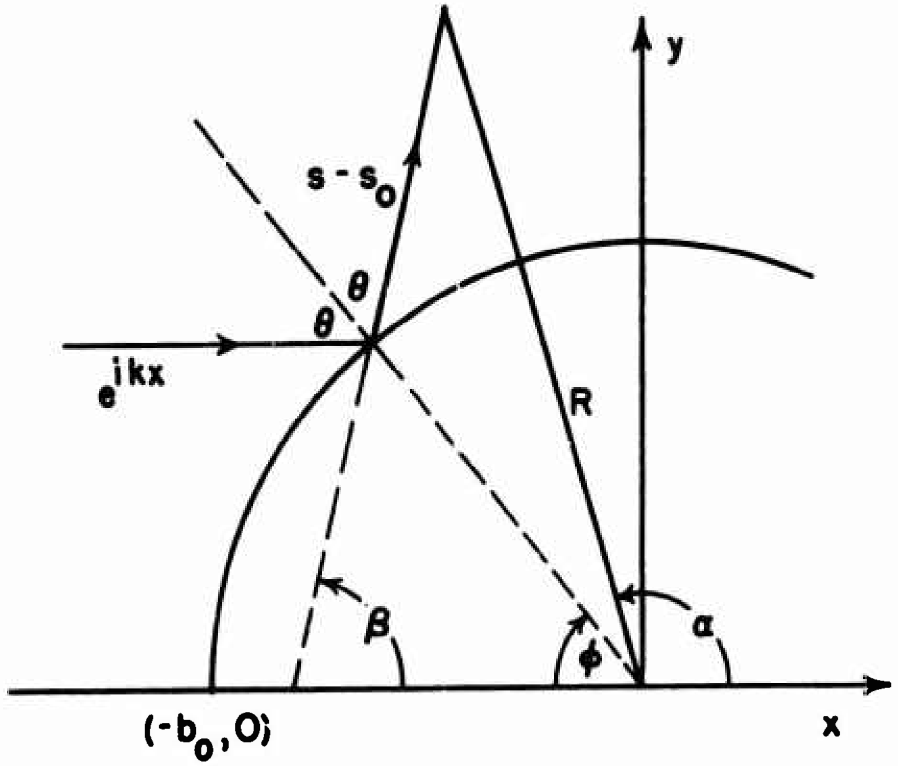

Fig. 3-2 The Geometry as Defined by Keller, Lewis, and Seckler (Ref. 8)

These authors use the scalar wave equation

$$(\nabla^2 + k^2)U(x,y) = 0$$

to determine a recursion relation to permit them to determine the $V_n(x,y)$ which occur in the expansion

$$U(x,y) \sim \exp[ik\psi(x,y)]\sum_{n=0}^{\infty} \frac{V_n(x,y)}{(ik)^n}$$

where $\psi(x,y)$ is the phase function which satisfies the eiconal equation of geometrical optics

$$(\nabla\psi)^2 = 1 \ .$$

These authors found it convenient to employ a set of coordinates (s,β), where $\underline{s}$ is the distance measured along a ray from the caustic, and β is the angle between a ray and the positive x-axis.

The origin of $\underline{s}$ is not shown in the figure. The caustic is a distance

$$s_o = (b_o/2) \cos \theta = (b_o/2) \sin (\beta/2)$$

measured back along the dashed line which is the extension to the interior of the cylinder of the reflected ray (i.e., the dashed line which is used to define the angle β).

In their Eq. (200), these authors express the field in the form

$$U(x,y) = U\ (s,\beta) \approx \exp(ikx) + D(s,\beta)F(s,\beta) \qquad (3\text{-}45)$$

where

$$D(s,\beta) = \frac{1}{2}\sqrt{\frac{b_o}{2s}\sin\frac{\beta}{2}}\ \exp\left\{ik\left(s - \frac{3}{2}b_o\sin\frac{\beta}{2}\right)\right\} \qquad (3\text{-}46)$$

and

$$F(s,\beta) = \sum_{n=0}^{\infty}\sum_{j=0}^{3n}\sum_{t=0}^{n} a_{jtn}\left(i16b_ok\sin\frac{\beta}{2}\right)^{-n}\left(\frac{b_o}{2s}\right)^{j}\left(\sin\frac{\beta}{2}\right)^{j-2t} \qquad (3\text{-}47)$$

Recursion relationships are given for the a_{jtn}, but we will refer the reader to Ref. 8 if he wishes to examine the intricate nature of these formulae. Our aim in this report is to present to the reader the explicit values for the a_{jtn} which we have obtained for the cases $j = 0(1)6$ for the boundary condition $U = 0$ (the so-called Dirichlet problem) and the boundary condition $\partial U/\partial \nu = 0$ (the so-called Neuman problem). We have chosen to employ only two subscripts, and hence we have made the definitions

$$a_{0jt} = A_{jt}\ , \qquad a_{4jt} = E_{jt}$$
$$a_{1jt} = B_{jt}\ , \qquad a_{5jt} = F_{jt}$$
$$a_{2jt} = C_{jt}\ , \qquad a_{6jt} = G_{jt}$$
$$a_{3jt} = D_{jt}\ ,$$

For the Dirichlet problem, the leading term (i.e., the coefficient which corresponds to that occuring in geometrical optics) is

$$a_{000} = A_{00} = -2 \tag{3-48}$$

In Table 3-3 we give the coefficients B_{jt} and C_{jt} for the Dirichlet problem. Tables 3-4 through 3-7 contain the other terms in the series for this problem which have been evaluated.

Table 3-3

COEFFICIENTS B_{jt} AND C_{jt} FOR THE DIRICHLET PROBLEM

B_{jt}			C_{jt}			
j \ t	0	1	j \ t	0	1	2
0	-6	16	0	15	-528	640
1	-6	2	1	-18	-354	560
2	-18	12	2	-87	-438	495
3	30	-30	3	-540	60	300
			4	945	-1260	210
			5	3150	-4410	1260
			6	-3465	6930	-3465

Table 3-4

COEFFICIENTS D_{jt} FOR THE DIRICHLET PROBLEM

j \ t	0	1	2	3
0	-105	23112	-90240	71680
1	-75	23985	-108144	91520
2	-540	18888	-81291	69960
3	-3060	6780	-62415	60315
4	-25830	27720	-69615	60270
5	43470	-34650	-73395	55125
6	346500	-471240	79695	41580
7	-180180	540540	-405405	45045
8	-945945	2162160	-1486485	270270
9	765765	-2297295	2297295	-765765

Table 3-5

COEFFICIENTS E_{jt} FOR THE DIRICHLET PROBLEM

j \ t	0	1	2	3	4
0	1181.25	-1200984	10507584.0	-22124544	13066240.0
1	2205.0	-1783299	17177688.0	-38025600	23152640.0
2	1822.5	-1338327	13367695.5	-30515976	18980160.0
3	-14895.0	-1077645	11616765.0	-26670435	16605000.0
4	-144821.25	-1532265	10799617.5	-23531445	14665901.25
5	-1638630.0	92610	7833420.0	-19866420	13229370.0
6	2671515.0	-3069990	5318775.0	-18724860	12969495.0
7	37567530.0	-54144090	24054030.0	-21351330	13243230.0
8	2331078.75	27297270	-22567545.0	-19729710	12499987.5
9	-204459255.0	487792305	-339999660.0	45945900	10720710.0
10	21824302.5	-189143955	327364537.5	-174594420	14549535.0
11	392837445.0	-1265809545	1440403965.0	-654729075	87297210.0
12	-250979478.75	1003917915	-1505876872.5	1003917915	-250979478.75

Table 3-6a

COEFFICIENTS F_{jt} FOR THE DIRICHLET PROBLEM

j \ t	0	1	2
0	-18191.25	72049590.0	-1189662912.0
1	-53156.25	140589546.75	-24939[illegible]76.0
2	-33941.25	115874419.5	-2224014304.5
3	-37408.25	113075966.25	-2109116207.5
4	-865856.25	112610925.0	-1878812775.0
5	-9485201.25	54580128.75	-1559640285.0
6	-130756106.25	163009192.5	-1571646037.5
7	204222768.75	-31114833.75	-1675876702.5
8	4466718506.25	-6389723340.0	1162431270.0
9	266[illegible]747131.25	-363546933.75	-2988015030.0
10	-36435673023.75	91072814332.5	-69088466947.5
11	-24170415018.75	30892300188.75	13334648827.5
12	132266185301.25	-454774815495.	549143099505.0
13	18823460906.25	18823460906.25	-175685635125.[illegible]
14	-207058069968.75	865879201687.5	-13929361070[illegible].5
15	109176073256.25	-545880366281.25	1091760732562.5

(Table is concluded on the next page.)

Table 3-6b

COEFFICIENTS F_{jt} FOR THE DIRICHLET PROBLEM

j \ t	3	4	5
0	4837567488.0	-6980886528.0	3280076800.0
1	10577896128.0	-15687469056.0	7513088000.0
2	9886569924.0	-15115684416.0	7394503680.0
3	9127981957.5	-13695202920.0	6609998400.0
4	7930033650.0	-11799834716.25	5680423350.0
5	6843403035.0	-10290725786.25	4995269923.75
6	6266154510.0	-9287917773.75	4547717212.5
7	5553170122.5	-8369484873.75	4240074588.75
8	4176752580.0	-7[illegible]28843722.5	4117563450.0
9	39[illegible]3509530.0	-7777492222.5	4225108367.5
10	18856197360.0	-8795193907.5	4379410035.0
11	-17263023277.5	-7049249707.5	4255738987.5
12	-259010822070.0	28360681098.75	4015671660.0
13	225881530875.0	-94117304531.25	6274486968.75
14	1054113810750.0	-357645757216.75	37546921812.5
15	-1091760732562.5	545880366281.25	-109176073256.25

Table 3-7a

COEFFICIENTS G_{jt} FOR THE DIRICHLET PROBLEM

j \ t	0	1	2
0	354729.375	-4899226590.0	137443571568.0
1	1400726.25	-11889983283.75	353103599970.0
2	1273269.375	-11381431736.25	369123784232.625
3	2082150.0	-11878860960.0	370580264955.0
4	2035687.5	-11008263420.0	339630798150.0
5	-52421985.0	-9304907535.0	311613956955.0
6	-758372422.5	-13367793285.0	301540337752.5
7	-12515933430.0	-2479456980.0	252746909415.0
8	18703866431.25	-11009178180.0	181892250540.0
9	591237584437.5	-873421221172.5	567608756715.0
10	721871541641.25	-583036241287.5	196495107558.75
11	-6435463023990.0	1673697029004 0.0	-12982622826175.0
12	-9653172711682.5	18387760531140.0	-6672038463090.0
13	32652430185375.0	-120696031330875.0	156084137834625.0
14	36705748767187.5	-100667868926625.0	76084428983062.5
15	-95201535879450.0	431463841508700.0	-748947862537875.0
16	-32152353573965.625	81226998502650.0	0.0
17	131993872566806.25	-680276112459693.75	1421472473796375.0
18	-592280197415l5.625	355368117449093.75	-888420296122734.375

(Table continues on next page.)

Table 3-7b

COEFFICIENTS G_{jt} FOR THE DIRICHLET PROBLEM

j \ t	3	4	5
0	-937260303360.0	2440841048064.0	-2679308943360.0
1	-2498278112832.0	6671675630592.0	-7457522688000.0
2	-2753248927788.0	7612975222944.0	-8721458457600.0
3	-2701466138812.5	7360090004100.0	-8345820571200.0
4	-2457927602392.5	6644016395060.625	-7477740833990.0
5	-2232804008452.5	5949249780835.25	-6620485137603.75
6	-2028862629832.5	5285640393061.875	-5827940150051.25
7	-1772818610062.5	4648813932262.5	-5153677211182.5
8	-1585806725002.5	4178670466595.625	-4664542970587.5
9	-1619620925422.5	3855431313483.75	-4306063901391.25
10	-1452705596842.5	3381430148971.875	-4001998734483.75
11	2066346785002.5	2706693644655.0	-3846096829575.0
12	-3040365405577.5	3217180448356.875	-4080926324475.0
13	-81229508297437.5	1579288370034 3.75	-4536454078406.25
14	3802339103062.5	-14785828541859.375	-3068224127718.75
15	608110728037312.5	-217260385779937.5	19870045332637.5
16	-16922291354 7187.5	17768405922454 6.875	-60920248876987.5
17	-1523006221924687.5	863036859090656.25	-23352762069511 8.75
18	1184560394830312.5	-888420296122734.375	355368118449093.75

(Table concluded on next page.)

Table 3-7c

COEFFICIENTS G_{jt} FOR THE DIRICHLET PROBLEM

j \ t	6
0	1045037056000.0
1	2948789043200.0
2	3512107315200.0
3	3335791488000.0
4	2969844006000.0
5	2607171340410.0
6	2288530498629.375
7	2040565008982.5
8	1874170592043.75
9	1772671312912.5
10	1726104090335.625
11	1749130548165.0
12	1840683497152.5
13	1932541986375.0
14	1929404742890.625
15	1965169318612.5
16	3384458270943.75
17	20306749625662.5
18	-59228019741515.625

The reader will observe that some of the entries in these tables contain as many as 18 significant figures. The author chose to stop the computer program with the coefficients G_{jt} since to continue to any higher-order terms would not permit the exact determination of the coefficients. The author is well aware of the fact that these coefficients already contain more significant figures than can be expected to be of significance in a practical

problem. However, the author feels that a scientist should be permitted to occasionally enjoy the "luxury" of obtaining exact results which have an esthetic value even if their practical value is questionable. However, in the particular case of the coefficients which have been presented above, it may be that the effective use of summation techniques in order to sum these series will actually require these exact results for the coefficients. Let us now turn to the second case of the Keller, Lewis, and Seckler recursion formula which was run on the IBM 7090.

For the Neuman problem, the leading term (i.e., the coefficient which corresponds to that occuring in geometrical optics) is

$$a_{000} = A_{00} = 2 \tag{3-49}$$

In Table 3-8 we give the coefficients B_{jt} and C_{jt} for the Neuman problem. Tables 3-9 through 3-12 contain the other terms in the series for this problem which have been evaluated.

Table 3-8

COEFFICIENTS B_{jt} AND C_{jt} FOR THE NEUMAN PROBLEM

B_{jt}			C_{jt}			
j \ t	0	1	j \ t	0	1	2
0	6	16	0	-15	-528	896
1	6	-2	1	18	-318	560
2	18	-12	2	87	-426	465
3	-30	30	3	540	-540	180
			4	-945	1260	-210
			5	-3150	4410	-1260
			6	3465	-6930	3465

Table 3-9

COEFFICIENTS D_{jt} FOR THE NEUMAN PROBLEM

j \ t	0	1	2	3
0	105	23112	-126336	114688
1	75	23535	-138096	128128
2	540	18600	-96789	86856
3	3060	5700	-65265	63525
4	25830	-2520	-71505	57330
5	-43470	105210	-97965	45675
6	-346500	526680	-190575	13860
7	180180	-540540	405405	-45045
8	945945	2162160	1486485	-270270
9	-765765	2297295	-2297295	765765

Table 3-10

COEFFICIENTS E_{jt} FOR THE NEUMAN PROBLEM

j \ t	0	1	2	3	4
0	-1181.25	-1200984	14474688.0	-34750464	22077440.0
1	-2205.0	-1775949	23133528.0	-57321600	37044224.0
2	-1822.5	-1327977	17333680.5	-43190280	28034496.0
3	14895.0	-1054515	14487555.0	-35566125	227411320.0
4	144821.25	-1535415	13210942.5	-29766555	18487218.75
5	1638630.0	-1755810	9070740.0	-23149980	14878710.0
6	-2671515.0	7283430	3329865.0	3329865	1297425.0
7	-37567530.0	57026970	-6756750.0	-24054030	11981970.0
8	-2331078.75	-38108070	65810745.0	-34324290	9121612.5
9	204459255.0	-500044545	37675380.0	-82702620	1531530.0
10	-21824302.5	189143955	-327364537.5	174594420	-14549535.0
11	-392837445.0	1265809545	-1440403965.0	654729075	-87297210.0
12	250979478.75	-1003917915	1505876872.5	-1003917915	250979478.75

Table 3-11a

COEFFICIENTS F_{jt} FOR THE NEUMAN PROBLEM

j \ t	0	1	2
0	18191.25	72049590.0	-1605837888.0
1	53156.25	140440709.25	-3370646760.0
2	33941.25	115532644.5	-2964851887.5
3	37406.25	112536753.75	-2779312432.5
4	865856.25	112393575.0	-2439376905.0
5	9485201.25	52560191.25	-1970244675.0
6	130756106.25	13 8967.5	-1954005322.5
7	-204222768.75	900303153.75	-2301056257.5
8	-4466718506.25	7104857760.0	-2896483590.0
9	-2868747131.25	-18050995 46.25	7435578150.0
10	36435673023.75	-94215513892.5	72114770227.5
11	24170415018.75	-28797167148.75	-26603824747.5
12	-132266185301.25	458790487155.0	-565205786145.0
13	-18823460906.25	-18823460906.25	175685635125.0
14	207058069968.75	-865879201687.5	1392936107062.5
15	-109176073256.25	545880366281.25	-1091760732562.5

(Table is concluded on the next page.)

Table 3-11b

COEFFICIENTS F_{jt} FOR THE NEUMAN PROBLEM

j \ t	3	4	5
0	7404675072.0	-11469545472.0	5641732096.0
1	16096961088.0	-25456091136.0	12694528000.0
2	14666724804.0	-23700901056.0	11992866816.0
3	13221905962.5	-20759495400.0	10283179200.0
4	11142795510.0	-17135697363.75	8380599990.0
5	9251998245.0	-14213492133.75	6925025756.25
6	8210393730.0	-12216371186.25	5883263347.5
7	7069519957.5	-10402388246.25	5035591811.25
8	4729184460.0	-8781950677.5	4408914510.0
9	1750538790.0	-8542491457.5	4081910332.5
10	-7332965640.0	-10759381132.5	3768329565.0
11	44499752797.5	-15997213732.5	2728037812.5
12	283104852030.0	-44423367738.75	0.0
13	-225881530875.0	94117304531.25	-6274486968.75
14	-1054113810750.0	357645757218.75	-37646921812.5
15	1091760732562.5	-545880366281.25	109176073256.25

Table 3-12a

COEFFICIENTS G_{jt} FOR THE NEUMAN PROBLEM

j \ t	0	1	2
0	-354729.375	-4899226590.0	181732327632.0
1	-1400726.25	-11886381416.25	471713468130.0
2	-1273269.375	-11369792643.75	493165448463.375
3	-2082150.0	-11861529660.0	493784995905.0
4	-2035687.5	-10987567920.0	449568654570.0
5	52421985.0	-9249823485.0	409455822825.0
6	758372422.5	-13433718375.0	394506450607.5
7	12515933430.0	-16615118520.0	327045483765.0
8	-18703866431.25	88663695120.0	202768986420.0
9	-591237584437.5	943451961952.5	106555433985.0
10	-721871541641.25	82823227987.5	1662935465441.25
11	6435463023990.0	-17576594855820.0	14383743048675.0
12	9653172711682.5	-17596673214120.0	2576053369890.0
13	-32652430185375.0	122804258952375.0	-162408820699125.0
14	-36705748767187.5	100366693552125.0	-71867973740062.5
15	95201535879450.0	-433210658680800.0	757681948398375.0
16	32152353573965.625	-81226998502650.0	0.0
17	-131993872566806.25	680276112459693.75	-1421472473796375.0
18	592280197415 15.625	-355368118449093.75	888420296122734.375

(Table continues on next page.)

Table 3-12b

COEFFICIENTS G_{jt} FOR THE NEUMAN PROBLEM

j \ t	3	4	5
0	-1395776937984.0	3889115394048.0	-4461070516224.0
1	-3754528644288.0	10699045797888.0	-12464072957952.0
2	-4120084137132.0	12105282445536.0	-14402448162816.0
3	-4006510659787.5	11534155406340.0	-13521037575360.0
4	-3595959838327.5	10208329844619.375	-11817369920790.0
5	-3217416555307.5	8938503496203.75	-10164946510136.25
6	-2876904578047.5	7739219430298.125	-8645108453808.75
7	-2456455358857.5	6590920773937.5	-7334426516917.5
8	-2133839285077.5	5722828821084.375	-6345237370972.5
9	-2183792928817.5	5128131594836.25	-5588995390728.75
10	-2391514342717.5	4311898373908.125	-4901269118996.25
11	-3503258861602.5	2864614297545.0	-4410473292225.0
12	+7296977365177.5	959118078043.125	-4613002819425.0
13	84040478459437.5	-7359973214343.75	-6004684029093.75
14	-17656406330062.5	34061052509859.375	-9279966226781.25
15	-625578899758312.5	234728557500937.5	-28604131193137.5
16	1692229135471 87.5	-1776840592245 46.875	609202488769 87.5
17	152300622192 4687.5	-86303685909 0656.25	23352762069 5118.75
18	-1184560394830312.5	888420296122734.375	-355368118449093.75

(Table concluded on next page.)

Table 3-12c

COEFFICIENTS G_{jt} FOR THE NEUMAN PROBLEM

j \ t	6
0	1794850684928.0
1	5071917154304.0
2	5951569096704.0
3	5525039063040.0
4	4776418491600.0
5	4051514138490.0
6	3408147408290.625
7	2885500155037.5
8	2496129902516.25
9	2201422370647.5
10	1969012214544.375
11	1811548053315.0
12	1725232936927.5
13	1581170716125.0
14	1082349002109.375
15	-218352146512.5
16	-3384458270943.75
17	-20306749625662.5
18	59228019741515.625

The reader may be interested to know that the "run time" for each of these problems on the IBM 7090 was 0.05 hour. Since the "cost" of computer time on such a computer is generally between $300 and $500 per hour, we see that the obtaining of these tables "cost" less than $50 in actual computer expenses. Needless to say, the true cost was far greater because of the time required to program the recursion formulae and the debugging of the programs. However, when one stops for a moment to realize the tremendous amount of

labour that would have been required to have obtained only the first fow terms in these series on a desk-type manual computer, one can begin to see vividly that the electronic computer has opened an era in which recursion formulae such as those derived a decade ago by Keller, Lewis, and Seckler may take on an entirely new importance. Granted that the series in their present forms are of limited usefulness because of their asymptotic nature, the fact that a summation scheme may be devised which will extend the value of these series to smaller value of ka makes these series stand out as a promising means of obtaining practical results for the scattering properties of smooth obstacles. The studies made by the present author have only involved the case of the circular cylinder and the two cases of the Dirichlet and the Neuman problem. However, in their classic paper, Keller, Lewis, and Seckler (Ref. 8) consider as their Example 24 the "Diffraction of a Place Wave by a Circular Cylinder ($\partial U/\partial \upsilon = ikZU$)." Time under the present research program was not available for the completion of a program based upon this case of the impedance boundary condition which would have led to the results of the Dirichlet and the Neuman problems for the special cases of $Z = \infty$ and $Z = 0$, respectively.

Although the author has devoted these efforts to the circular cylinder problem, it should be emphasized that the classic paper (Ref. 8) which was used to obtain Tables 3-3 through 3-12 also showed how one could apply similar methods to other obstacles. Even the case of a sphere (which was briefly considered as Examples 19 and 20 in Ref. 8) offers "rich rewards" to anyone who attempts to follow up the problem of working out the expansion in inverse powers of ka and then to attempt to find means of summing the series. It should also be borne in mind that with the coming of age of the computer that some of the actual algebra which is required when one follows the methods of Ref. 8 can be mechanized.

The present author has prepared several computer programs which are valuable for carrying out of tedious algebraic manipulations with polynomials. Although these have proven to be of considerable advantage, time has not permitted the application of these methods to attempting to extend the work on the finding of expansions in inverse powers of $k = 2\pi/\lambda$ for points in the lighted region. However, as an indication of what may lie just ahead in the field of carrying out tedious algebraic manipulations by means of electronic computers, the author wishes to call attention to a paper by Brown (Ref. 9) which describes the ALPAK system which has been under development at the Bell Telephone Laboratories. The present author readily admits that his own experience has not led him to be as optimistic as to lead him to agree with Brown's "...rule of thumb that one man-hour equals one 7090-second." However, once the particular algebra of a specific problem has been so organized as to be performed by the computer, then this rule of thumb is certainly very close to being consistent with the author's experience. The author believes that within the next decade that someone will rework the theory of Ref. 8 by employing routines on a digital computer to carry out the tedious algebra. The results will certainly lead to an enhancement of our understanding of the "breakdown" of geometrical optics as the wavelength increases and becomes comparable with the radii of curvature of the scattering obstacle.

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Section 4

ASYMPTOTIC EXPANSIONS FOR THE HEIGHT GAIN FUNCTION FOR THE CASE OF NORMAL REFRACTION

4.1 Introduction

This section deals with a number of aspects of the problem of obtaining and of employing asymptotic representations for the class of second-order differential equations which arise naturally when one attempts to solve the wave equation by the method of separation of variables. The choice of the material to be included is based upon the author's experience as to what results he has found useful (or interesting) in his work on the theory of the diffraction of waves by convex surfaces. The author has chosen to refer to the type of problem to be considered by the use of the term "height gain function" since the variable in the differential equation is generally related in some way to the height (or the distance) of the observation point from the convex surface. The restriction to the case of "normal refraction" is included in the title because all the asymptotic results to be discussed in this section arise from the consideration of differential equations which have only one "turning point." Therefore, we have excluded all problems in which the index of refraction above the convex (or the plane) surface varies in such a manner as to give rise to the phenomena of superrefraction, tropospheric waveguide modes, or similar complications of an already complex problem.

4.2 Problems Leading to the Height Gain Function

The height gain function $F(y,\lambda_s)$ with which we shall be concerned is the solution of the differential equation

$$\frac{d^2F}{dy^2} + [-\lambda_s + k^2 f(y)] F = 0 \tag{4-1}$$

for the case of "normal refraction" which we shall define by the requirements that $f(y) > 0$ and $f'(y) > 0$ for $y > 0$. The eigenvalues λ_s are to be defined by the impedance boundary condition

$$\left(\frac{dF}{dy} + ikZF \right)_{y=0} = 0 \tag{4-2}$$

The function $F(y, \lambda_s)$ occurs in many problems which involve the propagation of waves over convex surfaces or over plane surfaces above which the index of refraction is a function only of the height above the surface.

Let us cite several examples. The solution of the problem of propagation of axially symmetric waves in a horizontally stratified atmosphere above a plane surface is governed by the equation

$$\frac{\partial^2 G}{\partial x^2} + \frac{1}{x}\frac{\partial G}{\partial x} + \frac{\partial^2 G}{\partial y^2} + k^2 n^2(y) G = 0 . \tag{4-3}$$

The solution is of the form

$$G(x,y) = \sum_{s=1}^{\infty} \frac{i}{4} H_0^{(1)}(\mu_s x) \Phi_s(y) \Phi_s(y_0) \tag{4-4}$$

where $\Phi(y)$ is a solution of

$$\frac{d^2\Phi}{dy^2} + [-\mu_s^2 + k^2 n^2(y)]\Phi = 0 \tag{4-5}$$

which appears in the standard form of Eq. (4-1) with $\lambda_s = \mu_s^2$ and $f(y) = n^2(y)$.

As a second example, let us consider the problem of the propagation of two-dimensional waves over the surface of a circular cylinder

when the index of refraction is a function only of the distance from the center of the cylinder.

$$\frac{\partial^2 G}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial G}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 G}{\partial \varphi^2} + k^2 n^2(\rho) G = 0 \tag{4-6}$$

The solution is of the form

$$G(\rho,\varphi) = \sum_{s=1}^{\infty} \frac{i}{2} \frac{\exp(i\nu_s\varphi)}{\nu_s} \Phi_s(\rho)\Phi_s(\rho_0) \tag{4-7}$$

where $\Phi(\rho)$ is a solution of

$$\frac{d^2\Phi}{d\rho^2} + \frac{1}{\rho}\frac{d\Phi}{d\rho} + [-\frac{\nu_s^2}{\rho^2} + k^2 n^2(\rho)]\Phi = 0 \tag{4-8}$$

Eq. (4-8) does not immediately take on the appearance of the standard form which we have taken to be that of Eq. (4-1). However, if we make the transformations

$$\rho = a \exp(y/a) \quad , \quad y = a \log(\rho/a)$$

and define

$$\Phi(\rho) = F(y)$$

we will find that F(y) satisfies Eq. (4-1) provided we make the definitions

$$\lambda_s = (\nu_s/a)^2 \quad , \quad f(y) = [\rho n(\rho)/a]^2$$

Another good example is the problem of the propagation of axially symmetric waves over the surface of a sphere when the index of refraction is a function of the radial distance r alone.

$$\frac{\partial^2 G}{\partial r^2} + \frac{2}{r}\frac{\partial G}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial G}{\partial\theta}) + k^2 n^2(r) G = 0 \tag{4-9}$$

The solution is of the form

$$G(r,\theta) = \sum_{s=1}^{\infty} \frac{P_{\nu_s - 1/2}(-\cos\theta)}{4\cos(\nu_s \pi)} \Psi_s(r)\Psi_s(r_0) \qquad (4\text{-}10)$$

where $\Psi_s(r)$ is a solution of

$$\frac{d^2\Psi}{dr^2} + \frac{2}{r}\frac{d\Psi}{dr} + \left[- \frac{\nu_s^2 - 1/4}{r^2} + k^2 n^2(r) \right]\Psi = 0 \qquad (4\text{-}11)$$

Eq. (4-11) can be brought into the standard form of Eq. (4-1) by making the transformations

$$r = a \exp(y/a) \ , \quad y = a \log(r/a)$$

and the definition

$$\Psi(r) = \frac{1}{\sqrt{r}} F(y)$$

along with the identification of λ_s and $f(y)$ as being

$$\lambda_s = (\nu_s/a)^2 \ , \quad f(y) = [rn(r)/a]^2$$

There are several other classical problems which we could cite which would lead to an illustration of the fact that Eq. (4-1) is of paramount importance in the study of wave propagation problems. However, in order to focus more sharply on the subject matter of this Section, let us cite one more example which can serve as a model for most of the above problems. We will consider the height gain function $F(y,\lambda_s)$ as it arises in the solution of the separable wave equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + k^2 \left[f(y) + g(x) \right] U = - \delta(x - x_0)\, \delta(y - y_0) \qquad (4\text{-}12)$$

Barring certain pathological cases which we need not consider here, it is possible to obtain solutions of Eq. (4-12) in the form of a series of normal modes of the form

$$U(x, y; x_0, y_0) = \sum_{s=1}^{\infty} \frac{\Psi^+(x_>, \lambda_s)\, \Psi^-(x_<, \lambda_s)}{W_s} \; \frac{F(y, \lambda_s)\, F(y_0, \lambda_s)}{N_s} \tag{4-13}$$

where $\psi(x,\lambda)$ and $F(y,\lambda)$ are solutions of the ordinary differential equations

$$\frac{d^2\Psi}{dx^2} + \left[\lambda + k^2 g(x)\right] \Psi = 0 \tag{4-14}$$

$$\frac{d^2F}{dy^2} + \left[- \lambda + k^2 f(y)\right] F = 0 \tag{4-15}$$

If we assume an $\exp(-i\omega t)$ time dependance, we must require that the "distance" functions $\psi^{\pm}(x,\lambda_s)$ have the properties

$$\Psi^{\pm}(x, \lambda_s) \xrightarrow[k \to \infty]{} \frac{\text{constant}}{\sqrt[4]{k^2 g(x) + \lambda_s}} \exp\left[\pm i \int_0^x \sqrt{k^2 g(u) + \lambda_s}\; du\right] \tag{4-16}$$

and the "height gain" functions $F(y,\lambda_s)$ have the property

$$F(y, \lambda_s) \xrightarrow[k \to \infty]{} \frac{\text{constant}}{\sqrt[4]{k^2 f(y) - \lambda_s}} \exp\left[i \int_0^y \sqrt{k^2 f(v) - \lambda_s}\; dv\right] \tag{4-17}$$

The properties which have been prescribed in Eqs. (4-16) and (4-17) are asymptotic properties which are based upon the so-called WKB approximation for which there exists a very large literature. In order to complete the description of Eq. (4-13) we need to set down the fact that the eigenvalues λ_s are defined by the homogeneous boundary condition

$$\left[\frac{\partial F(y, \lambda_s)}{\partial y} + i\, k\, Z\, F(y, \lambda_s)\right]_{y=0} = 0 \tag{4-18}$$

The Wronskian W_s is defined by

$$W_s = W(\lambda_s) = \Psi^+ \frac{d\Psi^-}{dx} - \Psi^- \frac{d\Psi^+}{dx} \tag{4-19}$$

and the normalization constant is defined by

$$N_s = N(\lambda_s) = \int_0^\infty \left[F(y, \lambda_s)\right]^2 dy \qquad (4\text{-}20)$$

Although a long list of references could be cited for the benefit of the reader who is not familiar with results such as those which are summarized in Eqs. (4-12) through (4-20), the author wishes to acknowledge the fact that the summary given above follows that presented by Friedlander (Ref. 1) more closely than that of any other source with which the author is familiar. Although the monograph by Friedlander (Ref. 2) is generally more accessible to the general scientific public, the author highly commends the original research report to a reader who wishes to acquaint himself with this subject matter.

4.3 Survey of Some Previous Work on the Height Gain Function

Perhaps the most extensive papers on the height gain function and related problems are the two papers (Refs. 3 and 4) by Bremmer. There is a marked similiarity between many of the results obtained by Bremmer and results which will appear below. However, the work reported in this report has been undertaken independently of that of Bremmer and a detailed comparison of the asymptotic expansions have not been undertaken by the present author.*

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*Although many of the results in this report have been made available to Bremmer (and to his collaborators at the National Bureau of Standards at Boulder, Colorado) the present author has not been informed of any discrepancies between the results herein reported and those reported in the papers published by Bremmer. Since these two pieces of work have been carried out independently and follow somewhat different "lines of attack," it is hoped that in the near future such a comparison will be made.

The cylindrical (Bessel) functions $Z_\upsilon(kx)$ are solutions of

$$x^2 \frac{d^2Z}{dx^2} + x \frac{dZ}{dx} + (k^2x^2 - \upsilon^2)Z = 0 .$$

Since the transformation $y = a \log(x/a)$ leads to the operator

$$x^2 \frac{d^2}{dx^2} + x \frac{d}{dx} = a^2 \frac{d^2}{dy^2}$$

the cylindrical functions are the solutions

$$F(y) = Z_\upsilon[(ka)\exp(y/a)]$$

of the equation

$$\frac{d^2F}{dy^2} + [-(\upsilon/a)^2 + k^2\exp(2y/a)]F = 0$$

which is of the form of Eq. (4-1) with $\lambda = (\upsilon/a)^2$ and $f(y) = \exp(2y/a)$. Therefore, it is not surprising that much of the literature that is appropriate to the study of the asymptotic behavior of the height gain function for $k \to \infty$ has had its origin in the interest of the author in developing asymptotic formulae for the cylindrical functions. For a summary of work related to the asymptotic expansions for Bessel functions (and a good guide to the literature), the reader is advised to consult Olver's preface to a table of zeros of Bessel functions which was prepared by the Royal Society Mathematical Tables Committee (Ref. 5). Erdélyi (Ref. 6) has also recently written a survey of the literature related to the study of Eq. (4-1). However, these surveys fail to include some results by Pekeris (Ref. 7), Fock (Ref. 8), Imai (Refs. 9 and 10), and Friedlander (Ref. 1). In many respects, the research reported in Refs. 7 through 10 is more directly applicable to the propagation problems than the more general results which can be found in the other sources.

It will be convenient to start our discussion with a paper by

Pekeris (Ref. 7) in which he defined

$$\lambda_s = - k^2 \mu_s$$

and sought solutions of

$$\frac{d^2F}{dy^2} + k^2 \left[\mu_s + f(y)\right] F = 0 \tag{4-21}$$

He showed that one could express $F(y, -k^2\mu_s)$ in the form

$$F(y, -k^2\mu_s) \sim \frac{1}{\sqrt[4]{k^2\left[\mu_s + f(y)\right]}} \left[u^{1/2}H_{1/3}^{(2)}(u) + \frac{3A}{2k^{4/3}} u^{5/6} H_{-2/3}^{(2)}(u) - \frac{B}{2k^2} u^{3/2}H_{4/3}^{(2)}(u) + 0(k^{-8/3}) + \cdots\right] \tag{4-22}$$

where

$$u(y) = k \int_{y_1}^{y} \sqrt{\mu_s + f(h)}\, dh \tag{4-23}$$

and y_1 is the turning point defined by

$$\mu_s + f(y_1) = 0 \tag{4-24}$$

The constants in Eq. (4-22) are defined by

$$A = \frac{(1.5)^{4/3}}{315(\dot{f})^{2/3}} \left[9\left(\frac{\ddot{f}}{\dot{f}}\right)^2 - 10\left(\frac{\dddot{f}}{\dot{f}}\right)\right]$$

$$B = \frac{1}{450\,\dot{f}} \left[- 25\left(\frac{\ddddot{f}}{\dot{f}}\right) + 70\left(\frac{\ddot{f}\,\dddot{f}}{\dot{f}^2}\right) - 42\left(\frac{\ddot{f}}{\dot{f}}\right)^3\right]$$

where

$$\dot{f} = \frac{df(y_1)}{dy},$$

$$\ddot{f} = \frac{d^2f(y_1)}{dy^2}, \quad \text{etc.}$$

Pekeris showed that he could also express Eq. (4-22) in the form

$$F(y, -k^2\mu_s) \sim \frac{1}{\sqrt[4]{k^2\left[\mu_s + f(y)\right]}} u^{1/6}\left(u^{2/3} + A/k^{4/3}\right)^{\frac{1}{2}} H_{1/3}^{(2)}\left[\left(u^{2/3} + A/k^{4/3}\right)^{\frac{3}{2}}\right]$$

or

$$F(y, -k^2\mu_s) \approx \frac{1}{\sqrt[4]{k^2\left[\mu_s + f(y)\right]}} u^{1/6} h_2\left[\zeta + \left(\frac{3A^{3/2}}{2k^2}\right)^{2/3}\right] \tag{4-25}$$

where

$$\zeta = \left(\frac{3u}{2}\right)^{2/3} \tag{4-26}$$

The function $h_2(\chi)$ in Eq. (4-25) is defined by

$$h_2(\chi) = \left(\frac{2}{3}\right)^{1/3}\chi^{\frac{1}{2}} H_{1/3}^{(2)}\left(\frac{2}{3}\chi^{3/2}\right) = z^{1/3}H_{1/3}^{(2)}(z) \quad , \quad \chi \equiv \left(\frac{3z}{2}\right)^{2/3} \tag{4-27}$$

These are the Airy functions which were tabulated by the Harvard University Computation Laboratory (Ref. 11).

Imai (Ref. 9) rediscovered some of the results which had been derived by Pekeris. In a later paper, Imai (Ref. 10) found an extension of Eq. (4-25) which can be obtained by employing a result that was used by Cherry (Ref. 12). From Cherry's Lemma 1, we can show that $F(y, \lambda_s)$ can be expressed in the form

$$F(y, \lambda_s) = \left(\frac{d\eta}{dy}\right)^{\frac{1}{2}} h_2(\eta) \tag{4-28}$$

where $\eta(y)$ is a solution of the equation

$$\eta(\dot{\eta})^2 - \frac{3}{4}\left(\frac{\ddot{\eta}}{\dot{\eta}}\right)^2 + \frac{1}{2}\frac{\dddot{\eta}}{\dot{\eta}} = -\lambda_s + k^2 f(y) \tag{4-29}$$

where

$$\dot{\eta} = \frac{d\eta}{dy}, \ddot{\eta} = \frac{d^2\eta}{dy^2}, \dddot{\eta} = \frac{d^3\eta}{dy^3}$$

Eq. (4-25) is an example of Cherry's Lemma. Thus, let

$$\eta = \zeta + \left(\frac{3}{2}\right)^{2/3}\frac{A}{k^{4/3}} + O(k^{-6/3}) \tag{4-30}$$

so that

$$\frac{d\eta}{dy} \sim \frac{d\zeta}{dy} = \left(\frac{2}{3}\right)^{1/3} u^{-1/3} \frac{du}{dy} = \left(\frac{2}{3}\right)^{1/3} u^{-1/3} \sqrt{k^2 \ [\mu_s + f(y)]} \qquad (4\text{-}31)$$

Therefore, Eq. (4-25) can be expressed in the form

$$F\left(y_1, - k^2 \mu_s\right) \sim \left(\frac{2}{3}\right)^{1/6} \left(\frac{d\eta}{dy}\right)^{-1/2} h_2(\eta) \qquad (4\text{-}32)$$

where η is defined in Eq. (4-30). The term B/k^2 in Eq. (4-22) can be used to extend Eq. (4-30) to include a term in B/k^2. This step was apparently not carried out until the publication of Imai's 1956 paper (Ref. 10).

In order to appreciate Imai's results, we use Eq. (4-27) to express Eq. (4-28) in the form

$$F(y, \lambda_s) \sim \left(\frac{d\eta}{dy}\right)^{-1/2} Z^{1/3} H_{1/3}^{(2)}(Z) \quad , \quad Z = \frac{2}{3} \eta^{3/2} \qquad (4\text{-}33)$$

From Eq. (20) of Ref. 10, we find that Z has been approximated by

$$Z = \sqrt{k^2 + \lambda_o} \quad \left[\xi + \frac{1}{5} \alpha_1 \kappa^{-2} \xi^2\right]^{3/2} \qquad (4\text{-}34)$$

where

$$\xi = [u(y)/k]^{2/3} + \frac{\alpha}{\kappa^2}$$

$$u(y) = k \int_{y_1}^{y} \sqrt{\mu_s + f(h)} \ dh$$

$$\kappa^2 = k^2 + \alpha_o$$

The constants α, α_0, and α_1 will be defined below.

We have taken some liberties with Imai's notation in that we have substituted some of the notation employed by Pokeris. Actually, Imai (Ref. 10) studied the equation

$$\frac{d^2\Phi}{dx} + a^2 P(x) \Phi = 0 \qquad (4\text{-}35)$$

under the assumption that

$$P(x) = a_1 x \left(1 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots\right) \tag{4-36}$$

If we let

$$a^2 = k^2$$

$$\chi = y - y_1$$

$$a_1 = f'(y_1) = \left[\frac{df}{dy}\right]_{y = y_1} = \dot{f}$$

$$a_1 b_n = \frac{1}{n!} \left[\frac{d^{n+1} f}{dy^{n+1}}\right]_{y = y_1}$$

we can compare the papers of Imai and Pekeris. The constants α, α_0, and α_1 used by Imai are defined by the relations

$$\alpha = -\frac{3}{35}\left(\frac{3}{2} a_1\right)^{-2/3}\left(5b_2 - 3b_1^{\,2}\right) , \tag{4-37}$$

$$\alpha_o = -\frac{4}{75}\frac{1}{a_1}\left(25b_3 - 35b_1 b_2 + 14b_1^{\,3}\right) , \tag{4-38}$$

$$\alpha_1 = -\frac{27}{26950}\left(\frac{3}{2} a_1\right)^{-4/3}\left(6125b_4 - 4350b_2^{\,2} - 9800b_1 b_3 + 12080b_1^{\,2} b_2 - 3624b_1^{\,4}\right) . \tag{4-39}$$

We observe that the A and B of Eq. (4-22) are essentially the same quantities as the α and α_0 defined by Eqs. (4-37) and (4-38), respectively.

Let us use the definition of Z in Eq. (4-33) to write

$$\begin{aligned} \eta = \left(\frac{3}{2} Z\right)^{2/3} &\sim \left(\frac{3}{2}\right)^{2/3} \sqrt[3]{k^2 + \alpha_o} \left[\xi + \frac{1}{5}\alpha_1 \kappa^{-2} \xi^2\right] \\ &\sim \left(\frac{3k}{2}\right)^{2/3}\left[\xi + \frac{1}{3}\frac{\alpha_o}{k^2}\xi + \frac{1}{5}\alpha_1 k^{-2}\xi^2\right] \end{aligned} \tag{4-39}$$

From the definition of ξ which occured in Eq. (4-34), we find that

$$\eta \sim \left(\frac{3k}{2}\right)^{2/3} \left\{ \left[\frac{u(y)}{k}\right]^{2/3} + \frac{\alpha}{k^2} + \frac{1}{3} \frac{\alpha_o}{k^2} \left[\frac{u(y)}{k}\right]^{2/3} + \frac{1}{5} \alpha_1 k^{-2} \left[\frac{u(y)}{k}\right]^{4/3} \right\} \qquad (4\text{-}40)$$

$$= \left(\frac{3}{2}\right)^{2/3} \left\{ u^{2/3}(y) + \frac{\alpha}{k^{4/3}} + \frac{1}{3} \frac{\alpha_o}{k^{6/3}} u^{2/3}(y) + \frac{1}{5} \frac{\alpha_1}{k^{8/3}} u^{4/3}(y) \right\} \qquad (4\text{-}41)$$

If we recall the definition of ζ given in Eq. (4-26), we can write

$$\eta = \zeta + \left(\frac{3}{2}\right)^{2/3} \frac{\alpha}{k^{4/3}} + \frac{1}{3} \frac{\alpha_o}{k^2} \zeta + \frac{1}{5} \left(\frac{2}{3}\right)^{2/3} \frac{\alpha_1}{k^{8/3}} \zeta^2 + O\left(k^{-10/3}\right) \qquad (4\text{-}42)$$

When this result is compared with Eq. (4-30), we see that $\alpha = A$. The simplications we have made in Imai's results have led us to recognize that we have available two additional terms in Eq. (4-30). Imai was apparently not familiar with the work of Pekeris and Cherry so that the use of Eq. (4-42) in Eq. (4-28) to obtain a useful representation for $F(y, \lambda_s)$ is a new result. Further terms in Eq. (4-42) can be generated with the help of Eq. (4-29).

4.4 The Research Reported by Fock

In a classic paper on radiowave propagation, Fock (Ref. 8) studied the equation

$$\frac{d^2G}{du^2} + \left[-t + u + \beta_o u^2\right] G = 0 \qquad (4\text{-}43)$$

He showed that

$$G(u, t) = w(t-u) - \frac{\beta_o}{15}\left[(3u + 2t)\, w(t-u) + (3u^2 + 4ut + 8t^2)\, w'(t-u)\right] + O\left(\beta_o^2\right) \qquad (4\text{-}44)$$

where $w(z) = w_1(z)$ is Fock's Airy function

$$w(z) = \sqrt{\frac{\pi}{3}} \exp\left(i\frac{2\pi}{3}\right) (-z)^{1/2} H_{1/3}^{(1)} \left[\frac{2}{3}(-z)^{3/2}\right] \qquad (4\text{-}45)$$

Fock also used the form

$$G(u, t) \sim \left(-\frac{dX}{du}\right)^{-\frac{1}{2}} w(X) \qquad (4\text{-}46)$$

in which

$$X \sim (t-u) - \frac{\beta_o}{15}\ (3u^2 + 4tu + 8t^2) + O\left(\beta_o^2\right) \tag{4-47}$$

From Fock's paper, it is not clear whether Fock realized the importance of the form of Eq. (4-46) which is essentially the same as Cherry's form for the solution which we gave in Eq. (4-28). The relations between the two types of Airy functions are

$$w_2(-\eta) = \frac{\sqrt{\pi}}{\sqrt[6]{12}} \exp\left(-i\,\frac{2\pi}{3}\right) h_2\,(\eta) \tag{4-48a}$$

$$w_1\,(-\eta) = \frac{\sqrt{\pi}}{\sqrt[6]{12}} \exp\left(i\,\frac{2\pi}{3}\right) h_1\,(\eta) \tag{4-48b}$$

The expressions given in Eq. (4-28) and Eq. (4-46) are special cases of a more general result. Let F(y) and Z(x) be solutions of the equations

$$\frac{d^2F}{dy^2} + P(y)\ F = 0 \qquad , \qquad \frac{d^2Z}{dx^2} + Q(x)\ Z = 0 \tag{4-49}$$

We can then show that

$$F(y) = \left(\frac{d\eta}{dy}\right)^{-\frac{1}{2}} Z(\eta) \tag{4-50}$$

provided $\eta(y)$ is a solution of the non-linear equation

$$(\dot{\eta})^2\ Q(\eta) - \frac{3}{4}\left(\frac{\ddot{\eta}}{\dot{\eta}}\right)^2 + \frac{1}{2}\ \frac{\dddot{\eta}}{\dot{\eta}} = P(y) \tag{4-51}$$

Fock arrived at Eq. (4-47) by using the Langer (Ref. 13) asymptotic estimate (the first term in the Pekeris-Imai expansions)

$$\frac{2}{3}\ (-X)^{\frac{3}{2}} \sim \int_{u_1}^{u} \sqrt{v - t + \beta_o\, v^2}\ \ dv \tag{4-52}$$

where u_1 is the turning point which is defined by

$$u_1 - t + \beta_o\, u_1^{\ 2} = 0 \tag{4-53}$$

Fock did not indicate how one could obtain higher-order corrections of Eq. (4-47) or Eq. (4-52).

Eq. (4-47) has the advantage that it does not involve the turning point. This is a very desirable feature when one sets out to solve for the eigenvalues t_s which are defined by

$$\left[\frac{\partial G}{\partial u} + q\,G\right]_{u=0} = 0 \tag{4-54}$$

For example, Fock showed that

$$t_s(q) = \tau_s + \frac{\beta_o}{15}\left[8\tau_s^2 - \frac{3 + 4\,\tau_s q}{\tau_s - q^2}\right] + O\left(\beta_o^2\right) \tag{4-55}$$

where τ_s is a solution of

$$w_1'(\tau_s) - q w_1(\tau_s) = 0 \tag{4-56}$$

In order to see the advantages of Eq. (4-55), let us take $q = \infty$ and write

$$t_s(\infty) = \tau_s^\infty + \frac{8}{15}\beta_o\left[\tau_s^\infty\right]^2 + O\left(\beta_o^2\right) \tag{4-57}$$

where

$$w_1(\tau_s^\infty) = 0 \tag{4-58}$$

Let us now compare this with the procedure which involves the use of the turning point. Let us seek the roots λ_s defined by

$$F(0,\lambda_s) = 0 \tag{4-59}$$

Let us express $F(y,\lambda_s)$ in the form

$$F(y,\lambda_s) = \left(\frac{d\eta}{dy}\right)^{-1/2} w(-\eta) \tag{4-60}$$

where η is given by Eq. (4-42). We find that the roots of $F(0,\lambda_s) = 0$ are to be obtained from the relationship

$$-\tau_s^\infty = \zeta_o + \left(\frac{3}{2}\right)^{2/3} \frac{\alpha}{k^{4/3}} + \frac{1}{3}\,\frac{\alpha_o}{k^2}\,\zeta_o + \frac{1}{5}\left(\frac{2}{3}\right)^{2/3} \frac{\alpha_1}{k^{8/3}}\,\zeta_o^{\,2} + \cdots \tag{4-61}$$

where ζ_0 is defined by

$$\frac{2}{3}\zeta_o^{3/2} = u(0) = k\int_{y_1}^{0}\sqrt{\mu_s + f(h)}\;dh = \int_{y_1}^{0}\sqrt{k^2 f(h) - \lambda_s}\;dh \qquad (4\text{-}62)$$

and the turning point y_1 is defined by

$$k^2 f(y_1) - \lambda_s = 0$$

Since λ_s appears explicitly in the factor $\sqrt{k^2 f(h) - \lambda_s}$ in the integrand of Eq. (4-62), and implicitly in the integration limit y_1, the inversion of Eq. (4-61) to solve for λ_s requires some skillful analysis. A method for achieving this inversion was described by Pekeris (Ref. 7). A comparison of Fock's procedure with that used by Pekeris shows very clearly that Fock's method is more direct and hence it would be advantageous to extend Fock's analysis to include higher-order terms.

4.5 The Contributions of Friedlander

Friedlander (Ref. 1) considered the problem of determining the behavior, as $s \to \infty$, of the eigenvalues λ of the differential equation

$$\frac{d^2y}{dx^2} + \left[\lambda - s^2 g(x)\right] y = 0 \qquad (4\text{-}63)$$

subject to the conditions that $y \to 0$ as $x \to \infty$ and either $y = 0$ or $dy/dx = 0$ at $x = 0$. He assumed that $g(x)$ was characterized by the conditions $g(0) > 0$ and $g'(0) \gtrless 0$ and assumed that $g(x)$ could be represented in the form of a Taylor series

$$g(x) = g_0 + g_1 x + g_2 x^2 + g_3 x^3 + \cdots \quad , \quad g_m = \frac{1}{m!}\, g^{(m)}(0) \qquad (4\text{-}64)$$

Friedlander assumed that λ could be expressed in the form of an asymptotic series in inverse integral powers of $s^{\frac{2}{3}}$

$$\lambda = s^2 \sum_{m=0}^{\infty} c_m s^{-\frac{2m}{3}} \quad , \quad c_0 = g_0 \qquad (4\text{-}65)$$

and that $y(x)$ could be expressed in the form of a similar series

$$y(x) = \sum_{m=0}^{\infty} Y_m(\eta)\, s^{-\frac{2m}{3}} \quad , \quad \eta = s^{2/3} \tag{4-66}$$

Friedlander showed that the coefficients $Y_m(\eta)$ are determined by a system of recurrence relations

$$\frac{d^2Y_0}{d\eta^2} + (c_1 - g_1\eta)\, Y_0 = 0 \tag{4-67a}$$

$$\frac{d^2Y_1}{d\eta^2} + (c_1 - g_1\eta)\, Y_1 = (g_2\eta^2 - c_2)\, Y_0 \tag{4-67b}$$

$$\frac{d^2Y_2}{d\eta^2} + (c_1 - g_1\eta)\, Y_2 = (g_3\eta^3 - c_3)\, Y_0 + (g_2\eta^2 - c_2)\, Y_1 \tag{4-67c}$$

$$\frac{d^2Y_m}{d\eta^2} + (c_1 - g_1\eta)\, Y_m = \sum_{j=1}^{m} \left(g_{j+1}\eta^{j+1} - c_{j+1}\right) Y_{m-j} \tag{4-67d}$$

Friedlander showed that

$$Y_0(\eta) = L_0\, \mathrm{Ai}\,(g_1^{1/3}\eta - \alpha) \quad , \quad \alpha = c_1\, g_1^{-2/3} \tag{4-68}$$

where L_0 is a constant. He also showed that

$$Y_1(\eta) = \frac{\pi L_0}{g_1^{1/3}} \int_0^{\eta} \left[\mathrm{Ai}\left(g_1^{1/3}\xi - \alpha\right) \mathrm{Bi}\left(g_1^{1/3}\eta - \alpha\right) - \mathrm{Ai}\left(g_1^{1/3}\eta - \alpha\right) \mathrm{Bi}\left(g_1^{1/3}\xi - \alpha\right)\right] \left[g_2\xi^2 - c_2\right] \mathrm{Ai}\left(g_1^{1/3}\xi - \alpha\right) d\xi \tag{4-69}$$

and gave a similar result for $Y_m(\eta)$. However, Friedlander did not evaluate these integrals and hence $Y_0(\eta)$ is the only $Y_m(\eta)$ which is given in its final form.

If we put $s = \pm ik$ in Eq. (4-63), and replace x by y, $g(x)$ by $f(x)$, and $y(x)$ by $F(y,\lambda)$ we find that the function being studied (for

the pulse problem by Friedlander) is essentially the height gain function which was being studied by Fock.

$$\left[Ai\left(s^{2/3} g_1^{1/3} y - \alpha\right)\right]_{s = \mp ik} = \frac{\exp\left(\mp i\frac{\pi}{6}\right)}{2\sqrt{\pi}} w_{1,2}\left(\alpha e^{\pm i\frac{\pi}{3}} - k^{2/3} g_1^{1/3} y\right) \qquad (4\text{-}70)$$

Therefore, we can use Friedlander's theory as a starting point to obtain an extension of Fock's results. Let us return to the differential equation for $F(y, \lambda_s)$

$$\frac{d^2F}{dy^2} + \left[-\lambda_s + k^2 f(y)\right] F = 0 \qquad (4\text{-}71)$$

and express $f(y)$ in the form

$$f(y) = f_o + f_1 + f_2 y^2 + f_3 y^3 + \cdots = f_o + f_1\left[y + h_2 y^2 + h_3 y^3 + \cdots\right] \qquad (4\text{-}72)$$

where

$$h_n = \frac{f_n}{f_1} = \frac{1}{n!\, f_1}\left[\frac{d^n f(y)}{dy^n}\right]_{y=0} \qquad (4\text{-}73)$$

Let us define t_s by means of

$$\lambda_s = k^2 f_o + k^{4/3} f_1^{2/3} t_s \qquad (4\text{-}74)$$

and u by means of

$$u = k^{2/3} f_1^{1/3} y \qquad (4\text{-}75)$$

We then find that if we define the parameter β to be

$$\beta = k^{-2/3} f_1^{-1/3} \qquad (4\text{-}76)$$

that Eq. (4-71) [which is identical with Eq. (4-1)] takes the form

$$\frac{d^2G}{du^2} + \left[-t_s + u + \beta h_2 u^2 + \beta^2 h_3 u^3 + \beta^3 h_4 u^4 + \cdots\right] G = 0 \qquad (4\text{-}77)$$

where

$$G(u, t_s) = F(y, \lambda_s) \qquad (4\text{-}78)$$

A comparison of Eq. (4-78) with Eq. (4-43) reveals that if we let $\beta_0 = \beta h_2$ and neglect the terms in β^n for $n > 3$ in Eq. (4-77) that we have Fock's approximate differential equation.

Let us follow Friedlander [cf. Eq. (4-66)] and seek a representation for G(u,t) in the form

$$G(u, t) = \sum_{n=0}^{\infty} G_n(u, t)\,\beta^n \tag{4-79}$$

We then find that the $G_n(u,t)$ satisfy the equations

$$\left[\frac{d^2}{du^2} + (-t + u)\right] G_o = 0$$

$$\left[\frac{d^2}{du^2} + (-t + u)\right] G_1 = -h_2 u^2 G_o$$

$$\left[\frac{d^2}{du^2} + (-t + u)\right] G_2 = -h_2 u^2 G_1 - h_3 u^3 G_o$$

$$\left[\frac{d^2}{du^2} + (-t + u)\right] G_n = -\sum_{j=2}^{n+1} h_j u^j G_{n+1-j}$$

This set of recursive formulae is similar to Eq. (4-67). If we define

$$t = t(\tau) = \tau + \sum_{n=1}^{\infty} D_n(\tau)\,\beta^n = \sum_{n=0}^{\infty} D_n(\tau)\,\beta^n\ , \quad D_o = \tau \tag{4-80}$$

and

$$H(u, t) = G(u, t(\tau)) = \sum_{n=0}^{\infty} H_n(u, \tau)\,\beta^n \tag{4-81}$$

we find that

$$\left[\frac{d^2}{du^2} + (-\tau + u)\right] H_o = 0 \tag{4-82a}$$

$$\left[\frac{d^2}{du^2} + (-\tau + u)\right] H_1 = \left[-h_2 u^2 + D_1\right] H_o \tag{4-82b}$$

$$\left[\frac{d^2}{du^2} + (-\tau + u)\right] H_2 = \left[-h_2u^2 + D_1\right] H_1 + \left[-h_3u^3 + D_2\right] H_o \qquad (4\text{-}82c)$$

$$\left[\frac{d^2}{du^2} + (-\tau + u)\right] H_n = \sum_{j=2}^{n+1} \left[-h_ju^j + D_{j-1}\right] H_{n+1-j} \qquad (4\text{-}82d)$$

which is essentially the same as Eq. (4-67). Therefore, the theory given in Friedlander's paper for results such as those given in Eq. (4-68) and Eq. (4-69) can be used to determine the $H_n(u,\tau)$.

4.6 Determination of the $H_n(u,\tau)$

From the differential equation satisfied by the Airy function

$$\left[\frac{d^2}{du^2} + (-\tau + u)\right] w\,(\tau - u) = 0 \qquad (4\text{-}83)$$

we can show that

$$\left[\frac{d^2}{du^2} + (-\tau + u)\right] u^n\, w(\tau - u) = n(n-1)\, u^{n-2}\, w\,(\tau - u) - 2n\, u^{n-1}\, w'(\tau - u) \qquad (4\text{-}84)$$

and

$$\left[\frac{d^2}{du^2} + (-\tau + u)\right] u^n\, w'(\tau - u) = \left[(2n + 1)\, u^n - 2\tau n u^{n-1}\right] w(\tau - u) + n\,(n-1)\, u^{n-2}\, w'(\tau - u) \qquad (4\text{-}85)$$

These results can be used to construct solution of the inhomogeneous equations

$$\left[\frac{d^2}{du^2} + (-\tau + u)\right] A_n(u, \tau) = u^n\, w\,(\tau - u) \qquad (4\text{-}86)$$

$$\left[\frac{d^2}{du^2} + (-\tau + u)\right] B_n(u, \tau) = u^n\, w'\,(\tau - u) \qquad (4\text{-}87)$$

If $A_n(u,\tau)$ and $B_n(u,\tau)$ are known, we can solve Eq. (4-82). However, it may be more convenient to define

$$x = \tau - u \quad , \quad u = \tau + x \qquad (4\text{-}88)$$

and replace Eq. (4-82) by

$$\left[\frac{d^2}{dx^2} - x\right] H_n = \sum_{j=2}^{n+1} \left[-h_j\,(\tau + x)^j + D_{j-1}\right] H_{n+1-j} \tag{4-89}$$

In Table 4-1 we give some solutions of

$$\frac{d^2y}{dx^2} - x\,y = f(x)$$

which can be used to construct solutions of Eq. (4-89). For example, the solution of Eq. (4-82a) is

$$H_0(u,\tau) = c_0 w(\tau - u) \tag{4-90a}$$

where c_0 is a constant. We can then show that the solution of Eq. (4-82b) is

$$H_1(u,\tau) = c_1 w(\tau - u) + c_0 \left\{ -\frac{1}{5}\, h_2 u\, w\,(\tau - u) \right.$$
$$\left. - \left[\frac{1}{5}\, h_2 u^2 + \frac{4}{15}\, h_2\, \tau\, u \;+\; \frac{8}{15}\, \tau^2 h_2 - D_1\right] w'(\tau - u) \right\} \tag{4-90b}$$

where c_1 is a constant.

Before determining $H_2(u,\tau)$ it would be desirable to determine the constants c_0 and c_1 (or, at least determine the ratio c_1/c_0). Since the function $H(u,\tau)$ satisfies the homogeneous boundary condition derived from Eq. (4-54)

$$\left[\frac{\partial H}{\partial u} + qH\right]_{u=0} = 0 \tag{4-91}$$

we must require that $H_n(u,\tau_s)$ satisfy the equations

$$\left[\frac{\partial H_o}{\partial u} + q\,H_o\right]_{u=0} = 0 \tag{4-92a}$$

$$\left[\frac{\partial H_1}{\partial u} + q\,H_1\right]_{u=0} = 0 \tag{4-92b}$$

We observe that Eq. (4-90a) and Eq. (4-92a) lead to the equation

$$w'_{,}(\tau_s) - q\, w(\tau_s) = 0 \tag{4-93}$$

Table 4-1

PARTICULAR SOLUTIONS OF $\frac{d^2y}{dx^2} - xy = f(x)$

f(x)	y(x)
0	$w(x)$
$w(x)$	$w'(x)$
$w'(x)$	$\frac{1}{2} x\, w(x)$
$x\, w(x)$	$\frac{1}{3} x\, w'(x)$
$x\, w'(x)$	$\frac{1}{4} x^2\, w(x) - \frac{1}{2} w'(x)$
$x^2 w(x)$	$-\frac{1}{5} x\, w(x) + \frac{1}{5} x^2\, w'(x)$
$x^2\, w'(x)$	$\left[\frac{1}{6} x^3 \right] w(x) - x \frac{1}{3} w'(x)$
$x^3\, w(x)$	$\left[-\frac{3}{14} x^2 \right] w(x) + \left[\frac{3}{7} + x^3 \frac{1}{7} \right] w'(x)$
$x^3\, w'(x)$	$\left[\frac{3}{10} x + \frac{1}{8} x^4 \right] w(x) - \frac{3}{10} x^2\, w'(x)$
$x^4\, w(x)$	$\left[-\frac{2}{9} x^3 \right] w(x) + \left[\frac{4}{9} x + \frac{1}{9} x^4 \right] w'(x)$
$x^4\, w'(x)$	$\left[\frac{3}{7} x^2 + \frac{1}{10} x^5 \right] w(x) + \left[-\frac{6}{7} - \frac{2}{7} x^3 \right] w'(x)$
$x^5\, w(x)$	$\left[-\frac{6}{11} x - \frac{5}{22} x^4 \right] w(x) + \left[\frac{6}{11} x^2 + \frac{1}{11} x^5 \right] w'(x)$
$x^5\, w'(x)$	$\left[\frac{5}{9} x^3 + \frac{1}{12} x^6 \right] w(x) + \left[-\frac{10}{9} x - \frac{5}{18} x^4 \right] w'(x)$
$x^6\, w(x)$	$\left[-\frac{90}{91} x^2 - \frac{3}{13} x^5 \right] w(x) + \left[\frac{180}{91} + \frac{60}{91} x^3 + \frac{1}{13} x^6 \right] w'(x)$
$x^6\, w'(x)$	$\left[\frac{18}{11} x + \frac{15}{22} x^4 + \frac{1}{14} x^7 \right] w(x) + \left[-\frac{18}{11} x^2 - \frac{3}{11} x^5 \right] w'(x)$
$x^7\, w(x)$	$\left[-\frac{14}{9} x^3 - \frac{7}{30} x^6 \right] w(x) + \left[\frac{28}{9} x + \frac{7}{9} x^4 + \frac{1}{15} x^7 \right] w'(x)$
$x^7 w'(x)$	$\left[\frac{45}{13} x^2 + \frac{21}{26} x^5 + \frac{1}{16} x^8 \right] w(x) + \left[-\frac{90}{3} - \frac{30}{13} x^3 - \frac{7}{26} x^6 \right] w'(x)$
$x^8\, w(x)$	$\left[-\frac{1008}{187} x - \frac{420}{187} x^4 - \frac{4}{17} x^7 \right] w(x) + \left[\frac{1008}{187} x^2 + \frac{168}{187} x^5 + \frac{1}{17} x^8 \right] w'(x)$
$x^8\, w'(x)$	$\left[\frac{56}{9} x^3 + \frac{14}{15} x^6 + \frac{1}{18} x^9 \right] w(x) + \left[-\frac{112}{9} x - \frac{28}{9} x^4 - \frac{4}{15} x^7 \right] w'(x)$

which determines τ_s uniquely. If we observe that Eq. (4-90b) and Eq. (4-92b) lead to

$$\left[\frac{\partial H_1(u,\tau)}{\partial u} + q\, H_1\,(u,\tau)\right]_{u=0} = -\, c_1 \left[w'(\tau) - q\, w\,(\tau)\right] + c_o \left\{ \left[-\frac{1}{5} h_2 w(\tau)\right.\right.$$

$$\left.\left. -\frac{4}{15} h_2 \tau\, w'(\tau) + \frac{8}{15} h_2 \tau^3 w(\tau) - D_1 \tau\, w(\tau)\right] + q\left[-\frac{8}{15} h_2 \tau^2 + D_1\right] w'(\tau)\right\} = 0 \qquad (4\text{-}94)$$

from which it is seen that the value of D_1 is given by

$$(\tau - q^2)\, D_1 = h_2\left[-\frac{1}{5} - \frac{4}{15}\tau q + \frac{8}{15}\tau^2\,(\tau - q^2)\right] \qquad (4\text{-}95)$$

Therefore, we observe that the boundary condition given by Eq. (4-91) determines the constants D_n but leaves the values of c_n undetermined.

The c_n can be determined by requiring that the eigenfunctions $F(y,\lambda_s)$ have the property

$$\int_0^\infty F^2\,(y,\lambda_s)\,dy = 1 \qquad (4\text{-}96)$$

This leads to

$$\int_0^\infty H^2(u,\tau_s)\,du = k^{2/3}\, f_1^{\;1/3} = \beta^{-1} \qquad (4\text{-}97)$$

If we use the property

$$H^2 = \left[\sum_{n=0}^{\infty} H_n \beta^n\right]^2 = H_o^2 + 2H_o\;\, H_1\,\beta + (2H_o\;\, H_2 + H_1^2)\,\beta^2 + \cdots \qquad (4\text{-}98)$$

and require that

$$\int_0^\infty \left[H_o(u,\tau_s)\right]^2 du = c_o^2 \int_0^\infty w^2\,(\tau_s - u)\,du = k^{2/3}\, f_1^{\;1/3} \qquad (4\text{-}99)$$

we observe that Eq. (4-97) is satisfied provided

$$\int_0^\infty H_o(u,\tau_s)\, H_1(u,\tau_s)\,du = 0 \qquad (4\text{-}100a)$$

$$\int_0^\infty \left[2\, H_o(u,\tau_s)\, H_2(u,\tau_s) + H_1^2\,(u,\tau_s)\right] du = 0 \qquad (4\text{-}100b)$$

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along with similar equations involving H_3, H_4, etc., This procedure has the advantage that the ratio c_n/c_0 is independent of β. We can determine c_0 from Eq. (4-99) by observing that

$$\int_0^\infty w^2(\tau_s - u)\,du = \tau_s\left[w^2(\tau_s)\right] - \left[w'(\tau_s)\right]^2 \tag{4-101}$$

since

$$\frac{d}{du}\left\{(\tau - u)\left[w(\tau - u)\right]^2 - \left[w'(\tau - u)\right]^2\right\} = -\left[w(\tau - u)\right]^2$$

Therefore, we find that

$$c_o^{\,2} = \frac{k^{2/3}\,f_1^{\,1/3}}{\tau_s\left[w(\tau_s)\right]^2 - \left[w'(\tau_s)\right]^2} = \frac{k^{2/3}\,f_1^{\,1/3}}{\left[\tau_s - q^2\right]\left[w\,(\tau_s)\right]^2} \tag{4-102}$$

In order to determine c_1 we must evaluate the integrals that occur in the equation

$$\begin{aligned}\frac{c_1}{c_o}\int_0^\infty w^2(\tau - u)\,du &= \frac{c_1}{c_o}\left\{\tau\left[w(\tau)\right]^2 - \left[w'(\tau)\right]^2\right\} = -D_1\int_0^\infty w(\tau - u)w'(\tau - u)\,du \\ &+ \frac{h_2}{15}\int_0^\infty \left[3uw(\tau - u) + (3u^2 + 4u\tau + 8\tau^2)w'(\tau - u)\right]w(\tau - u)\,du \\ &= -\frac{D_1}{2}\left[w(\tau)\right]^2 + \frac{h_2}{15}\left[-3\int_{-\infty}^{\tau}(\zeta + \tau)\,w^2(\zeta)\,d\zeta\right. \\ &\left.+\int_{-\infty}^{\tau}(15\tau^2 - 10\zeta\tau + 3\zeta^2)\,w(\zeta)\,w'(\zeta)\,d\zeta\right]\end{aligned} \tag{4-103}$$

We can show that

$$3\int_{-\infty}^{\tau}(\zeta + \tau)\,w^2(\zeta)\,d\zeta = \left\{4\tau^2\left[w(\tau)\right]^2 - 4\tau\left[w'(\tau)\right]^2 + w(\tau)\,w'(\tau)\right\}$$

$$\int_{-\infty}^{\tau}(15\tau^2 - 10\zeta\tau + 3\zeta^2)\,w(\zeta)\,w'(\zeta)\,d\zeta = \left\{8\tau^2\left[w(\tau)\right]^2 - 4\tau\left[w'(\tau)\right]^2 - w(\tau)\,w'(\tau)\right\}$$

Since τ_s is a solution of $w'(\tau) = q\,w(\tau)$, we find that the integrals in Eq. (4-103) lead to the result

$$\frac{c_1}{c_o}(\tau - q^2) = -\frac{D_1}{2} + \frac{h_2}{15}\left\{4\tau^2 - 2q\right\} \qquad (4\text{-}104)$$

Because of the labour involved in the determination of the constants c_n which will occur in the terms $c_n w(\tau - u)$ which are to be added to the solutions of the functions $H_n(u,\tau)$ which satisfy Eq. (4-82), it would be most convenient to follow Friedlander's suggestion and take $c_0 = 1$ and $c_n = 0$ for $n \geq 1$. We should then replace Eq. (4-81) by

$$H(u, t) = G(u, t(\tau)) = c(\beta,\tau)\left[w(\tau - u) + \sum_{n=1}^{\infty} H_n(u,\tau)\,\beta^n\right] \qquad (4\text{-}105)$$

and determine $C(\beta,\tau)$ from Eq. (4-97). From Eq. (4-90b) we know that for this case

$$H_1(u,\tau) = -\frac{1}{5}h_2\,uw(\tau - u) - \left[\frac{1}{5}h_2u^2 + \frac{4}{15}h_2\tau u + \frac{8}{15}h_2\tau^2 - D_1\right]w'(\tau-u) \qquad (4\text{-}106)$$

From Eq. (4-105) and Eq. (4-97), we find that $C(\beta,\tau)$ is determined by the relation

$$\frac{1}{\beta} = C^2(\beta,\tau)\,w^2(\tau)\left\{(\tau - q^2) + \left[D_1 - \frac{4}{15}h_2(2\tau^2 - q)\right]\beta + O(\beta^2)\right\} \qquad (4\text{-}107)$$

We can also seek a solution in the form of Eq. (4-105) in which $H_1(u,\tau)$ is of the form

$$H_1(u,\tau) = (c_1 - \frac{1}{5}h_2u)\,w(\tau-u) - \left[\frac{1}{5}h_2u^2 + \frac{4}{15}h_2\tau u + \frac{8}{15}h_2\tau^2 - D_1\right]w'(\tau-u) \qquad (4\text{-}108)$$

where c_1 is a constant independent of β. For example, Eq. (4-44), which was taken from Fock's paper, is of the form of Eq. (4-108) if we take

$$c_1 = -\frac{2}{15}h_2\tau \qquad (4\text{-}109)$$

Let us now turn to the problem of the determination of $H_2(u,\tau)$. We have discussed the role played by the choice of c_1 in the determination of $H_2(u,\tau)$. For example, a term $c_1 w(\tau - y)$ in $H_1(u,\tau)$

leads to a term in $H_2(u,\tau)$ which is of the form

$$c_1\left\{-\frac{1}{5}h_2 u\, w(\tau-u)-\left[\frac{1}{5}h_2 u^2+\frac{4}{15}h_2\tau u+\frac{8}{15}h_2\tau^2-D_1\right]w'(\tau-u)\right\} \quad (4\text{-}110)$$

If we take c_1 to be defined by Eq. (4-109), we find that Eq. (4-82c), with the aid of the entries in Table 4-1, leads to

$$\begin{aligned} H_2(u,\tau) = c_2\, w(\tau-u) + \Big[& -\frac{1}{50}h_2^2u^5-\frac{1}{30}h_2^2\tau u^4-\frac{4}{45}h_2^2\tau^2u^3+\frac{1}{5}h_2D_1u^3 \\ & +\frac{1}{15}h_2D_1\tau u^2+\left(-\frac{3}{14}h_3+\frac{9}{70}h_2^2\right)u^2+\frac{4}{15}h_2D_1\tau^2u \\ & \left(-\frac{6}{35}h_3+\frac{22}{105}h_2^2\right)\tau u-\frac{1}{2}D_1^2u\Big]w(\tau-u)+\Big[-\frac{1}{7}h_3 \\ & -\frac{3}{35}h_2^2\Big)u^3-\left(\frac{6}{35}h_3-\frac{22}{105}h_2^2\right)\tau u^2-\Big(\frac{8}{35}h_3 \\ & -\frac{144}{315}h_2^2\Big)\tau^2u-\frac{7}{15}h_2D_1u-\left(\frac{6}{35}h_3-\frac{288}{315}h_2^2\right)\tau^3 \\ & -\frac{6}{5}h_2D_1\tau+\left(\frac{3}{7}h_3-\frac{9}{35}h_2^2+D_2\right)\Big]w'(\tau-u) \end{aligned} \quad (4\text{-}111)$$

In the next section we will show than an extension of Fock's re-results summarized in Eq. (4-46) and Eq. (4-47) suggests that a useful value for the arbitrary constant c_2 is

$$c_2=\frac{32}{225}h_2^2\tau^5-\frac{8}{15}h_2D_1\tau^3+\left(-\frac{4}{35}h_3+\frac{58}{315}h_2^2\right)\tau^2+\frac{1}{2}D_1^2\tau-\frac{2}{15}h_2D_1 \quad (4\text{-}112)$$

The obtaining of $H_3(u,\tau)$, $H_4(u,\tau)$, etc., by the above methods is exceedingly tedious although it is a very straight-foward process if one uses Table 4-1 along with Eq. (4-89).

4.7 Determination of $\eta(u,t)$

Let us follow Eq. (4-50) and express the solution of Eq. (4-77) in the form

$$G(u,t)=\left(\frac{d\eta}{du}\right)^{-\frac{1}{2}}w_{1,2}(-\eta)=\frac{\sqrt{\pi}}{\sqrt[6]{12}}\exp\left(\pm i\frac{2\pi}{3}\right)\left(\frac{d\eta}{du}\right)^{-\frac{1}{2}}h_{1,2}(\eta) \quad (4\text{-}113)$$

where $w(z)$ and $h(\eta)$ are solutions of the Airy differential equations

$$\frac{d^2w}{dz^2}-zw=0\ , \qquad \frac{d^2h}{d\eta^2}+\eta h=0$$

According to Eq. (4-51), the function $\eta(u,t)$ is a solution of

$$\eta\,(\dot\eta)^2 - \frac{3}{4}\left(\frac{\ddot\eta}{\dot\eta}\right)^2 + \frac{1}{2}\left(\frac{\dddot\eta}{\dot\eta}\right) = u - t_s + \sum_{n=2}^{\infty} h_n\, u^n\, \beta^{n-1} \qquad (4\text{-}114)$$

where the dot denotes differentiation with respect to u. We observe that Eq. (4-114) is essentially the same equation as Eq. (4-29) except that we have used Eq. (4-74) to define t_s and Eq. (4-75) to define u.

Although Eq. (4-114) is non-linear, and of the third order, it is simpler to work with than the linear, second order equation which defines $\ddot G(u,t)$. The simplicity comes from the fact that $\eta(u,t)$ can be expressed in the form

$$\eta(u,\ t) = (u - t) + \sum_{n=1}^{\infty} p_n\,(u,\ t)\,\beta^n \qquad (4\text{-}115)$$

where $p_n(u,t)$ is a sum of homogeneous polynomials in u and t for which the degree of the polynomials differ by three. Thus

$$p_1\,(u,\ t) \;=\; a_1\,u^2 + b_1\,u\,t + c_1\,t^2 \qquad (4\text{-}116a)$$

$$p_2\,(u,\ t) \;=\; a_2\,u^3 + b_2\,u^2\,t + c_2\,u\,t^2 + d_2\,t^3 + \alpha_2 \qquad (4\text{-}116b)$$

$$p_3\,(u,\ t) \;=\; a_3\,u^4 + b_3\,u^3\,t + c_3\,u^2\,t^2 + d_3\,u\,t^3 + e_3\,t^4 + \alpha_3\,u + \beta_3\,t \qquad (4\text{-}116c)$$

If we put Eq. (4-116) in Eq. (4-115), and then use Eq. (4-114), we can determine the constants $a_n, b_n, \ldots, \alpha_n, \beta_n, \ldots$, etc. The first several terms in the quantities in Eq. (4-114) are

$$\eta\left[\frac{d\eta}{du}\right]^2 = (u - t) + \beta\left[5a_1\,u^2 + (3b_1 - 4a_1)\,u\,t + (c_1 - 2b_1)\,t^2\right] + \cdots \qquad (4\text{-}117a)$$

$$-\frac{3}{4}\left[\frac{d\eta}{du}\right]^{-2}\left[\frac{d^2\eta}{du^2}\right]^2 = 4a_1{}^2\,\beta^2 + \cdots \qquad (4\text{-}117b)$$

$$\frac{1}{2}\left[\frac{d\eta}{du}\right]^{-1}\left[\frac{d^3\eta}{du^3}\right] = 6a_2\,\beta^2 + \cdots \qquad (4\text{-}117c)$$

In Eq. (4-118) we give the resulting expression for $\eta(u,t)$ for terms as high as those in β^4.

$$
\begin{aligned}
\eta(u,t) &= [u - t] \\
&+ \beta\left[\frac{1}{5} h_2 u^2 + \frac{4}{15} h_2 t u + \frac{8}{15} h_2 t^2\right] \\
&+ \beta^2\left[\left(\frac{1}{7} h_3 - \frac{8}{175} h_2{}^2\right) u^3 + \left(\frac{6}{35} h_3 - \frac{68}{525} h_2{}^2\right) tu^2 + \left(\frac{8}{35} h_3 - \frac{496}{1575} h_2{}^2\right) t^2 u + \left(\frac{16}{35} h_3 - \frac{1328}{1575} h_2{}^2\right) t^3 - \left(\frac{3}{7} h_3 - \frac{9}{35} h_2{}^2\right)\right] \\
&+ \beta^3\Bigg[\left(\frac{1}{9} h_4 - \frac{22}{315} h_2 h_3 + \frac{148}{7875} h_2{}^3\right) u^4 + \left(\frac{8}{63} h_4 - \frac{3212}{17325} h_2 h_3 + \frac{608}{7875} h_2{}^3\right) tu^3 + \Bigg(\frac{16}{105} h_4 - \frac{2288}{5775} h_2 h_3 \\
&\qquad + \frac{1808}{7875} h_2{}^3\Bigg) t^2 u^2 + \left(\frac{64}{315} h_4 - \frac{13376}{17325} h_2 h_3 + \frac{43712}{70875} h_2{}^3\right) t^3 u + \left(\frac{128}{315} h_4 - \frac{33088}{17325} h_2 h_3 + \frac{128512}{70875} h_2{}^3\right) t^4 \\
&\qquad - \left(\frac{4}{9} h_4 - \frac{28}{45} h_2 h_3 + \frac{56}{225} h_2{}^3\right) u - \left(\frac{80}{63} h_4 - \frac{1628}{693} h_2 h_3 + \frac{1768}{1575} h_2{}^3\right) t\Bigg] \\
&+ \beta^4\Bigg[\left(\frac{1}{11} h_5 - \frac{28}{495} h_2 h_4 - \frac{15}{539} h_3{}^2 + \frac{5464}{121275} h_2{}^2 h_3 + \frac{29584}{3031875} h_2{}^4\right) u^5 + \Bigg(\frac{10}{99} h_5 - \frac{1504}{10395} h_2 h_4 - \frac{1707}{24255} h_3{}^2 \\
&\qquad + \frac{321392}{1819125} h_2{}^2 h_3 - \frac{93584}{1819125} h_2{}^4\Bigg) tu^4 + \Bigg(\frac{80}{693} h_5 - \frac{2176}{7425} h_2 h_4 - \frac{16656}{121275} h_3{}^2 + \frac{888592}{1819125} h_2{}^2 h_3 \\
&\qquad - \frac{327808}{1819125} h_2{}^4\Bigg) t^2 u^3 + \left(\frac{32}{231} h_5 - \frac{8768}{17325} h_2 h_4 - \frac{3488}{13475} h_3{}^2 + \frac{708704}{606375} h_2{}^2 h_3 - \frac{2914624}{5457375} h_2{}^4\right) t^3 u^2 \\
&\qquad + \left(\frac{128}{693} h_5 - \frac{46336}{51975} h_2 h_4 - \frac{18176}{40425} h_3{}^2 + \frac{4738432}{1819125} h_2{}^2 h_3 - \frac{24237568}{16372125} h_2{}^4\right) t^4 u + \Bigg(\frac{256}{693} h_5 - \frac{109568}{51975} h_2 h_4 \\
&\qquad - \frac{42688}{40425} h_3{}^2 + \frac{38593536}{5457375} h_2{}^2 h_3 - \frac{371579392}{81860625} h_2{}^4\Bigg) t^5 - \Bigg(\frac{6}{11} h_5 - \frac{388}{495} h_2 h_4 - \frac{1044}{2695} h_3{}^2 + \frac{115372}{121275} h_2{}^2 h_3 \\
&\qquad - \frac{165512}{606375} h_2{}^4\Bigg) u^2 - \left(\frac{112}{99} h_5 - \frac{3568}{1485} h_2 h_4 - \frac{8828}{8085} h_3{}^2 + \frac{1330096}{363825} h_2{}^2 h_3 - \frac{2327168}{1819125} h_2{}^4\right) tu \\
&\qquad - \left(\frac{1808}{693} h_5 - \frac{71552}{10395} h_2 h_4 - \frac{24076}{8085} h_3{}^2 + \frac{12920064}{1091475} h_2{}^2 h_3 - \frac{25097808}{5457375} h_2{}^4\right) t^2\Bigg] + \cdots \qquad (4\text{-}118)
\end{aligned}
$$

In many applications it is more convenient to let t be defined by Eq. (4-80). If we recall the definition of $H(u,\tau)$ in Eq. (4-81) we can write

$$H(u, \tau) = G\left[u, t(\tau)\right] = \left(\frac{d\gamma}{du}\right)^{-\frac{1}{2}} w_{1,2}(-\gamma) \tag{4-119}$$

where γ is defined by

$$\begin{aligned}
\gamma(u, \tau) = \eta\left[u, t(\tau)\right] = (u - \tau) &+ \beta\left[\frac{1}{5} h_2 u^2 + \frac{4}{15} h_2 \tau u + \frac{8}{15} h_2 \tau^2 - D_1\right] \\
&+ \beta^2\Big[\left(\frac{1}{7} h_3 - \frac{8}{175} h_2^{\,2}\right) u^3 + \frac{6}{35} h_3 - \frac{68}{525} h_2^{\,2}\big) \tau u^2 + \big(\frac{8}{35} h_3 \\
&\quad - \frac{496}{1575} h_2^{\,2}\big) \tau^2 u + \frac{4}{15} h_2 D_1 u + \big(\frac{16}{35} h_3 \\
&\quad - \frac{1328}{1575} h_2^{\,2}\big) \tau^3 + \frac{16}{15} h_2 D_1 \tau - \big(\frac{3}{7} h_3 - \frac{9}{35} h_2^{\,2} + D_2\big)\Big] \\
&+ \beta^3\Big[\big(\frac{1}{9} h_4 - \frac{22}{315} h_2 h_3 + \frac{148}{7875} h_2^{\,3}\big) u^4 + \big(\frac{8}{63} h_4 \\
&\quad - \frac{3212}{17325} h_2 h_3 + \frac{608}{7875} h_2^{\,3}\big) \tau u^3 + \big(\frac{16}{105} h_4 \\
&\quad - \frac{2288}{5775} h_2 h_3 + \frac{1808}{7875} h_2^{\,3}\big) \tau^2 u^2 + \big(\frac{6}{35} h_3 \\
&\quad - \frac{68}{525} h_2^{\,2}\big) D_1 u^2 + \big(\frac{64}{315} h_4 - \frac{13376}{17325} h_2 h_3 \\
&\quad + \frac{43712}{70875} h_2^{\,3}\big) \tau^3 u - \big(\frac{4}{9} h_4 - \frac{28}{45} h_2 h_3 + \frac{56}{225} h_2^{\,3}\big) u \\
&\quad + \big(\frac{16}{35} h_3 \tau D_1 + \frac{4}{15} h_2 D_2 - \frac{992}{1575} h_2^{\,2} \tau D_1\big) u \\
&\quad + \big(\frac{128}{315} h_4 - \frac{33088}{17325} h_2 h_3 + \frac{128512}{70875} h_2^{\,3}\big) \tau^4 \\
&\quad + \big(\frac{48}{35} h_3 - \frac{1328}{525} h_2^{\,2}\big) \tau^2 D_1 - \big(\frac{80}{63} h_4 \\
&\quad - \frac{1628}{693} h_2 h_3 + \frac{1768}{1575} h_2^{\,3} - \frac{16}{15} h_2 D_2\big) \tau \\
&\quad + \big(\frac{8}{15} h_2 D_1^{\,2} - D_3\big)\Big] + O(\beta^4)
\end{aligned} \tag{4-120}$$

Let us turn now to the construction of the derivative of $F(y,\lambda)$ or $G(u,t)$. We begin by observing that

$$\frac{\partial F(y,\lambda)}{\partial y} = k^{2/3} f_1^{1/3} \frac{\partial G(u,t)}{\partial u} = \frac{1}{\beta}\frac{\partial G(u,t)}{\partial u} = \frac{1}{\beta}\frac{\partial H(u,\tau)}{\partial u} \tag{4-121}$$

From Eq. (4-113), we can write

$$\frac{\partial G}{\partial u} = \sqrt{\frac{d\eta}{du}}\left\{ w'_{1,2}(-\eta) + \frac{1}{2}\left[\frac{d\eta}{du}\right]^{-2}\frac{d^2\eta}{du^2} w_{1,2}(-\eta)\right\} \tag{4-122a}$$

or

$$\frac{\partial G}{\partial u} = \frac{\sqrt{\pi}}{\sqrt[6]{12}} \exp\left(\pm i\frac{2\pi}{3}\right)\sqrt{\frac{d\eta}{du}}\left\{ h'_{1,2}(\eta) - \frac{1}{2}\frac{\eta''}{(\eta')^2} h_{1,2}(\eta)\right\} \tag{4-122b}$$

Let us now represent the expression in curly braces in Eq. (4-122b) in the form

$$h'(\eta) - \frac{1}{2}\frac{\eta''}{[\eta']^2} h(\eta) = \left[1 + \alpha_1\beta + \alpha_2\beta^2 + \alpha_3\beta^3 + \cdots\right] h'(\eta + \gamma_1\beta + \gamma_2\beta^2 + \gamma_3\beta^3 + \cdots) \tag{4-123}$$

Let us first consider a more general result. Let us solve for the A_n and B_n which are defined in Eq. (4-124)

$$h'(\eta) - \Delta h(\eta) = [1 + \Delta A_1 + \Delta^2 A_2 + \Delta^3 A_3 + \cdots] h'[\eta + \Delta B_1 + \Delta^2 B_2 + \Delta^3 B_3 + \cdots] \tag{4-124}$$

In order to obtain this result, we observe that

$$\begin{aligned}
h^{(2)}(\eta) &= -\eta\, h(\eta) \\
h^{(3)}(\eta) &= -\eta\, h'(\eta) - h(\eta) \\
h^{(4)}(\eta) &= \eta^2 h(\eta) - 2h'(\eta) \\
h^{(5)}(\eta) &= 4\eta\, h(\eta) + \eta^2 h'(\eta) \\
h^{(6)}(\eta) &= (4 - \eta^3)\, h(\eta) + 6\eta\, h'(\eta) \\
h^{(7)}(\eta) &= (10 - \eta^3)\, h'(\eta) - 9\eta^2 h(\eta) \\
h^{(n)}(\eta) &= \frac{d}{d\eta^n} h(\eta)
\end{aligned} \tag{4-125}$$

and use the Taylor series expansion

$$
\begin{aligned}
&h'(\eta + \Delta B_1 + \Delta^2 B_2 + \Delta^3 B_3 + \Delta^4 B_4 + \cdot\cdot\cdot) \\
&= h'(\eta) - \left[\Delta B_1 + \Delta^2 B_2 + \Delta^3 B_3 + \Delta^4 B_4 + \cdot\cdot\cdot\right] \eta\, h(\eta) \\
&\quad - \frac{1}{2}\left[\Delta^2 B_1^{\,2} + 2\Delta^3 B_1 B_2 + (2B_1B_3 + B_2^{\,2})\,\Delta^4 + \cdot\cdot\cdot\right]\left[\eta h'(\eta) + h(\eta)\right] \\
&\quad + \frac{1}{6}\left[\Delta^3 B_1^{\,3} + 3\Delta^4 B_1^{\,2} B_2 + \cdot\cdot\cdot\right]\left[\eta^2\, h(\eta) - 2h'(\eta)\right] \\
&\quad + \frac{1}{24}\left[\Delta^4 B_1^{\,4} + \cdot\cdot\cdot\right]\left[4\eta\, h(\eta) + \eta^2\, h'(\eta)\right] + \cdot\cdot\cdot
\end{aligned}
\tag{4-126}
$$

in Eq. (4-124). By equating the coefficients of Δ on each side of Eq. (4-124) we find that

$$
\begin{aligned}
A_1 &= 0 & B_1 &= \frac{1}{\eta} \\
A_2 &= \frac{1}{2\eta} & B_2 &= -\frac{1}{2\eta^3} \\
A_3 &= -\frac{1}{6\eta^3} & B_3 &= \frac{1}{2}\frac{1}{\eta^5} - \frac{1}{3}\frac{1}{\eta^2} \\
A_4 &= \frac{9}{8}\frac{1}{\eta^5} - \frac{1}{8}\frac{1}{\eta^2} & B_4 &= -\frac{5}{8}\frac{1}{\eta^7} + \frac{5}{12}\frac{1}{\eta^4}
\end{aligned}
\tag{4-127}
$$

In order to use Eq. (4-123), we define

$$
\begin{aligned}
2\Delta = \left(\frac{d\eta}{du}\right)^{-2} \frac{d^2\eta}{du^2} = \Bigg\{ & \frac{2}{5} h_2 \beta + \beta^2 \left[+ \frac{12}{35} h_3 t - \frac{248}{525} h_2^{\,2} t\right] \\
& + \beta^3 \Bigg[\left(\frac{4}{3} h_4 - \frac{88}{105} h_2 h_3 + \frac{1960}{2625} h_2^{\,3}\right) u^2 + \left(\frac{16}{21} h_4\right. \\
& \quad \left. - \frac{6424}{5775} h_2 h_3 + \frac{3360}{2625} h_2^{\,3}\right) t\, u + \left(\frac{32}{105} h_4\right. \\
& \quad \left. - \frac{4576}{5775} h_2 h_3 + \frac{7360}{7875} h_2^{\,3}\right) t^2 \Bigg] \\
& + \beta^4 \Bigg[\left(\frac{20}{11} h_5 - \frac{1264}{495} h_2 h_4 + \frac{217696}{121275} h_2^{\,2} h_3 - \frac{300}{539} h_3^{\,2}\right. \\
& \quad \left. - \frac{2657920}{3031875} h_2^{\,4}\right) u^3 + \left(\frac{40}{33} h_5 - \frac{11648}{3465} h_2 h_4 - \frac{6828}{8085} h_3^{\,2}\right. \\
& \quad \left. + \frac{2366032}{606375} h_2^{\,2} h_3 - \frac{7118240}{3031875} h_2^{\,4}\right) t\, u^2 + \left(\frac{160}{231} h_5\right. \\
& \quad - \frac{137856}{51975} h_2 h_4 - \frac{33312}{40425} h_3^{\,2} + \frac{2905696}{606375} h_2^{\,2} h_3 \\
& \quad \left. - \frac{9343744}{3031875} h_2^{\,4}\right) t^2 u + \left(\frac{64}{231} h_5 - \frac{23168}{17325} h_2 h_4\right.
\end{aligned}
$$

(eq. concluded on following page)

$$
- \frac{6976}{13475} h_3^{\,2} + \frac{2048192}{606375} h_2^{\,2} h_3 - \frac{61579520}{27286875} h_2^{\,4}) t^3
+ (- \frac{12}{11} h_5 + \frac{952}{495} h_2 h_4 + \frac{2088}{2695} h_3^{\,2} - \frac{291112}{121275} h_2^{\,2} h_3
+ \frac{451760}{606375} h_2^{\,4}) \Big] + O(\beta^5) \Big\} \tag{4-128}
$$

From the definition of $\eta(u,t)$ in Eq. (4-118), we can obtain a representation for the reciprocals of η which occur in Eq. (4-127). We first define

$$
z = u - t \tag{4-129}
$$

and then find that

$$
\begin{aligned}
\frac{1}{\eta} = \frac{1}{z}\Big\{ 1 - \frac{\beta}{z}\Big[\frac{1}{5} h_2 u^2 + \frac{4}{15} h_2 t u + \frac{8}{15} h_2 t^2\Big] \\
+ \beta^2 \Big[- \frac{1}{z} \Big[(\frac{1}{7} h_3 - \frac{8}{175} h_2^{\,2}) u^3 + (\frac{6}{35} h_3 - \frac{68}{525} h_2^{\,2}) t u^2 + (\frac{8}{35} h_3 \\
- \frac{496}{1575} h_2^{\,2}) t^2 u + (\frac{16}{35} h_3 - \frac{1328}{1575} h_2^{\,2}) t^3 - (\frac{3}{7} h_3 - \frac{9}{35} h_2^{\,2}) \Big] \\
+ \frac{1}{z^2} \Big[\frac{1}{25} h_2^{\,2} u^4 + \frac{8}{75} h_2^{\,2} t u^3 + \frac{64}{225} h_2^{\,2} t^2 u^2 + \frac{64}{225} h_2^{\,2} t^3 u + \frac{64}{225} h_2^{\,2} t^4 \Big]\Big] \\
+ \beta^3 \Big[- \frac{1}{z} \Big[(\frac{1}{9} h_4 - \frac{22}{315} h_2 h_3 + \frac{148}{7875} h_2^{\,3}) u^4 + (\frac{8}{63} h_4 - \frac{3212}{17325} h_2 h_3 \\
+ \frac{608}{7875} h_2^{\,3}) t u^3 + (\frac{16}{105} h_4 - \frac{2288}{5775} h_2 h_3 + \frac{1808}{7875} h_2^{\,3}) t^2 u^2 \\
+ (\frac{64}{315} h_4 - \frac{13376}{17325} h_2 h_3 + \frac{43712}{70875} h_2^{\,3}) t^3 u + (\frac{128}{315} h_4 - \frac{33088}{17325} h_2 h_3 \\
+ \frac{128512}{70875} h_2^{\,3}) t^4 - (\frac{4}{9} h_4 - \frac{28}{45} h_2 h_3 + \frac{56}{225} h_2^{\,3}) u - (\frac{80}{63} h_4 - \frac{1628}{693} h_2 h_3 \\
+ \frac{1768}{1575} h_2^{\,3}) t \Big] + \frac{1}{z^2} \Big[(\frac{2}{35} h_2 h_3 - \frac{16}{875} h_2^{\,3}) u^5 + (\frac{8}{105} h_2 h_3 \\
- \frac{64}{2625} h_2^{\,3}) t u^4 + (\frac{1136}{2625} h_2 h_3 - \frac{1920}{7875} h_2^{\,3}) t^2 u^3 + (\frac{1472}{2625} h_2 h_3 \\
- \frac{15200}{23625} h_2^{\,3}) t^3 u^2 - (\frac{6}{35} h_2 h_3 - \frac{18}{175} h_2^{\,3}) u^2 + (\frac{256}{525} h_2 h_3 - \frac{18560}{23625} h_2^{\,3}) t^4 u \\
- (\frac{24}{105} h_2 h_3 - \frac{72}{525} h_2^{\,3}) t u + (\frac{256}{525} h_2 h_3 - \frac{21248}{23625} h_2^{\,3}) t^5 - (\frac{48}{105} h_2 h_3 \\
- \frac{144}{525} h_2^{\,3}) t^2 \Big] - \frac{1}{z^3} \Big[\frac{1}{125} h_2^{\,3} u^6 + \frac{4}{125} h_2^{\,3} t u^5 + \frac{120}{1125} h_2^{\,3} t^2 u^4 \\
+ \frac{640}{3375} h_2^{\,3} t^3 u^3 + \frac{960}{3375} h_2^{\,3} t^4 u^2 + \frac{768}{3375} h_2^{\,3} t^5 u + \frac{512}{3375} h_2^{\,3} t^6 \Big]\Big] + O(\beta^4) \Big\}
\end{aligned}
\tag{4-130}
$$

We can now use Eq. (4-129) and Eq. (4-130) to compute the argumant

$$\zeta(u,t) = \eta + \Delta B_1 + \Delta^2 B_2 + \Delta^3 B_3 + \cdots$$

$$= \eta + \gamma_1\beta + \gamma_2\beta^2 + \gamma_3\beta^3 + \cdots \qquad (4\text{-}131)$$

in Eq. (4-123). The execution of these tedious algebraic manipulations has been carried out only for the special case u = 0.

$$\begin{aligned}\zeta(0,\ t) = & - t + \beta\left[\frac{8}{15}h_2 t^2 - \frac{1}{5t}h_2\right] \\ & + \beta^2\left[\left(\frac{16}{35}t^3 - \frac{3}{5}\right)h_3 + \left(-\frac{1328}{1575}t^3 + \frac{406}{1050} + \frac{1}{50}\frac{1}{t^3}\right)h_2^2\right] \\ & + \beta^3\left[\left(\frac{128}{315}t^4 - \frac{448}{315}t\right)h_4 + \left(-\frac{33088}{17325}t^4 + \frac{47564}{17325}t + \frac{3}{25}\frac{1}{t^2}\right)h_2 h_3\right. \\ & \left. + \left(\frac{128512}{70875}t^4 - \frac{10648}{7875}t - \frac{182}{2625}\frac{1}{t^2} - \frac{1}{250}\frac{1}{t^5}\right)h_2^3\right]\end{aligned} \qquad (4\text{-}132)$$

Since the use of Eq. (4-123) involves a considerable amount of algebraic manipulation of such complex expressions as Eq. (4-130), it may be desirable in future work to seek a representation for the derivative of G(u,t) which is of the form

$$\frac{dG(u,\ t)}{du} = A(u,\ t)\, h'_{1,2}(\zeta) \qquad (4\text{-}133)$$

where ζ is defined by an expression similar to Eq. (4-114) and the amplitude function A(u,t) involves the derivative of ζ with respect to u. Let us indicate how this can be done by considering the derivative of $F(y,\lambda)$ with respect to y. Let us define $K(y,\lambda)$ by means of

$$\frac{dF(y,\ \lambda)}{dy} = \sqrt{k^2 f(y) - \lambda}\quad K(y,\ \lambda) \qquad (4\text{-}134)$$

Since
$$\left\{\frac{d^2}{dy^2} + \frac{k^2 f'(y)}{\lambda - k^2 f(y)}\ \frac{d}{dy} + \left[-\lambda + k^2 f(y)\right]\right\}\frac{dF(y,\ \lambda)}{dy} = 0$$

we can show that

$$\frac{d^2K}{dy^2} + \left\{\left[-\lambda + k^2 f(y)\right] - \frac{3}{4}\left[\frac{k^2 f'(y)}{\lambda - k^2 f(y)}\right]^2 - \frac{1}{2}\ \frac{k^2 f''(y)}{[\lambda - k^2 f(y)]}\right\}K = 0 \qquad (4\text{-}135)$$

For an "approximately identical" differential equation, we take

$$\frac{d^2y}{dx^2} + \left[x - \frac{3}{4}\frac{1}{x^2}\right] y = 0 \qquad (4\text{-}136)$$

which has the solution

$$y(x) = \frac{1}{\sqrt{x}}\, h'(x)$$

where h(x) is a solution of Airy's equation

$$\frac{d^2h}{dx^2} + x\,h = 0$$

We can then define $\xi(y,\lambda)$ by means of

$$K(y,\lambda) = [\partial\xi(y,\lambda)/\partial y]^{-1/2}\, y[\xi(y,\lambda)] \qquad (4\text{-}137)$$

and show that

$$\left(\xi - \frac{3}{4}\frac{1}{\xi^2}\right)(\dot{\xi})^2 - \frac{3}{4}\left(\frac{\ddot{\xi}}{\dot{\xi}}\right)^2 + \frac{1}{2}\frac{\dddot{\xi}}{\dot{\xi}} =$$

$$\left\{\left[-\lambda + k^2 f(y)\right] - \frac{3}{4}\left[\frac{k^2 f'(y)}{k^2 f(y) - \lambda}\right]^2 + \frac{1}{2}\frac{k^2 f''(y)}{\left[k^2 f(y) - \lambda\right]}\right\} \qquad (4\text{-}138)$$

where the dots denote differentiation with respect to y. We can then represent the derivative of $F(y,\lambda)$ in the form

$$\frac{dF(y,\lambda)}{dy} = \sqrt{\frac{k^2 f(y) - \lambda}{\xi}}\left(\frac{1}{k^{2/3} f_1^{1/3}}\frac{d\xi}{dy}\right)^{-1/2} h'_{1,2}(\xi) \qquad (4\text{-}139)$$

If we define t as in Eq. (4-74) and u as in Eq. (4-75), and use the property $F(y,\lambda) = G(u,t)$, we can show that

$$\frac{dG(u,t)}{du} = \sqrt{\frac{u - t + \beta h_2 u^2 + \beta^2 h_3 u^3 + \cdots}{\zeta}}\left(\frac{d\zeta}{du}\right)^{-1/2} h'_{1,2}(\zeta) \qquad (4\text{-}140)$$

where

$$\zeta = \zeta(u,t) = \xi(y,\lambda)$$

From Eq. (4-138), we find that ζ is a solution of non-linear differential equation

$$\left(\zeta - \frac{3}{4}\ \frac{1}{\zeta^2}\right)(\dot{\zeta})^2 - \frac{3}{4}\left(\frac{\ddot{\zeta}}{\dot{\zeta}}\right)^2 + \frac{1}{2}\ \frac{\dddot{\zeta}}{\dot{\zeta}} = \Bigg\{ \left[- t+u + \beta h_2 u^2 + \beta^2 h_3 u^3 + \cdots\right]$$

$$- \frac{3}{4}\left[\frac{1 + 2\beta h_2 u + 3\beta^2 h_3 u^2 + \cdots}{u - t + \beta h_2 u^2 + \beta^2 h_3 u^3 + \cdots}\right]^2$$

$$+ \frac{1}{2}\left[\frac{2\beta h_2 + 6\beta^2 h_3 u + 12\beta^3 h_4 u^2 + \cdots}{u - t + \beta h_2 u^2 + \beta^2 h_3 u^3 + \cdots}\right]\Bigg\} \qquad (4\text{-}141)$$

where the dots denote differentiation with respect to u.

We can solve for $\zeta(u,t)$ in much the same manner as we solve for $\eta(u,t)$ by using Eq. (4-114). However, we cannot express $\zeta(u,t)$ in form

$$\zeta(u,\ t) = u - t + \sum_{n=1}^{\infty} p_n\ (u,\ t)\ \beta^n$$

where $p_n(u,t)$ is the sum of homogeneous polynomials in u and t. Our studies of the behavior of of the coefficients of β^n in the expansion of $\zeta(u,t)$ led us to see that if we defined

$$z = u - t \qquad (4\text{-}142)$$

that we could express $\zeta(u,t)$ in the form

$$\zeta(u,\ t) = \zeta\ [z + t, t] = z + \sum_{n=1}^{\infty} q_n\ (z,\ t)\ \beta^n \qquad (4\text{-}143)$$

where $q_n(z,t)$ is the sum of homogeneous polynomials in z and t which are of the form

$$q_1(z,\ t) = a_1\ z^2 + b_1\ zt + c_1\ t^2 + \alpha_1\ z^{-1} + \beta_1\ t^{-1} \qquad (4\text{-}144a)$$

$$q_2(z,\ t) = a_2\ z^3 + b_2\ z^2\ t + c_2\ zt^2 + d_2\ t^3 + \alpha_2 + A_2\ z^{-3} + B_2\ z^{-2}\ t^{-1}$$

$$+ C_2\ z^{-1}\ t^{-2} + D_2\ t^{-3} \qquad (4\text{-}144b)$$

The coefficient $q_n(z,t)$ begins with a homogeneous polynomial of degree (n + 1), followed by homogeneous polynomials of degree (n - 2), (n - 2), (n - 5), (n - 8), ..., (-2n + 1).

If we reconsider Eq. (4-123), we find that $\zeta(u,t)$ is defined in terms of $\eta(u,t)$ by means of Eq. (4-131). Therefore, Eq. (4-132) is an example of the expansion defined by Eq. (4-143).

Let us conclude this discussion by observing that the expansion for γ in Eq. (4-120) can be used in Eq. (4-119) to obtain an expansion of the form of Eq. (4-81). Thus, if we define Ω by means of the relation

$$\gamma = u - \tau - \Omega \tag{4-145}$$

we find from Eq. (4-120) that

$$\Omega = \Omega(u,\ t) = \sum_{n=1}^{\infty} r_n(u,\ \tau)\, \beta^n \tag{4-146}$$

where $r_n(u,\tau)$ for $n < 3$ can be found in Eq. (4-120). Then we can write

$$\begin{aligned} w(-\gamma) = w\,(\tau - u) + \Omega\, w'(\tau-u) + \frac{1}{2}\,\Omega^2\,(\tau - u)\, w(\tau - u) \\ + \frac{1}{6}\,\Omega^3\,[w(\tau - u) + (\tau - u)\, w'(\tau - u)] + \cdots \end{aligned} \tag{4-147}$$

We then find that

$$\begin{aligned} w(-\gamma) = {} & w(\tau - u) + \beta r_1(u,\ \tau)\, w'(\tau - u) \\ & + \beta^2\left[\frac{1}{2}\,(\tau - u)\, r_1{}^2(u,\ \tau)\, w(\tau - u) + r_2(u,\ \tau)\, w'(\tau - u)\right] \\ & + \beta^3\left\{\left[(\tau - u)\, r_1(u,\ \tau)\, r_2(u,\ \tau) + \frac{1}{6}\, r_1{}^3(u,\ \tau)\right] w(\tau - u)\right. \\ & \left. + \left[r_3(u,\ \tau) + \frac{1}{6}\,(\tau - u)\, r_1{}^3(u,\ \tau)\right] w'(\tau - u)\right\} + O\,(\beta^4) \end{aligned}$$

$$\begin{aligned} \left(\frac{d\gamma}{du}\right)^{-\frac{1}{2}} & = \left[1 + \sum_{n=1}^{\infty} [dr_n(u,\tau)/du]\ \beta^n\right]^{-1/2} \\ & = 1 - \frac{\beta}{2}\,\frac{d\, r_1(u,\ \tau)}{du} - \frac{\beta^2}{2}\left\{\frac{d\, r_2(u,\ \tau)}{du} - \frac{3}{4}\left[\frac{d\, r_1(u,\ \tau)}{du}\right]^2\right\} \\ & \quad - \frac{\beta^3}{2}\left\{\frac{d\, r_3(u,\ \tau)}{du} - \frac{3}{2}\,\frac{d\, r_1(u,\ \tau)}{du}\,\frac{d\, r_2(u,\ \tau)}{du} + \frac{5}{8}\left[\frac{d\, r_1(u,\ \tau)}{du}\right]^3\right\} + O\,(\beta^4) \end{aligned}$$

We can use these results to show that

$$
\begin{aligned}
H(u, \tau) = G\left[u, t(\tau)\right] &= \left(\frac{d\gamma}{du}\right)^{-1/2} w(-\gamma) = w(\tau - u)\Bigg\{1 - \beta\left[\frac{1}{5} h_2 u + \frac{2}{15} h_2 \tau\right] \\
&+ \beta^2 \Big[-\frac{1}{50} h_2^{\,2} u^5 - \frac{1}{30} h_2^{\,2} \tau u^4 - \frac{4}{45} h_2^{\,2} \tau^2 u^3 + \frac{1}{5} h_2 D_1 u^3 \\
&+ \frac{1}{15} h_2 D_1 \tau u^2 + (-\frac{3}{14} h_3 + \frac{9}{70} h_2^{\,2}) u^2 + \frac{4}{15} h_2 D_1 \tau^2 u \\
&+ (-\frac{6}{35} h_3 + \frac{22}{105} h_2^{\,2}) \tau u - \frac{1}{2} D_1^{\,2} u + \frac{32}{225} h_2^{\,2} \tau^5 - \frac{8}{15} h_2 D_1 \tau^3 \\
&+ (-\frac{4}{35} h_3 + \frac{58}{315} h_2^{\,2}) \tau^2 + \frac{1}{2} D_1^{\,2} \tau - \frac{2}{15} h_2 D_1\Big] \\
&+ \beta^3 \Big[(-\frac{1}{35} h_2 h_3 + \frac{31}{2625} h_2^{\,3}) u^6 + (-\frac{23}{525} h_2 h_3 + \frac{173}{5250} h_2^{\,3}) \tau u^5 \\
&+ (-\frac{2}{21} h_2 h_3 + \frac{139}{1575} h_2^{\,3}) \tau^2 u^4 + (\frac{1}{7} h_3 - \frac{5}{42} h_2^{\,2}) D_1 u^4 \\
&+ (-\frac{8}{105} h_2 h_3 + \frac{2552}{14175} h_2^{\,3}) \tau^3 u^3 \\
&+ (\frac{1}{35} h_3 - \frac{19}{63} h_2^{\,2}) D_1 \tau u^3 + (-\frac{2}{9} h_4 + \frac{223}{630} h_2 h_3 \\
&- \frac{473}{3150} h_2^{\,3} + \frac{1}{5} h_2 D_2) u^3 + \frac{16}{675} h_2^{\,3} \tau^4 u^2 \\
&+ (\frac{2}{35} h_3 - \frac{78}{315} h_2^{\,2}) D_1 \tau^2 u^2 + (-\frac{4}{21} h_4 + \frac{572}{1155} h_2 h_3 \\
&+ \frac{1}{15} h_2 D_2 - \frac{146}{525} h_2^{\,3}) \tau u^2 + \frac{4}{15} h_2 D_1^{\,2} u^2 \\
&- \frac{32}{3375} h_2^{\,3} \tau^5 u + (\frac{8}{35} h_3 - \frac{16}{35} h_2^{\,2}) D_1 \tau^3 u \\
&+ (-\frac{16}{105} h_4 + \frac{748}{1155} h_2 h_3 - \frac{742}{1575} h_2^{\,3} + \frac{4}{15} h_2 D_2) \tau^2 u \\
&+ \frac{19}{30} h_2 D_1^{\,2} \tau u + (-\frac{3}{5} h_3 + \frac{7}{15} h_2^{\,2} - D_2) D_1 u \\
&+ (\frac{128}{525} h_2 h_3 - \frac{35008}{70875} h_2^{\,3}) \tau^6 + (-\frac{16}{35} h_3 + \frac{512}{315} h_2^{\,2}) D_1 \tau^4 \\
&+ (-\frac{32}{315} h_4 + \frac{704}{3465} h_2 h_3 + \frac{3404}{14175} h_2^{\,3} - \frac{8}{15} h_2 D_2) \tau^3 \\
&- \frac{7}{5} h_2 D_1^{\,2} \tau^2 + (\frac{1}{5} h_3 + \frac{1}{9} h_2^{\,2} + D_2) D_1 \tau \\
&+ (\frac{2}{9} h_4 - \frac{14}{45} h_2 h_3 + \frac{28}{225} h_2^{\,3} - \frac{2}{15} h_2 D_2 + \frac{1}{6} D_1^{\,3})\Big] + O(\beta^4)\Bigg\}
\end{aligned}
$$

(Eq. concluded on following page)

$$
\begin{aligned}
&+ w'(\tau - u)\Bigg\{ \beta \left[-\frac{1}{5} h_2 u^2 - \frac{4}{15} h_2 \tau u - \frac{8}{15} h_2 \tau^2 + D_1 \right] \\
&\quad + \beta^2 \Big[-(\frac{1}{7} h_3 - \frac{3}{35} h_2^{\,2}) u^3 - (\frac{6}{35} h_3 - \frac{22}{105} h_2^{\,2}) \tau u^2 - (\frac{8}{35} h_3 - \frac{144}{315} h_2^{\,2}) \tau^2 u \\
&\qquad - \frac{7}{15} h_2 D_1 u - (\frac{16}{35} h_3 - \frac{288}{315} h_2^{\,2}) \tau^3 - \frac{6}{5} h_2 D_1 \tau + (\frac{3}{7} h_3 - \frac{9}{35} h_2^{\,2} + D_2) \Big] \\
&\quad + \beta^3 \Big[\frac{1}{750} h_2^{\,3} u^7 + \frac{1}{250} h_2^{\,3} \tau u^6 + \frac{14}{1125} h_2^{\,3} \tau^2 u^5 - \frac{1}{50} h_2^{\,2} D_1 u^5 \\
&\qquad + \frac{28}{2025} h_2^{\,3} \tau^3 u^4 - \frac{1}{30} h_2^{\,2} D_1 \tau u^4 + (-\frac{1}{9} h_4 + \frac{89}{630} h_2 h_3 \\
&\qquad - \frac{169}{3150} h_2^{\,3}) u^4 + \frac{32}{2025} h_2^{\,3} \tau^4 u^3 - \frac{4}{45} h_2^{\,2} D_1 \tau^2 u^3 + (-\frac{8}{63} h_4 \\
&\qquad + \frac{104}{315} h_2 h_3 - \frac{292}{1575} h_2^{\,3}) \tau u^3 + \frac{1}{10} h_2 D_1^{\,2} u^3 - \frac{32}{3375} h_2^{\,3} \tau^5 u^2 \\
&\qquad + (-\frac{16}{105} h_4 + \frac{68}{105} h_2 h_3 - \frac{106}{225} h_2^{\,3}) \tau^2 u^2 + \frac{1}{30} h_2 D_1^{\,2} \tau u^2 \\
&\qquad + (-\frac{27}{70} h_3 + \frac{71}{210} h_2^{\,2}) D_1 u^2 - \frac{128}{10125} h_2^{\,3} \tau^6 u + (-\frac{64}{315} h_4 \\
&\qquad + \frac{704}{693} h_2 h_3 - \frac{14008}{14175} h_2^{\,3}) \tau^3 u + \frac{2}{15} h_2 D_1^{\,2} \tau^2 u + (-\frac{22}{35} h_3 \\
&\qquad + \frac{118}{105} h_2^{\,2}) D_1 \tau u + (\frac{4}{9} h_4 - \frac{223}{315} h_2 h_3 + \frac{473}{1575} h_2^{\,3} - \frac{7}{15} h_2 D_2 - \frac{1}{6} D_1^{\,3}) u \\
&\qquad - \frac{256}{10125} h_2^{\,3} \tau^7 + \frac{32}{225} h_2^{\,2} D_1 \tau^5 + (-\frac{128}{315} h_4 + \frac{1408}{693} h_2 h_3 \\
&\qquad - \frac{28688}{14175} h_2^{\,3}) \tau^4 - \frac{4}{15} h_2 D_1^{\,2} \tau^3 + (-\frac{52}{35} h_3 + \frac{922}{315} h_2^{\,2}) D_1 \tau^2 \\
&\qquad + (\frac{80}{63} h_4 - \frac{8338}{3465} h_2 h_3 + \frac{1822}{1575} h_2^{\,3} - \frac{6}{5} h_2 D_2 + \frac{1}{6} D_1^{\,3}) \tau \\
&\qquad + (-\frac{2}{3} h_2 D_1^{\,2} + D_3) \Big] + O(\beta^4) \Bigg\} \qquad (4\text{-}148)
\end{aligned}
$$

If we differentiate with respect to u, we find that

$$
\begin{aligned}
\frac{\partial H(u,\tau)}{\partial u} = w'(\tau - u) \Bigg\{ & -1 - \beta\left[\frac{1}{5} h_2 + \frac{2}{15} h_2 \tau\right] + \beta^2 \Bigg[\frac{1}{50} h_2{}^2 u^5 + \frac{1}{30} h_2{}^2 \tau u^4 + \frac{4}{45} h_2{}^2 \tau^2 u^3 \\
& - \frac{1}{5} h_2 D_1 u^3 - \frac{1}{15} h_2 D_1 \tau u^2 + \left(-\frac{3}{14} h_3 + \frac{9}{70} h_2{}^2\right) u^2 - \frac{4}{15} h_2 D_1 \tau^2 u \\
& + \left(-\frac{6}{35} h_3 + \frac{22}{105} h_2{}^2\right) \tau u + \frac{1}{2} D_1{}^2 u - \frac{32}{225} h_2{}^2 \tau^5 + \frac{8}{15} h_2 D_1 \tau^3 \\
& + \left(-\frac{4}{35} h_3 + \frac{86}{315} h_2{}^2\right) \tau^2 - \frac{1}{2} D_1{}^2 \tau - \frac{1}{3} h_2 D_1\Bigg] \\
& + \beta^3 \Bigg[\left(\frac{1}{35} h_2 h_3 - \frac{13}{5250} h_2{}^3\right) u^6 + \left(\frac{23}{525} h_2 h_3 - \frac{47}{5250} h_2{}^3\right) \tau u^5 \\
& + \left(\frac{2}{21} h_2 h_3 - \frac{41}{1575} h_2{}^3\right) \tau^2 u^4 \\
& + \left(-\frac{1}{7} h_3 + \frac{2}{105} h_2{}^2\right) D_1 u^4 + \left(\frac{8}{105} h_2 h_3 - \frac{1768}{14175} h_2{}^3\right) \tau^3 u^3 \\
& + \left(-\frac{1}{35} h_3 + \frac{53}{315} h_2{}^2\right) D_1 \tau u^3 + \left(-\frac{2}{9} h_4 + \frac{133}{630} h_2 h_3 - \frac{203}{3150} h_2{}^3 \right. \\
& \left. - \frac{1}{5} h_2 D_2\right) u^3 + \frac{16}{675} h_2{}^3 \tau^4 u^2 + \left(-\frac{2}{35} h_3 - \frac{2}{105} h_2{}^2\right) D_1 \tau^2 u^2 \\
& + \left(-\frac{4}{21} h_4 + \frac{572}{1155} h_2 h_3 - \frac{146}{525} h_2{}^3 - \frac{1}{15} h_2 D_2\right) \tau u^2 + \frac{1}{30} h_2 D_1{}^2 u^2 \\
& - \frac{32}{3375} h_2{}^3 \tau^5 u + \left(-\frac{8}{35} h_3 + \frac{16}{35} h_2{}^2\right) D_1 \tau^3 u \\
& + \left(-\frac{16}{105} h_4 + \frac{748}{1155} h_2 h_3 - \frac{742}{1575} h_2{}^3 - \frac{4}{15} h_2 D_2\right) \tau^2 u - \frac{17}{30} h_2 D_1{}^2 \tau u \\
& + \left(-\frac{6}{35} h_3 + \frac{22}{105} h_2{}^2 + D_2\right) D_1 u + \left(-\frac{128}{525} h_2 h_3 + \frac{34112}{70875} h_2{}^3\right) \tau^6 \\
& + \left(\frac{16}{35} h_3 - \frac{512}{315} h_2{}^2\right) D_1 \tau^4 + \left(-\frac{32}{315} h_4 + \frac{2816}{3465} h_2 h_3 \right. \\
& \left. - \frac{10604}{14175} h_2{}^3 + \frac{8}{15} h_2 D_2\right) \tau^3 + \frac{23}{15} h_2 D_1{}^2 \tau^2 + \left(-\frac{29}{35} h_3 \right. \\
& \left. + \frac{319}{315} h_2{}^2 - D_2\right) D_1 \tau + \left(\frac{2}{9} h_4 - \frac{25}{63} h_2 h_3 + \frac{277}{1575} h_2{}^3 \right. \\
& \left. - \frac{1}{3} h_2 D_2 - \frac{1}{3} D_1{}^3\right)\Bigg] + O(\beta^4) \Bigg\}
\end{aligned}
$$

(Eq. concluded on following page)

$$+ w(\tau - u)\Bigg\{\beta\Big[-\frac{1}{5}h_2u^3 - \frac{1}{15}h_2\tau u^2 - \frac{4}{15}h_2\tau^2u + D_1u + \frac{8}{15}h_2\tau^3 - \frac{1}{5}h_2 - D_1\tau\Big]$$

$$+ \beta^2\Big[u^4(-\frac{1}{7}h_3 - \frac{1}{70}h_2^2) + \tau u^3(-\frac{1}{35}h_3 - \frac{1}{105}h_2^2)$$

$$+ \tau^2u^2(-\frac{2}{35}h_3 - \frac{2}{105}h_2^2) + u^2(\frac{2}{15}h_2D_1) + \tau^3u(-\frac{8}{35}h_3 + \frac{144}{315}h_2^2)$$

$$+ \tau u(-\frac{3}{5}h_2D_1) + D_2u + \tau^4(\frac{16}{35}h_3 - \frac{288}{315}h_2^2) + \tau^2(\frac{22}{15}h_2D_1)$$

$$+ \tau(-\frac{3}{5}h_3 + \frac{7}{15}h_2^2 - D_2) - \frac{1}{2}D_1^2\Big]$$

$$+ \beta^3\Big[\frac{1}{750}h_2^3u^8 + \frac{1}{375}h_2^3\tau u^7 + \frac{19}{2250}h_2^3\tau^2u^6 - \frac{1}{50}h_2^2D_1u^6 + \frac{14}{10125}h_2^3\tau^3u^5$$

$$- \frac{1}{75}h_2^2D_1\tau u^5 + (-\frac{1}{9}h_4 - \frac{19}{630}h_2h_3 + \frac{271}{15750}h_2^3)u^5$$

$$+ \frac{4}{2025}h_2^3\tau^4u^4 - \frac{1}{18}h_2^2D_1\tau^2u^4 + (-\frac{1}{63}h_4 - \frac{19}{630}h_2h_3 + \frac{52}{1575}h_2^3)\tau u^4$$

$$+ \frac{1}{10}h_2D_1^2u^4 - \frac{256}{10125}h_2^3\tau^5u^3 + \frac{4}{45}h_2^2D_1\tau^3u^3 + (-\frac{8}{315}h_4 - \frac{4}{63}h_2h_3$$

$$+ \frac{106}{1575}h_2^3)\tau^2u^3 - \frac{1}{15}h_2D_1^2\tau u^3 + (\frac{13}{70}h_3 - \frac{29}{210}h_2^2)D_1u^3$$

$$- \frac{32}{10125}h_2^3\tau^6u^2 + (-\frac{16}{315}h_4 + \frac{484}{3465}h_2h_3 + \frac{326}{14175}h_2^3)\tau^3u^2$$

$$+ \frac{1}{10}h_2D_1^2\tau^2u^2 + (-\frac{11}{70}h_3 - \frac{5}{42}h_2^2)D_1\tau u^2 + (-\frac{2}{9}h_4 + \frac{223}{630}h_2h_3 - \frac{1}{6}D_1^3$$

$$- \frac{473}{3150}h_2^3 + \frac{2}{15}h_2D_2)u^2 - \frac{128}{10125}h_2^3\tau^7u + \frac{32}{225}h_2^2D_1\tau^5u$$

$$+ (-\frac{64}{315}h_4 + \frac{704}{693}h_2h_3 - \frac{14008}{14175}h_2^3)\tau^4u - \frac{2}{5}h_2D_1^2\tau^3u$$

$$+ (-\frac{26}{35}h_3 + \frac{412}{315}h_2^2)D_1\tau^2u + (\frac{4}{9}h_4 - \frac{2453}{3465}h_2h_3 + \frac{473}{1575}h_2^3 - \frac{3}{5}h_2D_2$$

$$+ \frac{1}{3}D_1^3)\tau u + (-\frac{2}{15}h_2D_1^2 + D_3)u + \frac{256}{10125}h_2^3\tau^8 - \frac{32}{225}h_2^2D_1\tau^6$$

$$+ (+\frac{128}{315}h_4 - \frac{1408}{693}h_2h_3 + \frac{142768}{70875}h_2^3)\tau^5 + \frac{4}{15}h_2D_1^2\tau^4$$

$$+ (\frac{12}{7}h_3 - \frac{1066}{315}h_2^2)D_1\tau^3 + (-\frac{64}{45}h_4 + \frac{962}{315}h_2h_3 - \frac{2564}{1575}h_2^3$$

$$+ \frac{22}{15}h_2D_2 - \frac{1}{6}D_1^3)\tau^2 + (\frac{13}{10}h_2D_1^2 - D_3)\tau + (-\frac{3}{5}h_3 + \frac{7}{15}h_2^2$$

$$- D_2)D_1\Big] + O(\beta^4)\Bigg\} \qquad (4\text{-}149)$$

A useful check on the coefficients in Eqs. (4-148) and (4-149) can be obtained from the Wronskian relation

$$W(H_1, H_2) = H_1 \frac{dH_2}{du} - H_2 \frac{dH_1}{du} = -2i \qquad (4\text{-}150)$$

This property is readily derived from the Wronskian for the Airy functions

$$W(w_1, w_2) = w_1(z) \frac{dw_2(z)}{dz} - w_2(z) \frac{dw_1(z)}{dz} = 2i \qquad (4\text{-}151)$$

by using the representations

$$H_{1,2}(u, \tau) = \left(\frac{d\eta}{du}\right)^{-1/2} w_{1,2}(-\eta) \qquad (4\text{-}152a)$$

$$\frac{dH_{1,2}(u, \tau)}{du} = -\left(\frac{d\eta}{du}\right)^{-1/2} w'_{1,2}(-\eta) - \frac{1}{2} \frac{\frac{d^2\eta}{du^2}}{\left[\frac{d\eta}{du}\right]^2} w_{1,2}(-\eta) \qquad (4\text{-}152b)$$

Let us express Eq. (4-148) in the form

$$H_{1,2} = w_{1,2}(\tau - u) + \sum_{n=1}^{\infty} \left[A_n(u, \tau)\, w_{1,2}(\tau - u) + B_n(u, \tau)\, w'_{1,2}(\tau - u)\right] \beta^n \qquad (4\text{-}153a)$$

and use Eq. (4-149) in the form

$$\frac{dH_{1,2}(u, \tau)}{du} = -w'_{1,2}(\tau - u) + \sum_{n=1}^{\infty} \Big[C_n(u, \tau)\, w_{1,2}(\tau - u) + D_n(u, \tau)\, w'_{1,2}(\tau - u)\Big] \beta^n \qquad (4\text{-}153b)$$

We can show that the coefficients A_n, B_n, C_n, and D_n have the property

$$\left(\sum_{n=1}^{\infty} A_n(u, \tau)\, \beta^n\right)\left(\sum_{n=1}^{\infty} D_n(u, \tau)\, \beta^n\right) = \left(\sum_{n=1}^{\infty} B_n(u, \tau)\, \beta^n\right)\left(\sum_{n=1}^{\infty} C_n(u, \tau)\, \beta^n\right) \qquad (4\text{-}154)$$

or $A_1 D_1 = B_1 C_1$ and

$$\sum_{j=1}^{n+1} A_j D_{n-j} = \sum_{j=1}^{n+1} B_j C_{n-j}\ , \quad n > 2 \qquad (4\text{-}155)$$

4.8 Application to Bessel Functions

An interest check on the accuracy of the expansion given in Eq. (4-118) can be obtained by making use of certain results for the Hankel functions which were obtained by Schobe (Ref. 14). The result which we need is

$$H_\nu^{(1)}(kx) \sim -\frac{1}{\sqrt{\pi}}\left(\frac{2}{kx}\right)^{1/3}\left\{\sum_{m=0}^{\infty}(-1)^m p_m(\xi)\left(\frac{2}{kx}\right)^{\frac{2m}{3}}\right\} w_1[T(\xi,kx)] \tag{4-156}$$

where the argument of the Airy function is

$$T(\xi,kx) = \sum_{n=0}^{\infty}(-1)^n q_n(\xi)\left(\frac{2}{kx}\right)^{\frac{2n}{3}}$$

where

$$\xi = \left(\frac{2}{kx}\right)^{2/3}(\nu - kx)$$

and

$$p_0(t) = 1 \qquad q_0(t) = t$$

$$p_1(t) = \frac{1}{75}t \qquad q_1(t) = \frac{1}{60}t^2$$

$$p_2(t) = \frac{13}{1260}t^2 \qquad q_2(t) = \frac{2}{1575}t^3 + \frac{1}{140}$$

$$p_3(t) = \frac{109}{56780}t^3 + \frac{1}{900} \qquad q_3(t) = \frac{41}{283500}t^4 + \frac{4}{1575}t$$

$$p_4(t) = \frac{203743}{523908000}t^4 + \frac{6761}{7276500}t \qquad q_4(t) = \frac{6553}{327442500}t^5 + \frac{1409}{1819125}t^2$$

In order to use Schobe's expansion, we must first observe that the Hankel function $H_\nu^{(1)}(kx)$ is a solution of

$$\frac{d^2H_\nu^{(1)}(kx)}{dx^2} + \frac{1}{x}\frac{dH_\nu^{(1)}(kx)}{dx} + \left[k^2 - \frac{\nu^2}{x^2}\right]H_\nu^{(1)}(kx) = 0 \tag{4-157}$$

Therefore, we can show that

$$H_\nu^{(1)}(kx) = F_1[a \log (x/a), (\nu/a)^2] \tag{4-158}$$

where $F_1(y,\lambda)$ is a solution of

$$\frac{d^2F_1(y,\lambda)}{dy^2} - \left[-\lambda + k^2 \exp\left(2\frac{y}{a}\right)\right]F_1(y,\lambda) = 0 \tag{4-159}$$

We start our comparison Schobe's expansion and our Eq. (4-118) by observing that $f(y) = \exp(2y/a)$ and therefore the result that corresponds to Eq. (4-72) is

$$f(y) = f_0 + f_1[y + h_2y^2 + h_3y^3 + \ldots]$$

$$= 1 + \frac{2}{a}\left[y + \frac{1}{a}y^2 + \frac{2}{3a^2}y^3 + \frac{1}{3a^3}y^4 + \ldots\right] \qquad (4\text{-}160)$$

Therefore, for this case we find that

$$f_0 = 1 \quad , \quad f_1 = \frac{2}{a} \qquad (4\text{-}161a)$$

and

$$h_2 = \frac{1}{a} \quad , \quad h_3 = \frac{2}{3a^2} \quad , \quad h_4 = \frac{1}{3a^3} \quad , \quad \text{etc.} \qquad (4\text{-}161b)$$

From Eq. (4-74) we find that t is defined by

$$\lambda = \frac{\nu^2}{a^2} = k^2 + k^{4/3}\left(\frac{2}{a}\right)^{2/3} t \qquad (4\text{-}162a)$$

or

$$t = \frac{\nu^2 - (ka)^2}{4\,(ka/2)^{4/3}} \qquad (4\text{-}162b)$$

From Eq. (4-75) we find that

$$u = k^{\frac{2}{3}} f_1^{\frac{1}{3}} y = k^{\frac{2}{3}} (2/a)^{\frac{1}{3}} a \log(x/a) = 2\,(ka/2)^{\frac{2}{3}} \log(x/a) \qquad (4\text{-}163)$$

From Eq. (4-76) we find that

$$\beta = \frac{1}{k^{2/3} f_1^{1/3}} = \frac{1}{k^{2/3}}\left(\frac{a}{2}\right)^{1/3} = \frac{a}{2}\left(\frac{2}{ka}\right)^{2/3} \qquad (4\text{-}164)$$

Therefore, we can express the Hankel function in the form

$$H_\nu^{(1)}(kx) \sim (\text{constant})\, F_1\left(a \log\frac{x}{a}\,,\ \frac{\nu^2}{a^2}\right) = (\text{constant})\, G_1(u,\, t) \qquad (4\text{-}165)$$

where $G_1(u,t)$ is defined by

$$G_1(u,\, t) = F_1(y,\, \lambda) = F_1\left(a \log\frac{x}{a}\,,\ \frac{\nu^2}{a^2}\right) \qquad (4\text{-}166)$$

where u is defined by Eq. (4-163), t is defined by Eq. (4-162b), and β is defined by Eq. (4-164).

From Eq. (4-113), we find that $G_1(u,t) = \sqrt{du/d\eta}\; w_1(-\eta)$ where η is defined by Eq. (4-118). Therefore, we can express Eq. (4-165) in the form

$$H_\nu^{(1)}(kx) = -\frac{i}{\sqrt{\pi}}\left(\frac{2}{ka}\right)^{1/3}\left(\frac{d\eta}{du}\right)^{-1/2} w_1(-\eta) \tag{4-167}$$

where the choice of the constant of the form

$$(\text{constant}) = -\frac{i}{\sqrt{\pi}}\left(\frac{2}{ka}\right)^{1/3}$$

follows from the result

$$H_\nu^{(1)}(kx) \sim -\frac{i}{\sqrt{\pi}}\left(\frac{2}{ka}\right)^{1/3} w_1(t-u) \tag{4-168}$$

which is valid when $kx \gg 1$ and $\nu \approx kx \approx ka$.

If we compare Eq. (4-167) with Eq. (4-156), we see that Schobe's expansion leads to

$$\eta = -\sum_{n=0}^{\infty} (-1)^n q_n(\xi)\left(\frac{2}{kx}\right)^{\frac{2n}{3}} \tag{4-169}$$

In order to compare Eq. (4-169) with Eq. (4-118), we need to express the right side of Eq. (4-169) in terms of u, t, and β. As a preliminary to this step, we need to express ξ and $(2/kx)$ in terms of u, t, and β. Since

$$x = a\,\exp(y/a) = a\,\exp[\beta u/a]$$

we find that

$$\left(\frac{2}{kx}\right)^{\lambda} = \left(\frac{2}{ka}\right)^{\lambda}\exp\left(-\frac{\lambda\beta u}{a}\right) = \left(\frac{2\beta}{a}\right)^{\frac{3\lambda}{2}}\left\{1-\beta\frac{\lambda u}{a}+\frac{1}{2}\beta^2\frac{\lambda^2 u^2}{a^2}+\cdots\right\} \tag{4-170}$$

The relationship of ξ to u, t, and β is somewhat more complex. From Eq. (4-156) we find that

$$\nu^2 = (kx)^2 + 4\left(\frac{kx}{2}\right)^{4/3}\xi + \left(\frac{kx}{2}\right)^{2/3}\xi^2 \tag{4-171}$$

whereas from Eq. (4-162a), we have

$$\nu^2 = (ka)^2 + 4\left(\frac{ka}{2}\right)^{4/3} t \tag{4-172}$$

If we use Eq. (4-170), we find that

$$(ka)^2 \exp\left(\frac{2\beta u}{a}\right)\left[1+\frac{1}{2}\left(\frac{2}{ka}\right)^{2/3}\xi \exp\left(-\frac{2\beta u}{3a}\right)\right]^2 = (ka)^2\left[1+\left(\frac{2}{ka}\right)^{2/3} t\right]$$

or

$$\exp\left(\frac{\beta u}{a}\right)\left[1+\frac{1}{2}\left(\frac{2}{ka}\right)^{2/3}\xi \exp\left(-\frac{2\beta u}{3a}\right)\right] = \sqrt{1+\left(\frac{2}{ka}\right)^{2/3} t}$$

Therefore

$$\xi = \xi(u,\ t) = \frac{a}{\beta}\ \exp\left(-\frac{\beta u}{3a}\right)\left[\sqrt{1+\frac{2\beta}{a}t} - \exp\left(\frac{\beta u}{a}\right)\right]$$

$$= (t-u) + \sum_{n=1}^{\infty} E_n(u,\ t)\left(\frac{\beta}{a}\right)^n \qquad (4\text{-}173)$$

where the coefficients $E_n(u,t)$ are defined by

$$E_1(u,\ t) = -\left[\frac{1}{2}t^2 + \frac{1}{3}tu + \frac{1}{6}u^2\right]$$

$$E_2(u,\ t) = -\left[\frac{1}{2}t^3 + \frac{1}{6}t^2u + \frac{1}{18}tu^2 - \frac{1}{18}u^3\right]$$

$$E_3(u,\ t) = \left[-\frac{5}{8}t^4 - \frac{1}{6}t^3u - \frac{1}{36}t^2u^2 - \frac{1}{162}tu^3 - \frac{5}{648}u^4\right]$$

$$E_4(u,\ t) = \left[\frac{7}{8}t^5 + \frac{5}{24}t^4u + \frac{1}{36}t^3u^2 + \frac{1}{324}t^2u^3 + \frac{1}{1944}tu^4 - \frac{11}{9720}u^5\right]$$

By the use of Eq. (4-170) and Eq. (4-173) in Eq. (4-168), we have verified the correctness of all the coefficients in Eq. (4-118)

The Correctness of the coefficients in the expansion of $\zeta(0,t)$ given in Eq. (4-132) can be verified in a similar manner if we use an expansion for the derivative which is in the form of Eq. (4-140), namely,

$$H_\nu^{(1)'}(kx) \sim \frac{1}{\sqrt{\pi}}\left(\frac{2}{kx}\right)^{2/3}\left\{1+\sum_{n=1}^{\infty} B_n(\xi)\left(\frac{2}{kx}\right)^{\frac{2n}{3}}\right\} w_1'(-\zeta) \qquad (4\text{-}174a)$$

$$-\zeta = \xi + \sum_{n=1}^{\infty} C_n(\xi)\left(\frac{2}{kx}\right)^{\frac{2n}{3}} \qquad (4\text{-}174b)$$

The coefficient $C_1(\xi)$ was obtained by Scott (Ref. 15). A search

of the literature failed to turn up further terms. Therefore, we have extended Schobe's paper by showing that

$$B_1(t) = \frac{1}{15}\, t \quad , \qquad C_1(t) = \frac{6 - t^3}{60\, t}$$

$$B_2(t) = -\frac{37}{6300}\, t^2 - \frac{1}{200}\, \frac{1}{t} \quad , \qquad C_2(t) = \frac{2}{1575}\, t^3 - \frac{39}{1800} - \frac{1}{200}\, \frac{1}{t^3}$$

$$B_3(t) = \frac{239}{283500}\, t^3 + \frac{943}{252000} + \frac{1}{6000}\, \frac{1}{t^3} \quad ,$$

$$C_3(t) = -\frac{41}{283500}\, t^4 + \frac{1447}{252000}\, t + \frac{29}{12000}\, \frac{1}{t^2} + \frac{1}{2000}\, \frac{1}{t^5}$$

$$B_4(t) = -\frac{53863}{374220000}\, t^4 - \frac{43993}{23760000}\, t - \frac{79}{720000}\, \frac{1}{t^2} - \frac{1}{8000}\, \frac{1}{t^5}$$

$$C_4(t) = \frac{6553}{327442500}\, t^5 - \frac{260737}{99792000}\, t^2 - \frac{1679}{1108800}\, \frac{1}{t} - \frac{7}{20000}\, \frac{1}{t^4} - \frac{5}{80000}\, \frac{1}{t^7}$$

Since we have not carried out the algebraic details for $\zeta(u,t)$ defined in Eq. (4-131) except for the case $u = 0$, we are limited to considering only the expansion for $\zeta(0,t)$ which was given in Eq. (4-132). For the case of the derivative of the Hankel function as defined by Eq. (4-174), we would need the **results**

$$\left(\frac{2}{kx}\right) = \left(\frac{2\beta}{a}\right)^{\frac{3\lambda}{2}}$$

$$\xi = \frac{a}{\beta}\, \sqrt{1 + \frac{2\beta}{a} t} - 1$$

$$= t - \frac{1}{2} t^2 \left(\frac{\beta}{a}\right) + \frac{1}{2} t^3 \left(\frac{\beta}{a}\right)^2 - \frac{5}{8} t^4 \left(\frac{\beta}{a}\right)^3 + \frac{7}{8} t^5 \left(\frac{\beta}{a}\right)^4 + \ldots$$

along with the values of h_n which were defined in Eq. (4-161b).

Additional confidence in the correctness of the coefficients in Eq. (4-148) and Eq. (4-149) has been acquired by using Eq. (4-170) and Eq. (4-173) in Schobe's expansions for $H_\nu^{(1)}(kx)$ and $H_\nu^{(1)\prime}(kx)$ which are of the form

$$
\begin{aligned}
H_\nu^{(1)}(kx) \approx & -\frac{i}{\sqrt{\pi}}\left(\frac{2}{kx}\right)^{1/3} w_1(\xi)\left\{1 - \left(\frac{1}{15}\xi\right)\left(\frac{2}{kx}\right)^{2/3} + \left(\frac{1}{7200}\xi^5 + \frac{13}{1260}\xi^2\right)\left(\frac{2}{kx}\right)^{4/3}\right. \\
& - \left(\frac{283}{9072000}\xi^6 + \frac{463}{226800}\xi^3 + \frac{1}{900}\right)\left(\frac{2}{kx}\right)^{6/3} \\
& + \left(\frac{1}{311040000}\xi^{10} + \frac{599}{95256000}\xi^7 + \frac{26897}{59875200}\xi^4 \right. \\
& \left.\left. + \frac{7939}{8316000}\xi\right)\left(\frac{2}{kx}\right)^{8/3} + \dots\right\} \\
& -\frac{i}{\sqrt{\pi}}\left(\frac{2}{kx}\right)^{1/3} w_1'(\xi)\left\{0 - \left(\frac{1}{60}\xi^2\right)\left(\frac{2}{kx}\right)^{2/3} + \left(\frac{1}{420}\xi^3 + \frac{1}{140}\right)\left(\frac{2}{kx}\right)^{4/3}\right. \\
& - \left(\frac{1}{1296000}\xi^7 + \frac{13}{32400}\xi^4 + \frac{19}{6300}\xi\right)\left(\frac{2}{kx}\right)^{6/3} \\
& + \left(\frac{17}{72576000}\xi^8 + \frac{2521}{33264000}\xi^5 + \frac{359}{346500}\xi^2\right)\left(\frac{2}{kx}\right)^{8/3} \\
& \left. + \dots\right\}
\end{aligned}
\tag{4-175}
$$

and

$$
\begin{aligned}
H_\nu^{(1)'}(kx) \approx & \frac{i}{\sqrt{\pi}}\left(\frac{2}{kx}\right)^{2/3} w_1'(\xi)\left\{1 + \left(\frac{2}{kx}\right)^{2/3}\frac{1}{15}\xi + \left(\frac{2}{kx}\right)^{4/3}\left(\frac{1}{7200}\xi^5 - \frac{19}{2520}\xi^2\right)\right. \\
& - \left(\frac{2}{kx}\right)^{6/3}\left(\frac{61}{4536000}\xi^6 - \frac{283}{226800}\xi^3 - \frac{23}{12600}\right) \\
& + \left(\frac{2}{kx}\right)^{8/3}\left(\frac{\xi^{10}}{311040000} + \frac{443\xi^7}{381024000} - \frac{1483\xi^4}{5987520}\right. \\
& \left.\left. - \frac{9809\xi}{8316000}\right) + \dots\right\} \\
& + \frac{i}{\sqrt{\pi}}\left(\frac{2}{kx}\right)^{2/3} w_1(\xi)\left\{0 - \left(\frac{2}{kx}\right)^{2/3}\left(\frac{\xi^3}{60} - \frac{1}{10}\right) + \left(\frac{2}{kx}\right)^{4/3}\left(\frac{1}{3360}\xi^4 - \frac{1}{60}\xi\right)\right. \\
& - \left(\frac{2}{kx}\right)^{6/3}\left(\frac{1}{1296000}\xi^8 - \frac{181}{4536000}\xi^5 - \frac{103}{25200}\xi^2\right) \\
& + \left(\frac{2}{kx}\right)^{8/3}\left(\frac{\xi^9}{7257600} - \frac{13\xi^6}{887040} - \frac{3821\xi^3}{1782000} - \frac{947}{2772000}\right) \\
& \left. + \dots\right\}
\end{aligned}
\tag{4-176}
$$

In order to emphasize the generality of the new expansions

described by Eq. (4-148) and Eq. (4-149), let us observe that the terms up to and including $(kx/2)^{-2}$ in Schobe's results described by Eq. (4-175) and Eq. (4-176) are obtained from the new results by using the following definitions

$$D_n = 0 \quad \text{for all } n$$

$$u = 0$$

$$\beta = \frac{x}{2}\left(\frac{2}{kx}\right)^{2/3}$$

$$t = \xi + \frac{1}{4}\left(\frac{2}{kx}\right)^{2/3} \xi^2$$

and

$$h_2 = \frac{1}{x} \;, \quad h_3 = \frac{2}{3x^2} \;, \quad h_4 = \frac{1}{3x^3} \;, \text{ etc.}$$

4.9 Numerical Aspects of Employing These Expansions

If a numerical anlyst were faced with the problem of evaluating the function $F(y,\lambda)$ which satisfies Eq. (4-1), he would probably find in Ref. 5 (and in the references which have been cited by Olver) a number of methods which he would prefer when they were compared with the expansions which we have expounded upon in the "forty-plus" pages of this Section. The author will readily agree that he himself tends to favor the basing of future work on more computer work and less algebraic work. However, the expansions in this Section are the outgrowths of earlier studies by Fock, Pekeris, and Friedlander. The expansions are also very similar to those in the recent paper by Bremmer (Ref. 2). The author feels that the type of expansions which have been the subject of this section have not outlived their usefulness. Even as the more erudite methods become developed into computer programs which are "de-bugged" and become operational, these expansions will continue to provide some valuable checks and guidelines.

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Section 5

ASYMPTOTIC EXPANSIONS FOR THE PROPAGATION CONSTANTS AND THE NORMALIZATION INTEGRAL

5.1 The Eigenvalue Problem

The eigenvalues λ_s or t_s are defined in the radiowave propagation problems by the homogeneous boundary conditions (sometimes referred to as "impedance boundary conditions") which has been given above as Eq. (4-54), namely

$$\left[\frac{dG(u, t_s)}{du} + q\, G(u, t_s)\right]_{u=0} = 0 \tag{5-1a}$$

We can also write

$$\left[\frac{\partial H(u, \tau_s)}{\partial u} + q\, H(u, \tau_s)\right]_{u=0} = 0 \tag{5-1b}$$

and

$$\left[\frac{\partial F(y, \lambda_s)}{\partial y} + k^{2/3} f_1^{\,1/3}\, q\, F(y, \lambda_s)\right]_{y=0} = 0 \tag{5-1c}$$

We have found the relations between λ_s, t_s, and τ_s to be

$$\lambda_s = k^2 f_o + k^{4/3} f_1^{\,2/3} t_s = k^2 f_o + k^{4/3} f_1^{\,2/3}\left[\tau_s + \sum_{n=1}^{\infty} D_n(\tau_s, q)\, \beta^n\right] \tag{5-2}$$

where

$$\beta = k^{-\frac{2}{3}} f_1^{\,-\frac{1}{3}}$$

Eqs. (4-148) and (4-149) are ideally suited for the determination of the coefficients $D_n(\tau_s, q)$ for n = 1, 2, and 3.

The case $q = \infty$ is quite easy. Since

$$H(0, \tau_s^{\infty}) = 0 \tag{5-3a}$$

we find that τ_s^∞ is defined by

$$w(\tau_s^\infty) = 0 \tag{5-3b}$$

From Eq. (4-148), we find that in order to satisfy Eq. (5-3a) we must have

$$D_1 = \frac{8}{15} h_2 \tau^2 \tag{5-4a}$$

$$D_2 = \left(\frac{16}{35} h_3 - \frac{48}{175} h_2^{\,2}\right)\tau^3 - \left(\frac{3}{7} h_3 - \frac{9}{35} h_2^{\,2}\right) \tag{5-4b}$$

$$D_3 = \left(\frac{128}{315} h_4 - \frac{11968}{17325} h_2 h_3 + \frac{22912}{70875} h_2^{\,3}\right)\tau^4 - \left(\frac{80}{63} h_4 - \frac{6556}{3465} h_2 h_3 + \frac{1336}{1575} h_2^{\,3}\right)\tau \tag{5-4c}$$

$$D_4 = \left(\frac{256}{693} h_5 - \frac{41984}{51975} h_2 h_4 - \frac{17344}{40425} h_3^{\,2} + \frac{7521792}{5457375} h_2^{\,2} h_3 - \frac{41575168}{81860625} h_2^{\,4}\right)\tau^5 + \left(-\frac{1808}{693} h_5 + \frac{50432}{10395} h_2 h_4 + \frac{19324}{8085} h_3^{\,2} - \frac{8047680}{1091475} h_2^{\,2} h_3 + \frac{14141280}{5457375} h_2^{\,4}\right)\tau^2 \tag{5-4d}$$

where all the τ are to be assumed to be τ_s^∞. The roots of

$$w_{1,2}(\tau_s^\infty) = 0 \tag{5-5a}$$

are well known (See Section 2). They can be expressed in the form

$$\tau_s^\infty = \alpha_s \exp(\pm\tfrac{1}{3}\pi) \tag{5-5b}$$

where α_s denotes the roots of the Airy function

$$Ai(-\alpha_s) = 0$$

In Table 2-1 we have listed some 15 decimal values which were computed for the author by G. F. Miller and P. H. Haines at the National Physical Laboratory (Teddington, England) in 1958. Sherry (Ref. 1) has obtained 25 decimal values for these constants.

For the case q = 0, we must have

$$\frac{\partial H(0, \tau_s^0)}{\partial u} = 0 \tag{5-6a}$$

This leads to

$$w'(\tau_s^0) = 0 \tag{5-6b}$$

as the condition which defines τ_s^0. From Eq. (4-149), we find that in order to satisfy Eq. (5-6) we must have

$$D_1 = \left(\frac{8}{15}\tau^2 - \frac{1}{5\tau}\right) h_2 \tag{5-7a}$$

$$D_2 = \left(\frac{16}{35} h_3 - \frac{48}{175} h_2^{\ 2}\right) \tau^3 - \left(\frac{3}{5} h_3 - \frac{7}{25} h_2^{\ 2}\right) - \frac{1}{50} h_2^{\ 2} \frac{1}{\tau^3} \tag{5-7b}$$

$$\begin{aligned} D_3 = {} & \left(\frac{128}{315} h_4 - \frac{11968}{17325} h_2 h_3 + \frac{22912}{70875} h_2^{\ 3}\right) \tau^4 + \left(- \frac{448}{315} h_4 + \frac{33308}{17325} h_2 h_3 \right. \\ & \left. - \frac{12176}{15750} h_2^{\ 3}\right) \tau - \frac{1}{375} h_2^{\ 3} \frac{1}{\tau^2} - \frac{1}{250} h_2^{\ 3} \frac{1}{\tau^5} \end{aligned} \tag{5-7c}$$

where all τ are to be taken to be τ_s^0. The roots of

$$w'_{1,2}(\tau_s^0) = 0 \tag{5-8a}$$

are well known. They can be expressed in the form

$$\tau_s^0 = \beta_s \exp\left(\pm\tfrac{1}{3}\pi\right) \tag{5-8b}$$

where β_s denotes the roots of the derivative of the Airy function

$$Ai'(-\beta_s) = 0$$

In Table 2-2 we list the 15 decimal values by Miller and Haines. Values which are given to 25 decimals have been obtained by Sherry (Ref. 1).

For the case of arbitrary q, we can use Eq. (4-153a) and Eq. (4-153b) to express Eq. (5-1b) in the form

$$\begin{aligned} & - w'(\tau_s) + \sum_{n=1}^{\infty} \left[C_n(0, \tau_s)\, w(\tau_s) + D_n(0, \tau_s)\, w'(\tau_s)\right] \beta^n \\ & + q\, w(\tau_s) + q \sum_{n=1}^{\infty} \left[A_n(0, \tau_s)\, w(\tau_s) + B_n(0, \tau_s) w'(\tau_s)\right] \beta^n = 0 \end{aligned} \tag{5-9}$$

Since the coefficient of each power of β must vanish, we find that τ_s must be defined by

$$w'(\tau_s) - q\,w(\tau_s) = 0 \qquad (5\text{-}10)$$

We can use Eq. (5-10) to express the coefficient of β^n in Eq. (5-9) in the form

$$q\left[D_n(0,\tau_s) + q\,B_n(0,\tau_s)\right] + \left[C_n(0,\tau_s) + q\,B_n(0,\tau_s)\right] = 0 \qquad (5\text{-}11)$$

For n = 1, Eq. (5-11) takes the form

$$q\left\{-\frac{2}{15}h_2\tau + q\left[-\frac{8}{15}h_2\tau^2 + D_1\right]\right\} + \left\{\left[\frac{8}{15}h_2\tau^3 - \frac{1}{5}h_2 - D_1\tau\right] + q\left[-\frac{2}{15}h_2\tau\right]\right\} = 0 \quad (5\text{-}12)$$

We can solve Eq. (5-12) for D_1 and find that

$$D_1 = \frac{h_2}{15}\left[8\tau^2 - \frac{3 + 4\tau q}{\tau - q^2}\right] \qquad (5\text{-}13)$$

For n = 2, Eq. (5-11) takes the form

$$\begin{aligned}
q&\left\{\left[-\frac{32}{225}h_2^{\,2}\tau^5 + \frac{8}{15}h_2D_1\tau^3 + \left(-\frac{4}{35}h_3 + \frac{86}{315}h_2^{\,2}\right)\tau^2 - \frac{1}{2}D_1^{\,2}\tau - \frac{1}{3}h_2D_1\right]\right.\\
&\left.+ q\left[-\left(\frac{16}{35}h_3 - \frac{288}{315}h_2^{\,2}\right)\tau^3 - \frac{6}{5}h_2D_1\tau + \left(\frac{3}{7}h_3 - \frac{9}{35}h_2^{\,2} + D_2\right)\right]\right\}\\
&+\left\{\left[\tau^4\left(\frac{16}{35}h_3 - \frac{288}{315}h_2^{\,2}\right) + \tau^2\left(\frac{22}{15}h_2D_1\right) + \tau\left(-\frac{3}{5}h_3 + \frac{7}{15}h_2^{\,2} - D_2\right) - \frac{1}{2}D_1^{\,2}\right]\right.\\
&\left.+ q\left[\frac{32}{225}h_2^{\,2}\tau^5 - \frac{8}{15}h_2D_1\tau^3 + \left(-\frac{4}{35}h_3 + \frac{58}{315}h_2^{\,2}\right)\tau^2 + \frac{1}{2}D_1^{\,2}\tau - \frac{2}{15}h_2D_1\right]\right\} = 0
\end{aligned} \qquad (5\text{-}14)$$

If we combine the similar terms in Eq. (5-14) we find that D_2 is given by

$$\begin{aligned}
(\tau - q^2)\,D_2 = {}&(\tau - q^2)\,\tau^3\left(\frac{16}{35}h_3 - \frac{288}{315}h_2^{\,2}\right) + \frac{22}{15}h_2D_1\tau^2 + \left(-\frac{3}{5}h_3 + \frac{7}{15}h_2^{\,2}\right)\tau\\
&- \frac{1}{2}D_1^{\,2} + q\left[\left(-\frac{8}{35}h_3 + \frac{16}{35}h_2^{\,2}\right)\tau^2 - \frac{7}{15}h_2D_1\right] + q^2\left[-\frac{6}{5}h_2D_1\tau\right.\\
&\left.+ \left(\frac{3}{7}h_3 - \frac{9}{35}h_2^{\,2}\right)\right]
\end{aligned} \qquad (5\text{-}15)$$

If we use Eq. (5-13), we can eliminate D_1 from Eq. (5-15) to obtain the form

$$(\tau - q^2)^2 D_2 = (\tau - q^2)\,\tau^3 \left(\frac{16}{35} h_3 - \frac{288}{315} h_2^{\,2}\right) + \frac{16}{25} h_2^{\,2} \tau^4 - \frac{14}{225} h_2^{\,2} \tau^2 \left(\frac{3+4\tau q}{\tau - q^2}\right)$$
$$+ \tau \left(-\frac{3}{5} h_3 + \frac{7}{15} h_2^{\,2}\right) - \frac{1}{450} h_2^{\,2} \left(\frac{3 + 4\tau q}{\tau - q^2}\right)^2$$
$$+ q\left[\left(-\frac{8}{35} h_3 + \frac{328}{1575} h_2^{\,2}\right) \tau^2 + \frac{7}{225} h_2^{\,2} \left(\frac{3 + 4\tau q}{\tau - q^2}\right)\right]$$
$$+ q^2 \left[-\frac{16}{25} h_2^{\,2} \tau^3 + \left(\frac{3}{7} h_3 - \frac{9}{35} h_2^{\,2}\right) + \frac{2}{25} h_2^{\,2} \tau \left(\frac{3+4\tau q}{\tau + q^2}\right)\right] \qquad (5\text{-}16)$$

For n = 3, Eq. (5-11) takes the form

$$q\Bigg\{\Bigg[\left(-\frac{128}{525} h_2 h_3 + \frac{34112}{70875} h_2^{\,3}\right) \tau^6 + \left(\frac{16}{35} h_3 - \frac{512}{315} h_2^{\,2}\right) D_1 \tau^4 + \left(-\frac{32}{315} h_4\right.$$
$$\left. + \frac{2816}{3465} h_2 h_3 - \frac{10604}{14175} h_2^{\,3} + \frac{8}{15} h_2 D_2\right) \tau^3 + \frac{23}{15} h_2 D_1^{\,2} \tau^2 + \left(-\frac{29}{35} h_3\right.$$
$$\left. + \frac{319}{315} h_2^{\,2} - D_2\right) D_1 \tau + \left(\frac{2}{9} h_4 - \frac{25}{63} h_2 h_3 + \frac{277}{1575} h_2^{\,3} - \frac{1}{3} h_2 D_2 - \frac{1}{3} D_1^{\,3}\right)\Bigg]$$
$$+ q \Bigg[-\frac{256}{10125} h_2^{\,3} \tau^7 + \frac{32}{225} h_2^{\,2} D_1 \tau^5 + \left(-\frac{128}{315} h_4 + \frac{1408}{693} h_2 h_3 - \frac{28688}{14175} h_2^{\,3}\right) \tau^4$$
$$- \frac{4}{15} h_2 D_1^{\,2} \tau^3 + \left(-\frac{52}{35} h_3 + \frac{922}{315} h_2^{\,2}\right) D_1 \tau^2 + \left(\frac{80}{63} h_4 - \frac{8338}{3465} h_2 h_3\right.$$
$$\left. + \frac{1822}{1575} h_2^{\,3} - \frac{6}{5} h_2 D_2 + \frac{1}{6} D_1^{\,3}\right) \tau + \left(-\frac{2}{3} h_2 D_1^{\,2} + D_3\right)\Bigg]\Bigg\}$$
$$+ \Bigg\{\Bigg[\frac{256}{10125} h_2^{\,3} \tau^8 - \frac{32}{225} h_2^{\,2} D_1 \tau^6 + \left(+\frac{128}{315} h_4 - \frac{1408}{693} h_2 h_3 + \frac{142768}{70875} h_2^{\,3}\right) \tau^5$$
$$+ \frac{4}{15} h_2 D_1^{\,2} \tau^4 + \left(\frac{12}{7} h_3 - \frac{1066}{315} h_2^{\,2}\right) D_1 \tau^3 + \left(+\frac{962}{315} h_2 h_3 - \frac{2564}{1575} h_2^{\,3} - \frac{64}{45} h_4\right.$$
$$\left. + \frac{22}{15} h_2 D_2 - \frac{1}{6} D_1^{\,3}\right) \tau^2 + \left(\frac{13}{10} h_2 D_1^{\,2} - D_3\right) \tau + \left(-\frac{3}{5} h_3 + \frac{7}{15} h_2^{\,2} - D_2\right) D_1\Bigg]$$
$$+ q \Bigg[\left(\frac{128}{525} h_2 h_3 - \frac{35008}{70875} h_2^{\,3}\right) \tau^6 + \left(-\frac{16}{35} h_3 + \frac{512}{315} h_2^{\,2}\right) D_1 \tau^4 + \left(-\frac{32}{315} h_4\right.$$
$$\left. + \frac{704}{3465} h_2 h_3 - \frac{3404}{14175} h_2^{\,3} - \frac{8}{15} h_2 D_2\right) \tau^3 - \frac{7}{5} h_2 D_1^{\,2} \tau^2 + \left(\frac{1}{5} h_3 + \frac{1}{9} h_2^{\,2}\right.$$
$$\left. + D_2\right) D_1 \tau + \left(\frac{2}{9} h_4 - \frac{14}{45} h_2 h_3 + \frac{28}{225} h_2^{\,3} - \frac{2}{15} h_2 D_2 + \frac{1}{6} D_1^{\,3}\right)\Bigg]\Bigg\} = 0 \qquad (5\text{-}17)$$

If we solve for D_3, and combine similar terms, we find that

$$
\begin{aligned}
(\tau - q^2)\, D_3 = {} & \frac{256}{10125} h_2^{\,3} \tau^8 - \frac{32}{225} h_2^{\,2} D_1 \tau^6 + \left(\frac{128}{315} h_4 - \frac{1408}{693} h_2 h_3 + \frac{142768}{70875} h_2^{\,3}\right) \tau^5 \\
& + \frac{4}{15} h_2 D_1^{\,2} \tau^4 + \left(\frac{12}{7} h_3 - \frac{1066}{315} h_2^{\,2}\right) D_1 \tau^3 + \left(- \frac{64}{45} h_4 + \frac{962}{315} h_2 h_3\right. \\
& \left. - \frac{2564}{1575} h_2^{\,3} + \frac{22}{15} h_2 D_2 - \frac{1}{6} D_1^{\,3}\right) \tau^2 + \frac{13}{10} h_2 D_1^{\,2} \tau + \left(- \frac{3}{5} h_3 + \frac{7}{15} h_2^{\,2} - D_2\right) D_1 \\
& + q \left[- \frac{128}{10125} h_2^{\,3} \tau^6 + \left(- \frac{64}{315} h_4 + \frac{704}{693} h_2 h_3 - \frac{14008}{14175} h_2^{\,3}\right) \tau^3 + \frac{2}{15} h_2 D_1^{\,2} \tau^2 \right. \\
& \left. + \left(- \frac{22}{35} h_3 + \frac{354}{1575} h_2^{\,2}\right) \tau + \left(\frac{4}{9} h_4 - \frac{223}{315} h_2 h_3 + \frac{473}{1575} h_2^{\,3} - \frac{7}{15} h_2 D_2 - \frac{1}{6} D_1^{\,3}\right) \right] \\
& + q^2 \left[- \frac{256}{10125} h_2^{\,3} \tau^7 + \frac{32}{225} h_2^{\,2} D_1 \tau^5 + \left(- \frac{128}{315} h_4 + \frac{1408}{693} h_2 h_3 - \frac{28688}{14175} h_2^{\,3}\right) \tau^4 \right. \\
& - \frac{4}{15} h_2 D_1^{\,2} \tau^3 + \left(- \frac{52}{35} h_3 + \frac{922}{315} h_2^{\,2}\right) D_1 \tau^2 + \left(\frac{80}{63} h_4 - \frac{8338}{3465} h_2 h_3\right. \\
& \left. \left. + \frac{1822}{1575} h_2^{\,3} - \frac{6}{5} h_2 D_2 + \frac{1}{6} D_1^{\,3}\right) \tau - \frac{2}{3} h_2 D_1^{\,2} \right]
\end{aligned}
\qquad (5\text{-}18)
$$

Because of the complexity of Eq. (5-18) in its present form, we have not used Eq. (5-13) and Eq. (5-16) to eliminate D_1 and D_2. The algebra has become so tedious that we have not attempted to obtain D_4 except for the $q = \infty$ (See. Eq. (5-4d).

5.2 Another Expansion for λ_s

The expansions for $\tau_s(q)$ given in Eq. (2-21) and Eq. (2-22) can be used in Eq. (5-13), Eq. (5-16), and Eq. (5-18) to obtain representations for the solutions of Eq. (5-1c) in the form

$$
\lambda_s = k^2 f_o + k^{4/3} f_1^{\,2/3} \left[\tau_s^o + \sum_{n=1}^{\infty} P_n\left(\tau_s^o, \frac{q}{\beta}\right) \beta^n \right] \qquad (5\text{-}19)
$$

or

$$
\lambda_s = k^2 f_o + k^{4/3} f_1^{\,2/3} \left[\tau_s^{\infty} + \sum_{n=1}^{\infty} Q_n\left(\tau_s^{\infty}, \frac{1}{q\beta}\right) \beta^n \right] \qquad (5\text{-}20)
$$

where the coefficients $P_n(\tau_s^o, \frac{q}{\beta})$ and $Q_n(\tau_s^{\infty}, \frac{1}{q\beta})$ are of the form

$$P_n(\tau_s^o, \frac{q}{\beta}) = D_n(\tau_s^o, 0) + \sum_{m=1}^{\infty} a_m^n (\tau_s^o) (\frac{q}{\beta})^m \tag{5-21}$$

$$Q_n(\tau_s^\infty, \frac{1}{q\beta}) = D_n(\tau_s^\infty, \infty) + \sum_{m=1}^{\infty} b_m^n (\tau_s^\infty) \frac{1}{(\beta q)^m} \tag{5-22}$$

where $D_n(\tau_s^o, 0)$ is given by Eq. (5-7) and $D_n(\tau_s^\infty, \infty)$ is given by Eq. (5-4).

It is perhaps more convenient to determine $P_n(\tau_s^o, \frac{q}{\beta})$ and $Q_n(\tau_s^\infty, \frac{1}{\beta q})$ by expressing the boundary condition in the form

$$\frac{\partial H(0, \tau)}{\partial u} + \beta Q H(0, \tau) = 0 \quad , \quad q = \beta Q \tag{5-23}$$

when q is small, and in the form

$$H(0, \tau) + \beta P \frac{\partial H(0, \tau)}{\partial u} = 0 \quad , \quad \frac{1}{q} = \beta P \tag{5-24}$$

when q is large.

For the case of small q, Eq. (5-23) leads to

$$w'(\tau_s^o) = 0 \tag{5-25a}$$

$$c_1(0, \tau_s^o) + Q = 0 \tag{5-25b}$$

$$c_2(0, \tau_s^o) + QA_1(0, \tau_s^o) = 0 \tag{5-25c}$$

$$c_{n+1}(0, \tau_s^o) + QA_n(0, \tau_s^o) = 0 \ , \ n \geq 1 \tag{5-25d}$$

where the $C_n(u,\tau)$ and $A_n(u,\tau)$ are defined in Eq. (4-153). From Eq. (5-25) we find that

$$P_1(\tau, Q) = (\frac{8}{15}\tau^2 - \frac{1}{5\tau}) h_2 + \frac{Q}{\tau} \tag{5-26a}$$

$$P_2(\tau, Q) = (\frac{16}{35} h_3 - \frac{48}{175} h_2^2) \tau^3 - (\frac{3}{5} h_3 - \frac{7}{25} h_2^2) - \frac{1}{50} h_2^2 \frac{1}{\tau^3} + \frac{4}{5} h_2 Q + \frac{1}{5} h_2 \frac{Q}{\tau^3} - \frac{1}{2} \frac{Q^2}{\tau^3} \tag{5-26b}$$

$$P_3(\tau, Q) = (\frac{128}{315} h_4 - \frac{11968}{17325} h_2h_3 + \frac{22912}{70875} h_2^{\ 3}) \tau^4 + (- \frac{448}{315} h_4 + \frac{33308}{17325} h_2h_3$$

$$- \frac{12176}{15750} h_2^{\ 3}) \tau - \frac{1}{375} h_2^{\ 3} \frac{1}{\tau^2} - \frac{1}{250} h_2^{\ 3} \frac{1}{\tau^5}$$

$$+ Q \left[- \frac{272}{315} h_2^{\ 2} \tau + \frac{8}{7} h_3 \tau + \frac{1}{25} h_2^{\ 2} \frac{1}{\tau^2} + \frac{3}{50} h_2^{\ 2} \frac{1}{\tau^5} \right]$$

$$+ Q^2 \left[- \frac{1}{5} h_2 \frac{1}{\tau^2} - \frac{3}{10} h_2 \frac{1}{\tau^5} \right] + Q^3 \left[\frac{1}{3} \frac{1}{\tau^2} + \frac{1}{2} \frac{1}{\tau^5} \right] \qquad (5\text{-}26c)$$

For the case of large q, Eq. (5-24) leads to

$$w(\tau_s^\infty) = 0 \qquad (5\text{-}27a)$$

$$B_1(0, \tau_s^\infty) - P = 0 \qquad (5\text{-}27b)$$

$$B_2(0, \tau_s^\infty) + P\, D_1(0, \tau_s^\infty) = 0 \qquad (5\text{-}27c)$$

$$B_{n+1}(0, \tau_s^\infty) + P\, D_n(0, \tau_s^\infty) = 0 \quad , \quad n \geq 1 \qquad (5\text{-}27d)$$

where the $B_n(u,\tau)$ and $D_n(u,\tau)$ are defined by Eq. (4-153).* From Eq. (5-27), we find that

$$Q_1(\tau, P) = \frac{8}{15} h_2 \tau^2 + P \qquad (5\text{-}28a)$$

$$Q_2(\tau, P) = (\frac{16}{35} h_3 - \frac{48}{175} h_2^{\ 2}) \tau^3 - (\frac{3}{7} h_3 - \frac{9}{35} h_2^{\ 2}) + \frac{4}{3} h_2 \tau P \qquad (5\text{-}28b)$$

$$Q_3(\tau, P) = (\frac{128}{315} h_4 - \frac{11968}{17325} h_2h_3 + \frac{22912}{70875} h_2^{\ 3}) \tau^4 - (\frac{80}{63} h_4 - \frac{6556}{3465} h_2h_3$$

$$+ \frac{1336}{1575} h_2^{\ 3}) \tau + (\frac{8}{5} h_3 - \frac{32}{45} h_2^{\ 2}) \tau^2 P + h_2 P^2 + \frac{\tau}{3} P^3 \qquad (5\text{-}28c)$$

Some unpublished work of the present author can be used as a numerical check on the coefficients in Eq. (5-26). It has been shown that the roots ν_s of the Hankel function combination

$$H_{\nu_s}^{(1,2)'}(ka) + (\frac{2}{ka}) Z\, H_{\nu_s}^{(1,2)}(ka) = 0 \qquad (5\text{-}29)$$

---

*The reader is cautioned to avoid confusing the $D_n(\tau,q)$ defined by Eq. (5-2) with the $D_n(u,\tau)$ defined by Eq. (4-153). The context in which these functions are used should provide the means to distinguish between them.

can be expressed in the form

$$\nu_s = ka + \left(\frac{ka}{2}\right)^{1/3} \Bigg\{ \tau + \left(\frac{2}{ka}\right)^{2/3} \left[\frac{1}{60}\tau^2 - \frac{1}{10}\frac{1}{\tau} + \frac{Z}{\tau}\right]$$

$$+ \left(\frac{2}{ka}\right)^{4/3} \left[\left(-\frac{1}{1400}\tau^3 + \frac{1}{50} - \frac{1}{200}\frac{1}{\tau^3}\right) + Z\left(\frac{1}{10}\frac{1}{\tau^3} - \frac{1}{10}\right) - Z^2\left(\frac{1}{2}\frac{1}{\tau^3}\right)\right]$$

$$+ \left(\frac{2}{ka}\right)^{6/3} \Bigg[\left(\frac{281}{4536000}\tau^4 - \frac{611}{126000}\tau - \frac{1}{3000}\frac{1}{\tau^2} - \frac{1}{2000}\frac{1}{\tau^5}\right)$$

$$+ Z\left(\frac{41}{2520}\tau + \frac{1}{100}\frac{1}{\tau^2} + \frac{3}{200}\frac{1}{\tau^5}\right) - Z^2\left(\frac{1}{10}\frac{1}{\tau^2} + \frac{3}{20}\frac{1}{\tau^5}\right)$$

$$+ Z^3\left(\frac{1}{3}\frac{1}{\tau^2} + \frac{1}{2}\frac{1}{\tau^5}\right)\Bigg]$$

$$+ \left(\frac{2}{ka}\right)^{8/3} \Bigg[\left(-\frac{73769}{10478160000}\tau^5 + \frac{56299}{24948000}\tau^2 + \frac{1679}{1108800}\frac{1}{\tau}\right.$$

$$\left. - \frac{7}{120000}\frac{1}{\tau^4} - \frac{1}{16000}\frac{1}{\tau^7}\right) + Z\left(-\frac{97}{32400}\tau^2 - \frac{803}{42000}\frac{1}{\tau}\right.$$

$$\left. + \frac{7}{3000}\frac{1}{\tau^4} + \frac{1}{400}\frac{1}{\tau^7}\right) + Z^2\left(\frac{2017}{25200}\frac{1}{\tau} - \frac{7}{200}\frac{1}{\tau^4} - \frac{3}{80}\frac{1}{\tau^7}\right)$$

$$+ Z^3\left(-\frac{11}{90}\frac{1}{\tau} + \frac{7}{30}\frac{1}{\tau^4} + \frac{1}{4}\frac{1}{\tau^7}\right) + Z^4\left(-\frac{7}{12}\frac{1}{\tau^4} - \frac{5}{8}\frac{1}{\tau^7}\right)\Bigg]$$

$$+ \cdots \Bigg\} \tag{5-30}$$

where

$$\tau = \tau_s^o = \beta_s \exp(\pm i\,\pi/3) \tag{5-31}$$

where the + sign is used with $H_\nu^{(1)}(ka)$ and the - sign with $H_\nu^{(2)}(ka)$. If we express ν_s^2 in the form

$$\left(\frac{\nu_s}{ka}\right)^2 = 1 + \left(\frac{2}{ka}\right)^{2/3} \left[\tau_s^o + \sum_{n=1}^{\infty} E_n(\tau_s^o, Z)\left(\frac{2}{ka}\right)^{2n/3}\right] \tag{5-32}$$

we find that

$$E_1(\tau, Z) = \frac{4}{15}\tau^2 - \frac{1}{10}\frac{1}{\tau} + \frac{Z}{\tau} \tag{5-33a}$$

$$E_2(\tau, Z) = \frac{4}{525}\tau^3 - \frac{3}{100} - \frac{1}{200}\frac{1}{\tau^3} + Z\left(\frac{1}{10}\frac{1}{\tau^3} + \frac{2}{5}\right) - \frac{Z^2}{2\tau^3} \tag{5-33b}$$

$$E_3(\tau, Z) = -\frac{16}{70875}\tau^4 + \frac{34}{7875}\tau - \frac{1}{3000}\frac{1}{\tau^2} - \frac{1}{2000}\frac{1}{\tau^5} + Z\left(-\frac{8}{315}\tau + \frac{1}{100}\frac{1}{\tau^2} + \frac{3}{200}\frac{1}{\tau^5}\right) - Z^2\left(\frac{1}{10}\frac{1}{\tau^2} + \frac{3}{20}\frac{1}{\tau^5}\right) + Z^3\left(\frac{1}{3}\frac{1}{\tau^2} + \frac{1}{2}\frac{1}{\tau^5}\right) \quad (5\text{-}33c)$$

$$E_4(\tau, Z) = \frac{1472}{81860625}\tau^5 + \frac{859}{24948000}\tau^2 + \frac{53}{173250}\frac{1}{\tau} - \frac{21}{360000}\frac{1}{\tau^4} - \frac{1}{16000}\frac{1}{\tau^7} + Z\left(\frac{8}{2025}\tau^2 + \frac{3}{1750}\frac{1}{\tau} + \frac{7}{3000}\frac{1}{\tau^4} + \frac{1}{400}\frac{1}{\tau^7}\right) + Z^2\left(-\frac{38}{1575}\frac{1}{\tau} - \frac{7}{200}\frac{1}{\tau^4} - \frac{3}{80}\frac{1}{\tau^7}\right) + Z^3\left(\frac{2}{45}\frac{1}{\tau} + \frac{7}{30}\frac{1}{\tau^4} + \frac{1}{4}\frac{1}{\tau^7}\right) + Z^4\left(-\frac{7}{12}\frac{1}{\tau^4} - \frac{5}{8}\frac{1}{\tau^7}\right) \quad (5\text{-}33d)$$

Let us recall [from Eq. (4-161b) and Eq. (4-164)] that for the Hankel functions

$$\beta = \left(\frac{a}{2}\right)\left(\frac{2}{ka}\right)^{2/3}, \quad h_2 = \frac{1}{a}, \quad h_3 = \frac{2}{3a^2}, \quad h_4 = \frac{1}{3a^3} \quad .$$

If we let

$$Q = \left(\frac{2}{a}\right) Z$$

we find that we can check the numerical coefficients in Eq. (5-26) by a comparison with Eq. (5-33) since

$$\left(\frac{a}{2}\right)^n P_n(\tau, Q) = E_n\left(\tau, \frac{a}{2} Q\right) \quad (5\text{-}34)$$

We have also shown that the roots ν_s defined by

$$H^{(1,2)}_{\nu_s}(ka) + \left(\frac{2}{ka}\right)^{1/3} \Gamma H^{(1,2)}_{\nu_s}(ka) = 0 \quad (5\text{-}35)$$

can be represented by an asymptotic expansion of the form

$$\nu_s = ka + \left(\frac{ka}{2}\right)^{1/3}\left\{\tau + \left(\frac{2}{ka}\right)^{2/3}\left[\frac{1}{60}\tau^2 + \Gamma\right] + \left(\frac{2}{ka}\right)^{4/3}\left[-\frac{1}{1400}\tau^3 - \frac{1}{140} + \frac{1}{6}\tau\Gamma\right] + \left(\frac{2}{ka}\right)^{6/3}\left[\frac{281}{4536000}\tau^4 + \frac{29}{12600}\tau - \frac{1}{360}\tau^2\Gamma + \frac{1}{4}\Gamma^2 + \frac{1}{3}\tau\Gamma^3\right] + \cdots\right\} \quad (5\text{-}36)$$

where

$$\tau = t_s^{\infty} = \alpha_s \exp(\pm i\,\pi/3) \quad (5\text{-}37)$$

where the + sign is used with $H_\nu^{(1)}(ka)$. If we express ν_s^2 in the form

$$\left(\frac{\nu_s}{ka}\right)^2 = 1 + \left(\frac{2}{ka}\right)^{2/3}\left[\tau_s^\infty + \sum_{n=1}^{\infty} F_n(\tau_s^\infty, \Gamma)\left(\frac{2}{ka}\right)^{2n/3}\right] \tag{5-38}$$

we find that

$$F_1(\tau, \Gamma) = \frac{4}{15}\tau^2 + \Gamma \tag{5-39a}$$

$$F_2(\tau, \Gamma) = \frac{4}{525}\tau^3 - \frac{1}{140} + \frac{2}{3}\tau\Gamma \tag{5-39b}$$

$$F_3(\tau, \Gamma) = -\frac{16}{70875}\tau^4 - \frac{2}{1575}\tau + \frac{4}{45}\tau^2\Gamma + \frac{1}{4}\Gamma^2 + \frac{1}{3}\tau\Gamma^3 \tag{5-39c}$$

If we set

$$P = \left(\frac{2}{a}\right)\Gamma$$

we can use the relation

$$\left(\frac{a}{2}\right)^n Q_n(\tau, P) = F_n\left(\tau, \frac{a}{2}P\right) \tag{5-40}$$

to obtain a method of verifying the accuracy of the coefficients in Eq. (5-28).

5.3 The Turning Points

In the studies by Pekeris (Ref. 2), Imai (Refs. 3 and 4), Bremmer (Ref. 5) and other authors, one needs the values of the turning point y_1 defined by Eq. (4-24), namely

$$\lambda_s + k^2 f(y_1) = k^2[\mu_s + f(y_1)] = 0 \tag{5-41}$$

If we recall the steps which we took in Eqs. (4-74) through (4-77), we see that we can define

$$y_1 = \beta u_1 = [k^{\frac{2}{3}} f_1^{\frac{1}{3}}]^{-1} u_1 \tag{5-42}$$

where u_1 is defined by

$$-t_s + u_1 + \beta h_2 u_1^2 + \beta^2 h_3 u_1^3 + \beta^3 h_4 u_1^4 + \dots = 0$$

If we invert this expression, we find that

$$u_1 = t_s - \beta h_2 t_s^{\,2} - \beta^2 (h_3 - 2h_2^{\,2})\, t_s^{\,3} - \beta^3 (h_4 - 5h_2h_3 + 5h_2^{\,3})\, t_s^{\,4}$$
$$- \beta^4 (h_5 - 6h_2h_4 - 3h_3^{\,2} + 21h_2^{\,2}h_3 - 14h_2^{\,4})\, t_s^{\,5} + O(\beta^5) \qquad (5\text{-}43)$$

From Eq. (5-2) and Eq. (5-13), we obtain the result

$$t_s = \tau_s + \beta h_2 \frac{1}{15}\left[8\tau_s^{\,2} - \frac{3 + 4q\tau_s}{\tau_s - q^2}\right] + O(\beta^2) \qquad (5\text{-}44)$$

Therefore, we find that

$$u_1 = \beta\tau_s - \beta^2 h_2 \frac{1}{15}\left[7\tau_s^{\,2} + \frac{3 + 4q\tau_s}{\tau_s - q^2}\right] + O(\beta^3) \qquad (5\text{-}45)$$

For q = 0, we can use the results in Eq. (5-7) to show that

$$u_1 = \beta\Bigg\{\tau - \beta\left[\frac{7}{15}h_2\tau^2 + \frac{1}{5\tau}h_2\right] - \beta^2\left[\left(\frac{19}{35}h_3 - \frac{346}{525}h_2^{\,2}\right)\tau^3 + \left(\frac{3}{5}h_3 - \frac{17}{25}h_2^{\,2}\right.\right.$$
$$\left.\left. + \frac{1}{50}h_2^{\,2}\right)\frac{1}{\tau^3}\right] - \beta^3\left[\left(\frac{187}{315}h_4 - \frac{31097}{17325}h_2h_3 + \frac{85943}{70875}h_2^{\,3}\right)\tau^4\right.$$
$$\left. + \left(\frac{448}{315}h_4 - \frac{64493}{17325}h_2h_3 + \frac{36536}{15750}h_2^{\,3}\right)\tau + \frac{17}{750}h_2^{\,3}\frac{1}{\tau^2} + \frac{1}{250}h_2^{\,3}\frac{1}{\tau^5}\right]$$
$$+ O(\beta^4)\Bigg\} \qquad (5\text{-}46)$$

where all τ are to be taken to be τ_s^0.

For q = ∞, we can use the results in Eq. (5-4) to show that

$$u_1 = \beta\Bigg\{\tau - \frac{7}{15}h_2\tau^2\beta - \beta^2\left[\left(\frac{19}{35}h_3 - \frac{346}{525}h_2^{\,2}\right)\tau^3 + \left(\frac{3}{7}h_3 - \frac{9}{35}h_2^{\,2}\right)\right]$$
$$- \beta^3\left[\left(\frac{187}{315}h_4 - \frac{31097}{17325}h_2h_{[\text{illegible}]} + \frac{85943}{70875}h_2^{\,3}\right)\tau^4 + \left(\frac{80}{63}h_4 - \frac{9526}{3465}h_2h_3\right.\right.$$
$$\left.\left. + \frac{2146}{1575}h_2^{\,3}\right)\tau\right]$$
$$- \beta^4\left[\left(\frac{437}{693}h_5 - \frac{116746}{51975}h_2h_4 + \frac{194059}{40425}h_3^{\,2} - \frac{200019867}{5457375}h_2^{\,2}h_3\right.\right.$$
$$+ \left.\frac{2084790958}{81860625}h_2^{\,4}\right)\tau^5 + \left(\frac{1808}{693}h_5 - \frac{76832}{10395}h_2h_4 - \frac{29719}{8085}h_3^{\,2}\right.$$
$$\left.\left. + \frac{15327645}{1091475}h_2^{\,2}h_3 - \frac{30322830}{54573[\text{illegible}]5}h_2^{\,4}\right)\tau^2\right] + O(\beta^5)\Bigg\} \qquad (5\text{-}47)$$

where all τ are to be taken to be τ_s^∞.

5.4 The Propagation Constants

In many applications we need representations for $\sqrt{\lambda_s}$. For example, in the case of the Hankel functions $H_\nu^{(1)}(ka)$ we defined

$$\lambda = (\nu/a)^2 \tag{5-48}$$

in Eq. (4-162a). In order to treat these problems it will be convenient to define

$$\lambda = \kappa^2 \tag{5-49}$$

We will call κ the <u>propagation constant</u>. It will also be convenient to define

$$f_0 = n_0^2 \tag{5-50}$$

and call n_o the <u>index of refraction</u> at $y = 0$.

We can then express Eq. (5-2) in the form

$$\kappa_s^2 = (kn_o)^2 + k^{4/3} f_1^{2/3} \tau_s + k^{4/3} f_1^{2/3} \sum_{n=1}^{\infty} D_n(\tau_s, q) \beta^n \tag{5-51}$$

If we assume that κ_s can be expressed in the form

$$\kappa_s = kn_o + \frac{1}{2} \frac{k^{1/3} f_1^{2/3}}{n_o} \left[\tau_s + \sum_{n=1}^{\infty} N_n(\tau_s, q) \beta^n\right] \tag{5-52}$$

we can show that

$$\begin{aligned}\kappa_s^2 = (kn_o)^2 + k^{4/3} f_1^{2/3} \tau_s + k^{4/3} f_1^{2/3} \Bigg\{ \left[N_1 + \frac{f_1}{4f_o} \tau^2\right] \beta \\ + \left[N_2 + \frac{f_1}{4f_o} 2\tau N_1\right] \beta^2 + \left[N_3 + \frac{f_1}{4f_o} (2\tau N_2 + N_1^2)\right] \beta^3 + \ldots \Bigg\}\end{aligned} \tag{5-53}$$

Therefore, we find that

$$N_1(\tau, q) = D_1(\tau, q) - \frac{f_1}{4f_o} \tau^2 \tag{5-54a}$$

$$N_2(\tau, q) = D_2(\tau, q) - \frac{f_1}{4f_o} 2\tau N_1(\tau, q) \tag{5-54b}$$

$$N_3(\tau, q) = D_3(\tau, q) - \frac{f_1}{4f_o} \left[2\tau N_2(\tau, q) + N_1^2(\tau, q)\right] \tag{5-54c}$$

The expansions for ν_s which are given in Eq. (5-30) and Eq. (5-36) can be used to verify the correctness of the coefficients in Eq. (5-54).

5.5 The Characteristic Wave Numbers

Up to this point, we have always considered k to be known and we have sought to determine the eigenvalues λ_s. However, in pulse diffraction problems we encounter the case in which λ is known and we must find the characteristic wave numbers k_s for which either

$$F(0,\lambda) = 0 \tag{5-55a}$$

or

$$[dF(y,\lambda)/dy]_{y=0} = 0 \tag{5-55b}$$

The more general problem

$$[dF/dy + i\, k_s\, Z\, F]_{y=0} = 0 \tag{5-55c}$$

will not be considered because the impedance Z is usually a function of k_s.

We can determine k_s for the Dirichlet problem defined by Eq. (5-55a) by using Eq. (5-2) with D_n defined by Eq. (5-4). Let us consider Eq. (5-2) as the definition of a relation between λ and k which makes $F(y,\lambda)$ vanish for y = 0. We can then define the characteristic wave numbers k_s by means of Eq. (5-2) which we write in the form

$$\lambda = k_s^{\,2} f_o + k_s^{\,4/3} f_1^{\,2/3} \tau_s + D_1(\tau_s)\, k_s^{\,2/3} f_1^{\,1/3} + \sum_{n=2}^{\infty} D_n(\tau_s)\, k_s^{\frac{4-2n}{3}} f_1^{\frac{2-n}{3}} \tag{5-56}$$

If we seek a representation for k_s of the form

$$f_o k_s^{\,2} = \lambda - f_1^{\,2/3} \tau_s \left(\frac{\lambda}{f_o}\right)^{2/3} + \sum_{n=1}^{\infty} F_n(\tau_s)\, f_1^{\frac{2-n}{3}} \left(\frac{\lambda}{f_o}\right)^{\frac{4-2n}{3}} \tag{5-57}$$

we can determine $F_n(\tau_s)$ by inserting Eq. (5-57) in Eq. (5-56) and

equate the coefficients of λ on each side of the equation.

The above suggestion leads to an expansion for k_s^2 whereas in practice we generally want k_s. However, if we express k_s in the form

$$k_s = \sqrt{\frac{\lambda}{f_o}} - \frac{1}{2f_o} f_1^{2/3} \tau_s \left(\frac{\lambda}{f_o}\right)^{1/6} + \sum_{n=1}^{\infty} M_n(\tau_s) f_1^{\frac{2-n}{3}} \left(\frac{\lambda}{f_o}\right)^{\frac{1-2n}{6}} \tag{5-58}$$

and use this in Eq. (5-56) we can relate the $M_n(\tau_s)$ to the $D_n(\tau_s)$.

For the special case of the k_s defined by the Hankel function combination

$$H_\nu^{(1,2)'}(k_s a) + \left(\frac{i}{ka}\right) Z\, H_\nu^{(1,2)}(k_s a) = 0 \tag{5-59}$$

we have shown that

$$k_s a = \nu - \left(\frac{\nu}{2}\right)^{1/3} \tau + \left(\frac{2}{\nu}\right)^{1/3} R_1(\tau, Z) + \left(\frac{2}{\nu}\right)^{3/3} R_2(\tau, Z) + \left(\frac{2}{\nu}\right)^{5/3} R_3(\tau, Z)$$
$$+ \left(\frac{2}{\nu}\right)^{7/3} R_4(\tau, Z) \tag{5-60}$$

where

$$R_1(\tau, Z) = + \frac{3\tau^2}{20} + \frac{1}{10}\frac{1}{\tau} - \frac{Z}{\tau} \tag{5-61a}$$

$$R_2(\tau, Z) = + \frac{\tau^3}{1400} - \frac{1}{50} + \frac{1}{200}\frac{1}{\tau^3} - Z\left(\frac{1}{10}\frac{1}{\tau^3} - \frac{1}{10}\right) + \frac{1}{2}\frac{Z^2}{\tau^3} \tag{5-61b}$$

$$R_3(\tau, Z) = - \frac{479}{504000}\tau^4 - \frac{509}{126000}\tau + \frac{1}{3000}\frac{1}{\tau^2} + \frac{1}{2000}\frac{1}{\tau^5} - Z\left(-\frac{11}{280}\tau + \frac{1}{100}\frac{1}{\tau^2}\right.$$
$$\left. + \frac{3}{200}\frac{1}{\tau^5}\right) - Z^2\left(-\frac{1}{10}\frac{1}{\tau^2} - \frac{3}{20}\frac{1}{\tau^5}\right) - Z^3\left(\frac{1}{3}\frac{1}{\tau^2} + \frac{1}{2}\frac{1}{\tau^5}\right) \tag{5-61c}$$

$$R_4(\tau, Z) = - \frac{20231}{129360000}\tau^5 - \frac{20879}{24948000}\tau^2 - \frac{2006}{11088000}\frac{1}{\tau} + \frac{7}{120000}\frac{1}{\tau^4}$$
$$+ \frac{5}{80000}\frac{1}{\tau^7} - Z\left(-\frac{5}{226800}\tau^2 + \frac{819}{126000}\tau^2 + \frac{391}{126000}\frac{1}{\tau} + \frac{28}{12000}\frac{1}{\tau^4}\right.$$
$$\left. + \frac{1}{400}\frac{1}{\tau^7}\right) - Z^2\left(-\frac{783}{25200} - \frac{21}{600}\frac{1}{\tau^4} - \frac{111}{400}\frac{1}{\tau^7}\right) - Z^3\left(\frac{1}{10}\frac{1}{\tau} + \frac{1}{10}\frac{1}{\tau^4}\right.$$
$$\left. + \frac{1}{4} + \frac{1}{\tau^7}\right) + Z^4\left(\frac{7}{12}\frac{1}{\tau^4} + \frac{5}{8}\frac{1}{\tau^7}\right) \tag{5-61d}$$

In these results, τ is defined as in Eq. (5-31).

The coefficients $R_1(\tau,0)$, $R_2(\tau,0)$, and $R_3(\tau,0)$ have been previously given by Olver [See Eq. 9.6 of Ref. 6]. The coefficients $R_1(\tau,-1/4)$, $R_2(\tau,-1/4)$, and $R_3(\tau,-1/4)$ have been previously obtained by Miller (G.F.) and Olver [See Eq. (1.39) of Ref. 7].

For the case $Z = \infty$ when k_s is defined by

$$H_\nu^{(1,2)}(k_s a) = 0 \tag{5-62}$$

we have shown that k_s can be represented by an expansion of the form of Eq. (5-60) in which τ is defined by Eq. (5-37) and

$$B_1(\tau, \infty) = \frac{3}{20}\tau^2 \tag{5-63a}$$

$$B_2(\tau, \infty) = \frac{1}{1400}\tau^3 + \frac{1}{140} \tag{5-63b}$$

$$B_3(\tau, \infty) = -\frac{479}{504000}\tau^4 + \frac{1}{12600}\tau \tag{5-63c}$$

$$B_4(\tau, \infty) = -\frac{20231}{129360000}\tau^5 - \frac{551}{1293600} \tag{5-63d}$$

These coefficients were previosuly obtained by Olver (Ref. 6).

These expansions for k_s can be used to extend the treatment of pulse diffraction by a sphere which was recently the subject of a paper by Weston (Ref. 8). Weston derived the coefficients $B_1(\tau_s^\infty,\infty)$ and $B_1(\tau_s^0,1/4)$. The cases $Z = \infty$ and $Z = 1/4$ were of interest to Weston because he sought to obtain results for pulse diffraction problems by expressing the results in the form

$$U(t) = \frac{c}{2\pi}\int_{-\infty}^{\infty} A(k)\, u(k) \exp(-i\, k\, c\, t)\, dk \tag{5-64}$$

where c denotes the velocity of propagation, A(k) denotes the frequency spectrum, and u(k) denotes the solution of the boundary value problem for an $\exp(-ikct) = \exp(-i\omega t)$ time dependance. For the case of the plane wave

$$\underset{\approx}{E}_i = \hat{\underset{\approx}{x}} \exp[-ik(z + ct)] \tag{5-65}$$

illuminating a perfectly conducting sphere of radius a, the

secondary field is given by

$$\underset{\approx}{E}_s(k) = \frac{i}{k}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}(-i)^n\left\{\frac{\psi_n'(ka)}{\zeta_n^{(1)'}(ka)}\,\text{curl curl}\left[\underset{\approx}{r}\,h^{(1)}(kr)\,P_n^1(\cos\theta)\cos\phi\right]\right.$$

$$\left.+\frac{\psi_n(ka)}{\zeta_n^{(1)}(ka)}\,\text{curl}\left[\underset{\approx}{\dot{r}}\,h_n^{(1)}(kr)\,P_n^{\;1}(\cos\theta)\sin\phi\right]\right\} \qquad (5\text{-}66)$$

where the Riccati Bessel functions are defined by

$$\psi_n(x) = x\,j_n(x) = \sqrt{\frac{\pi x}{2}}\;J_{n+\frac{1}{2}}(x) \qquad (5\text{-}67a)$$

$$\zeta_n^{(1)}(x) = x\,h_n(x) = \sqrt{\frac{\pi x}{2}}\;H^{(1)}_{n+\frac{1}{2}}(x) \qquad (5\text{-}67b)$$

When Eq. (5-66) is used in Eq. (5-64), one finds that U(t) can be evaluated in the form of a series of residues derived from the zeros k_s of the functions

$$\zeta^{(1)}_{\nu-1/2}(k_s a) = \sqrt{\frac{\pi k_s a}{2}}\;H_\nu^{(1)}(k_s a) = 0 \qquad (5\text{-}68a)$$

$$\zeta^{(1)'}_{\nu-1/2}(k_s a) = \sqrt{\frac{\pi k_s a}{2}}\;\left[H_\nu^{(1)'}(k_s a) + \frac{1}{2k_s a}\,H_{\nu_s}^{(1)}(k_s a)\right] = 0 \qquad (5\text{-}68b)$$

The roots k_s of Eq. (5-68a) depend upon the coefficients $B_n(\tau_s^\infty,\infty)$ given in Eq. (5-63) The roots k_s of Eq. (5-68b) depend upon the coefficients $B_n(\tau_s^0,1/4)$ which can be derived from Eq. (5-61). The explicit form for the solution of Eq. (5-68b) is

$$k_s a = \nu - \left(\frac{\nu}{2}\right)^{1/3}\tau + \left(\frac{2}{\nu}\right)^{1/3}\left(\frac{3}{20}\tau^2 - \frac{3}{20}\frac{1}{\tau}\right) + \left(\frac{2}{\nu}\right)^{3/3}\left(\frac{\tau^3}{1400} + \frac{1}{200} + \frac{9}{800}\frac{1}{\tau^3}\right)$$

$$+\left(\frac{2}{\nu}\right)^{5/3}\left(-\frac{479}{504000}\tau^4 + \frac{1457}{252000}\tau - \frac{9}{8000}\frac{1}{\tau^2} - \frac{27}{16000}\frac{1}{\tau^5}\right)$$

$$+\left(\frac{2}{\nu}\right)^{7/3}\left(-\frac{20231}{129360000}\tau^5 + \frac{3287}{49896000}\tau^2 - \frac{12801}{22176000}\frac{1}{\tau} + \frac{4567}{1920000}\frac{1}{\tau^4}\right.$$

$$\left.+\frac{19605}{1280000}\frac{1}{\tau^7}\right) + \cdots \qquad (5\text{-}69)$$

In acoustic diffraction problems involving a rigid sphere of radius

a, we find that the secondary field is of the form

$$u_s(k) = - \sum_{n=0}^{\infty} (2n+1)(-i)^n \frac{j_n'(ka)}{h_n^{(1)'}(ka)} h_n^{(1)}(kr)\, P_n(\cos\theta) \tag{5-70}$$

where the spherical Bessel functions are defined in Eq. (5-67). If we seek to study a pulse diffraction problem for this case we find that k_s is defined by

$$h_n^{(1)'}(k_s a) = \sqrt{\frac{\pi}{k_s a}} \left[H_\nu^{(1)'}(k_s a) - \frac{1}{2k_s a} H_\nu^{(1)}(k_s a) \right] = 0 \tag{5-71}$$

We observe that this case requires the coefficients $B_n(\tau_s^0, -1/4)$ which can be derived from Eq. (5-61). The explicit form for the solution for this case is

$$\begin{aligned} k_s a = \nu - \left(\frac{\nu}{2}\right)^{1/3} \tau + \left(\frac{2}{\nu}\right)^{1/3} \left(\frac{3}{20}\tau^2 + \frac{7}{20}\frac{1}{\tau}\right) + \left(\frac{2}{\nu}\right)^{3/3} \left(\frac{\tau^2}{1400} - \frac{9}{200} + \frac{49}{800}\frac{1}{\tau^3}\right) \\ + \left(\frac{2}{\nu}\right)^{5/3} \left(-\frac{479}{504000}\tau^4 - \frac{499}{36000}\tau + \frac{343}{24000}\tau^2 + \frac{343}{16000}\frac{1}{\tau^5}\right) \\ + \left(\frac{2}{\nu}\right)^{7/3} \left(-\frac{20231}{129360000}\tau^5 - \frac{86803}{49896000}\tau^2 + \frac{90907}{22176000}\frac{1}{\tau} \right. \\ \left. + \frac{12807}{1920000}\frac{1}{\tau^4}\right) + \cdots \end{aligned} \tag{5-72}$$

Similar asymptotic expansions for the characteristic frequencies k_s can be found from Eq. (5-52) for diffraction problems involving circular cylinders, elliptic cylinders, parabolic cylinders, ellipsoids of revolution, and paraboloids of revolution.

Before we conclude this discussion, we should emphasize the fact that these results are useful also in the extension of Olver's studies (Ref. 7) of the zeros of Bessel functions. For example, if we replace Eq. (5-59) by

$$J_\nu'(x_s) + \left(\frac{2}{x_s}\right) Z\, J_\nu(x_s) = 0 \tag{5-73}$$

we can use Eq. (5-60) and Eq. (5-61) to Solve for x_s by letting $\tau_s = \beta_s$, where $Ai'(-\beta_s) = 0$. If we consider the Neumann functions

$Y_\nu(x)$ [sometimes denoted by $N_\nu(x)$] and seek to solve for x_s defined by

$$Y_\nu{}'(x_s) + \left(\frac{2}{x_s}\right) Z\, Y_\nu(x_s) = 0 \qquad (5\text{-}74)$$

we again use Eq. (5-60) and Eq. (5-61), but now we must let

$$\tau = \delta_s \qquad (5\text{-}75)$$

where δ_s denotes the roots of

$$Bi'(-\delta_s) = 0 \qquad (5\text{-}76)$$

Tables of these roots have been prepared by Miller (Ref. 9) and Olver (Ref. 7).

5.6 The Normalization Integral

Let us now see what representations we can obtain for the normalization integral

$$N_s = N(\lambda_s) = \int_0^\infty \left[F(y,\ \lambda_s)\right]^2 dy \qquad (5\text{-}77)$$

which was defined in Eq. (4-20). For complex λ, the function $F(y,\lambda)$ will increase as y increases and the integral of the square of $F(y,\lambda)$ taken over y from 0 to ∞ will diverge. However, for certain assumptions on $f(y)$ in the complex plane, $F(y,\lambda)$ will behave like $w_{1,2}(t - k^{\frac{2}{3}} f_1^{\frac{1}{3}} y)$ for complex y. Under these conditions, the integral will converge on the ray $y = \ y \ \exp(\pm\frac{1}{3}\pi)$. Therefore, we can express the normalization integral in the form

$$N_s = \int_0^{\infty \exp(\pm i\,\pi/3)} [F\,(y,\lambda)]^2\, dy \qquad (5\text{-}78)$$

where the $\pm$ sign is used with $\exp(\mp ikct)$ time dependance (i.e., with $w_{1,2}(t - u)$. However, if we assume that k has a small imaginary part (for example, Im $k > 0$ for $\exp(-ikct)$ time dependance), then we can continue to use Eq. (5-77).

The techniques for the evaluation of N_s are well known. The classic method consists of considering two solutions $F(y,\lambda_i)$ and $F(y,\lambda_j)$ which are associated with different eigenvalues λ_s. We then assume that $i \neq j$ and write

$$\int_0^\infty \frac{d}{dy}\left[F(y,\lambda_i)\frac{d}{dy}F(y,\lambda_j) - F(y,\lambda_j)\frac{d}{dy}F(y,\lambda_i)\right]dy$$

$$= \left[F(y,\lambda_i)\frac{d}{dy}F(y,\lambda_j) - F(y,\lambda_j)\frac{d}{dy}F(y,\lambda_i)\right]_0^\infty$$

$$= \int_0^\infty \left[F(y,\lambda_i)\frac{d^2}{dy^2}F(y,\lambda_j) - F(y,\lambda_j)\frac{d^2}{dy^2}F(y,\lambda_i)\right]dy$$

$$= (\lambda_j - \lambda_i)\int_0^\infty F(y,\lambda_i)\,F(y,\lambda_j)\,dy \qquad (5\text{-}79)$$

We assume that the behavior of $f(y)$ is such that the evaluation at $y = \infty$ can be replaced by zero. We then have

$$(\lambda_i - \lambda_j)\int_0^\infty F(y,\lambda_i)\,F(y,\lambda_j)\,dy = F(0,\lambda_i)\frac{dF(0,\lambda_j)}{dy} - F(0,\lambda_j)\frac{dF(0,\lambda_i)}{dy} \qquad (5\text{-}80)$$

Since we assume that the boundary condition on $F(y,\lambda_s)$ is of the form given in Eq. (5-1c), namely

$$\frac{dF(0,\lambda)}{dy} + k^{2/3} f_1^{1/3}\, q\, F(0,\lambda) = 0$$

the right side of Eq. (5-80) is equal to zero and hence we have

$$\int_0^\infty F(y,\lambda_i)\,F(y,\lambda_j)\,dy = 0 \qquad (5\text{-}81)$$

provided $i \neq j$.

For $i = j$ we can use L'Hospital's rule and arrive at

$$N_s = \int_0^\infty \left[F(y, \lambda_s)\right]^2 dy = \left[\frac{\partial F(0, \lambda_i)}{\partial \lambda i} \frac{\partial F(0, \lambda_j)}{\partial y} - F(0, \lambda_j) \frac{\partial^2 F(0, \lambda_i)}{\partial \lambda_i \, \partial y}\right]_{\lambda_i = \lambda_j = \lambda_s}$$

$$= \frac{\partial F(0, \lambda_s)}{\partial \lambda_s} \frac{\partial F(0, \lambda_s)}{\partial y} - F(0, \lambda_s) \frac{\partial^2 F(0, \lambda_s)}{\partial \lambda_s \, \partial y} \qquad (5\text{-}82)$$

If we recall that we have defined

$$F(y, \lambda_s) = G(u, t_s)$$

$$y = \beta u = (k^{2/3} f_1^{1/3})^{-1} u$$

$$\lambda_s = k^2 f_o + k^{4/3} f_1^{2/3} t_s$$

we can also express Eq. (5-82) in the form

$$N_s = \left[k^{2/3} f_1^{1/3}\right]^{-1} \int_0^\infty \left[G(u, t_s)\right]^2 du \qquad (5\text{-}83)$$

where

$$\int_0^\infty \left[G(u, t_s)\right]^2 du = \frac{\partial G(0, t_s)}{\partial t_s} \frac{\partial G(0, t_s)}{\partial y} - G(0, t_s) \frac{\partial^2 G(0, t_s)}{\partial t_s \, \partial y} \qquad (5\text{-}84)$$

Fock (Ref.10) has shown how to arrive at the result given in Eq. (5-82) without the use of a result of the form of Eq. (5-80). Let us observe that from

$$\frac{d^2 F}{dy^2} + \left[- \lambda_s + k^2 f(y)\right] F = 0 \qquad (5\text{-}85)$$

it follows that

$$\frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial \lambda_s}\right) + \left[- \lambda_s + k^2 f(y)\right] \frac{\partial F}{\partial \lambda_s} = F \qquad (5\text{-}86)$$

If we multiply Eq. (5-85) by $\partial F/\partial\lambda_s$ and multiply Eq. (5-86) by F, we can form the difference of the two results and show that

$$F^2(y, \lambda_s) = F(y, \lambda_s)\frac{\partial^2}{\partial y^2}\left(\frac{\partial F(y, \lambda_s)}{\partial \lambda_s}\right) - \frac{\partial F(y, \lambda_s)}{\partial \lambda_s}\frac{\partial^2 F(y, \lambda_s)}{\partial y^2}$$

$$= \frac{\partial}{\partial y}\left[F(y, \lambda_s)\frac{\partial^2 F(y, \lambda_s)}{\partial y\, \partial \lambda_s} - \frac{\partial F(y, \lambda_s)}{\partial \lambda_s}\frac{\partial F(y, \lambda_s)}{\partial y}\right] \qquad (5\text{-}87)$$

The result given in Eq. (5-82) follows immediately from Eq. (5-87).

Fock (Ref. 10) also showed that Eq. (5-82) could be expressed in the form

$$N_s = -F^2(0, \lambda_s)\frac{\partial}{\partial \lambda_s}\left\{\frac{\dfrac{\partial F(0, \lambda_s)}{\partial y}}{F(o, \lambda_s)}\right\} \qquad (5\text{-}88)$$

From Eq. (5-1c) we find that the logarithmic derivative in Eq. (5-88) is equal to $-k^{\frac{2}{3}}f_1^{\frac{1}{3}}q$. Therefore, we can write

$$N_s = k^{2/3} f_1^{1/3} F^2(0, \lambda_s)\left[\frac{\partial \lambda_s}{\partial q}\right]^{-1} \qquad (5\text{-}89)$$

From Eq. (5-2) and the property

$$\frac{d\tau}{dq} = \frac{1}{\tau - q^2} \qquad (5\text{-}90)$$

we find that

$$\frac{\partial \lambda_s}{\partial q} = \frac{k^{4/3} f_1^{2/3}}{\tau_s - q^2}\left[1 + \sum_{n=1}^{\infty}\frac{\partial D_n(\tau_s, q)}{\partial \tau_s}\beta^n\right] \qquad (5\text{-}91)$$

This result is most useful when y and y_0 are zero. For this case we can express Eq. (4-13), i.e.,

$$U(x, y; x_o, y_o) = \sum_{s=1}^{\infty}\frac{\Psi^+(x_>, \lambda_s)\,\Psi^-(x_<, \lambda_s)}{W_s}\,\frac{F(y, \lambda_s)\,F(y_o, \lambda_s)}{N_s} \qquad (5\text{-}92)$$

$$U(x,0;x_0,0) = \sum_{s=1}^{\infty} \frac{\Psi^{+}(x_{>}, \lambda_s)\,\Psi^{-}(x_{<}, \lambda_s)}{k^{\frac{2}{3}} f_1^{\frac{1}{3}}\, W_s(\tau_s - q^2)} \left[1 + \sum_{n=1}^{\infty} \frac{\partial D_n(\tau_s, q)}{\partial \tau_s} \beta^n\right] \quad (5\text{-}93)$$

We can obtain another useful expression for N_s by using Eq. (4-18), i.e.,

$$\left[\frac{\partial F(y, \lambda_s)}{\partial y} + i\,k\,Z\,F(y, \lambda_s)\right]_{y=0} = 0 \quad (5\text{-}94)$$

to write

$$N_s = -\,F_1(0, \lambda_s)\,\frac{\partial}{\partial \lambda_s}\left[\frac{\partial F(0, \lambda_s)}{\partial y} + i\,k\,Z\,F(o, \lambda_s)\right] \quad (5\text{-}95)$$

If we express $F(y, \lambda_s)$ in the form

$$F(y, \lambda_s) = F_1(y, \lambda_s) = G(u, t_s) = H(u, t_s) = \left(\frac{d\eta}{du}\right)^{-1/2} w_1(-\eta) \quad (5\text{-}96)$$

we find that by defining

$$F_2(y, \lambda_s) = \left(\frac{d\eta}{du}\right)^{-1/2} w_2(-\eta) \quad (5\text{-}97)$$

and use of the Wronskian relation

$$w_1(-\eta)\,w_2'(-\eta) - w_1'(-\eta)\,w_2(-\eta) = 2i \quad (5\text{-}98)$$

that we have

$$F_1(y, \lambda)\,\frac{dF_2(y, \lambda)}{dy} - F_2(y, \lambda)\,\frac{dF_1(y, \lambda)}{dy} = -\,2i\,\frac{du}{dy} = -\,2i\,k^{2/3}\,f_1^{\,1/3} \quad (5\text{-}99)$$

Therefore, if

$$\left[\frac{dF_1}{dy} + i\,k\,Z\,F_1\right]_{y=0} = 0 \quad (5\text{-}100)$$

it follows that

$$\frac{dF_2(0, \lambda_s)}{dy} + i\,k\,Z\,F_2(0, \lambda_s) = -\frac{2i\,k^{2/3}\,f_1^{\,1/3}}{F_1(0, \lambda_s)} \tag{5-101}$$

If we use Eq. (5-101) in Eq. (5-95), we find that

$$\frac{1}{N_s} = \frac{1}{2i\,k^{2/3}\,f_1^{\,1/3}}\;\frac{\dfrac{\partial F_2(0, \lambda_s)}{\partial y} + i\,k\,Z\,F_2(0, \lambda_s)}{\dfrac{\partial}{\partial \lambda_s}\left[\dfrac{\partial F_1(0, \lambda_s)}{\partial y} + i\,k\,Z\,F_1(0, \lambda_s)\right]} \tag{5-102}$$

If we use Eq. (5-96) and Eq, (5-97) and define

$$q = i\left(\frac{k}{f_1}\right)^{1/3} Z$$

we can express Eq. (5-102) in the form

$$\frac{1}{N_s} = \frac{1}{2i\,k^{2/3}\,f_1^{\,1/3}}\;\frac{-\left(\dfrac{d\eta}{du}\right)^{1/2}\left[w_2'(-\eta) + \dfrac{1}{2}\dfrac{\dot\eta}{\dot\eta^2}\,w_2(-\eta) - \dfrac{q}{\dot\eta}\,w_2(-\eta)\right]}{\dfrac{\partial}{\partial \lambda_s}\left\{-\left(\dfrac{d\eta}{du}\right)^{1/2}\left[w_1'(-\eta) + \dfrac{1}{2}\dfrac{\dot\eta}{\dot\eta^2}\,w_1(-\eta) - \dfrac{q}{\dot\eta}\,w_1(-\eta)\right]\right\}} \tag{5-103}$$

where the dots denote differentiation with respect to u. From Eq. (5-100), we have

$$w_1'(-\eta) + \frac{1}{2}\frac{\dot\eta}{\dot\eta^2}\,w_1(-\eta) - \frac{q}{\dot\eta}\,w_1(-\eta) = 0 \tag{5-104}$$

This property permits us to express Eq. (5-103) in the form

$$\frac{1}{N_s} = \frac{1}{2i\,k^{2/3}\,f_1^{\,1/3}}\;\frac{w_2'(-\eta) + \dfrac{1}{2}\dfrac{\dot\eta}{\dot\eta^2}\,w_2(-\eta) - \dfrac{8}{\dot\eta}\,w_2(-\eta)}{\dfrac{\partial}{\partial \lambda_s}\left[w_1'(-\eta) + \dfrac{1}{2}\dfrac{\dot\eta}{\dot\eta^2}\,w_1(-\eta) - \dfrac{8}{\dot\eta}\,w_1(-\eta)\right]} \tag{5-105}$$

From the Wronskian property expressed by Eq. (5-98) we see that for the cases of $q = 0$ and $q = \infty$ that the reciprocal of N_s can be expressed in the elegant forms

$$\left[\frac{1}{N_s}\right]_{q=0} = \frac{1}{k^{2/3} f_1^{1/3}} \left[\frac{1}{\eta\left[w_1(-\eta)\right]^2} \frac{\partial\lambda_s}{\partial\eta}\right]_{\eta=-\tau_s^o}$$

$$= -\frac{1}{k^{2/3} f_1^{1/3}} \frac{1}{\tau_s^o\left[w_1(\tau_s^o)\right]^2} \frac{\partial\lambda_s}{\partial\tau_s^o}$$

$$= -\frac{k^{2/3} f_1^{1/3}}{\tau_s^o\left[w_1(\tau_s^o)\right]^2} \frac{\partial\lambda_s}{\partial\tau_s^o} \tag{5-106a}$$

and

$$\left[\frac{1}{N_s}\right]_{q=\infty} = \frac{1}{k^{2/3} f_1^{1/3}} \left[\frac{1}{\left[w'(-\eta)\right]^2} \frac{\partial\lambda_s}{\partial\eta}\right]_{\eta=-\tau_s^\infty}$$

$$= \frac{1}{k^{2/3} f_1^{1/3}} \frac{1}{\left[w_1'(\tau_s^\infty)\right]^2} \frac{\partial\lambda_s}{\partial\tau_s^\infty} \tag{5-106b}$$

We can evaluate $w_1(\tau_s^0)$ and $w_1'(\tau_s^\infty)$ by using the values of $Ai(-\beta_s)$ and $Ai'(-\alpha_s)$ given in Tables 2-1 and 2-2. The relations which we need are

$$w_1'(\tau_s^\infty) = -2\sqrt{\pi}\exp(-i\,\pi/6)\,Ai'(-\alpha_s) \tag{5-107a}$$

$$w_1(\tau_s^0) = 2\sqrt{\pi}\exp(i\,\pi/6)\,Ai(-\beta_s) \tag{5-107b}$$

$$\tau_s^\infty = \alpha_s \exp(i\,\pi/3) \tag{5-107c}$$

$$\tau_s^o = \beta_s \exp(i\,\pi/3) \tag{5-107d}$$

We can then write

$$\left[\frac{1}{N_s}\right]_{q=0} = \frac{1}{k^{2/3} f_1^{1/3}} \frac{1}{4\pi} \frac{1}{\beta_s\left[Ai(-\beta_s)\right]^2} \frac{\partial\lambda_s}{\partial\beta_s} \tag{5-108a}$$

$$\left[\frac{1}{N_s}\right]_{q=\infty} = \frac{1}{k^{2/3} f_1^{1/3}} \frac{1}{4\pi} \frac{1}{\left[Ai'(-\alpha_s)\right]^2} \frac{\partial\lambda_s}{\partial\alpha_s} \tag{5-108b}$$

Olver (Ref. 11) has shown that the roots α_s and β_s of the Airy functions possess an interesting and a useful property. Since <u>s</u> takes on only integral values, we will have to give a meaning to the <u>Olver relations</u>

$$\frac{d\beta_s}{ds} = \frac{1}{\beta_s\left[Ai(-\beta_s)\right]^2} \qquad (5\text{-}109a)$$

$$\frac{d\alpha_s}{ds} = \frac{1}{\left[Ai'(-\alpha_s)\right]^2} \qquad (5\text{-}109b)$$

Let $\alpha(s)$ be a continuous function of s, and let the Airy function have $\alpha(s)$ as its argument in the following sense

$$y(s) = Ai[-\alpha(s)]$$

Then the property displayed as Eq. (5-109b) is to be interpreted in the sense

$$\left[\frac{d\alpha(s)}{ds}\right]_{s=n} = \frac{1}{\{[-dy/d\alpha]^2\}_{s=n}} = \frac{1}{\{Ai'[-\alpha(n)]\}^2}$$

where n is an integer.

The Olver relations permit us to express both Eqs. (5-106a) and (5-106b) in the single relation which includes both the case $q = 0$ and the case $q = \infty$.

$$\frac{1}{N_s} = \frac{1}{4\pi k^{2/3} f_1^{1/3}} \frac{\partial\lambda_s}{\partial s} \qquad (5\text{-}110)$$

The Olver relation can be generalized to include the case of the roots τ_s defined by

$$w_1'(\tau_s) - q\, w_1(\tau_s) = 0$$

It can be shown that the generalized Olver relation for these roots is of the form

$$\frac{\partial \tau_s}{\partial s} = \frac{4\pi}{\left[w_1'(\tau_s)\right]^2 - \tau_s\left[w_1(\tau_s)\right]^2} \tag{5-111}$$

If we use Eq. (5-111) in the representation

$$\frac{1}{N_s} = \frac{1}{k^{2/3} f_1^{1/3}} \frac{1}{\left[w_1'(\tau_s)\right]^2 - \tau_s\left[w_1(\tau_s)\right]^2} \frac{\partial \lambda_s}{\partial \tau_s} \tag{5-112}$$

we see that Eq. (5-110) holds for arbitrary values of q.

If we use Eq. (5-2) along with Eq, (5-112), we can express Eq. (5-92) in the form

$$U(x, y; x_0, y_0) = k^{2/3} f_1^{1/3} \sum_{s=1}^{\infty} \left\{ \frac{\Psi^+(x_>, \lambda_s)\,\Psi^-(x_<, \lambda_s)}{W_s} \right.$$

$$\left. \frac{F(y, \lambda_s)\,F(y_0, \lambda_s)}{\left[w'(\tau_s)\right]^2 - \tau_s\left[w(\tau_s)\right]^2} \left[1 + \sum_{n=1}^{\infty} \frac{\partial D_n(\tau_s, q)}{\partial \tau_s} \beta^n\right] \right\} \tag{5-113}$$

where $F(y,\lambda_s)$ is assumed to be defined as in Eq. (5-96)

Eq. (5-110) can be used to express $U(x,y,;x_0,y_0)$ in the form

$$U(x, y; x_0, y_0) = \frac{1}{4\pi k^{2/3} f_1^{1/3}} \sum_{s=1}^{\infty} \left\{ \frac{\Psi^+(x_>, \lambda_s)\,\Psi^-(x_<, \lambda_s)}{W_s} \right.$$

$$\left. F(y, \lambda_s)\,F(y_0, \lambda_s) \frac{\partial \lambda_s}{\partial s} \right\} \tag{5-114}$$

This form is very convenient for use with the Euler-Maclaurin summation formula

$$\left(\sum_{s=N}^{\infty} - \int_{s=N}^{\infty} ds\right) f(s) = \frac{1}{2} f(N) - \frac{1}{12}\frac{d}{dN} f(N) + \frac{1}{720}\frac{d^3}{dN^3} f(N) - \frac{1}{30240}\frac{d^5}{dN^5} f(N)$$

$$+ \cdots (-1)^n \frac{B_n}{(2n)!} \frac{d^{2n-1}}{dN^{2n-1}} f(N) + \cdots \tag{5-115}$$

where B_n denotes the Bernoulli numbers

$$B_1 = \frac{1}{6}, \; B_2 = \frac{1}{30}, \; B_3 = \frac{1}{42}, \; B_4 = \frac{1}{30}, \; B_5 = \frac{5}{66}, \; B_6 = \frac{691}{2730}, \; B_7 = \frac{7}{6}, \; \ldots$$

or the Gregory formula of numerical integration

$$\begin{aligned}\left(\sum_{s=N}^{\infty} - \int_{s=N}^{\infty} ds\right) f(s) &= \frac{1}{2} f(N) - \frac{1}{12}\Delta f(N) + \frac{1}{24}\Delta^2 f(N) - \frac{19}{720}\Delta^3 f(N) \\ &+ \frac{3}{160}\Delta^4 f(N) - \frac{863}{60480}\Delta^5 f(N) + \frac{275}{24192}\Delta^6 f(N) \\ &- \frac{33953}{3628800}\Delta^7 f(N) + \frac{8183}{1036800}\Delta^8 f(N) \\ &- \frac{3250433}{479001600}\Delta^9 f(N) + \frac{4671}{788480}\Delta^{10} f(N) \\ &- \frac{13695779093}{2615348736000}\Delta^{11} f(N) + \cdots \end{aligned} \tag{5-116}$$

where the foward differences are defined by

$$\Delta f(N) = f(N+1) - f(N)$$

$$\Delta^2 f(N) = f(N+2) - 2f(N+1) + f(N)$$

$$\Delta^{n+1} f(N) = \Delta^n f(N+1) - \Delta^n f(N)$$

Either of Eqs. (5-115) or (5-116) can be used to express the wave function $U(x,y;x_0,y_0)$ in the form

$$\begin{aligned} U(x,y;\, x_o,y_o) &= \frac{1}{4\pi k^{2/3} f_1^{1/3}} \int_{k^2 f_o}^{\infty} \frac{\Psi^+(x_>,\lambda)\,\Psi^-(x_<,\lambda)}{W(\lambda)} F(y,\lambda)\, F(y_o,\lambda)\, d\lambda \\ &- \frac{1}{4\pi k^{2/3} f_1^{1/3}} \int_{k^2 f_o}^{\lambda_N} \frac{\Psi^+(x_>,\lambda)\,\Psi^-(x_<,\lambda)}{W(\lambda)} F(y,\lambda)\, F(y_o,\lambda)\, d\lambda \\ &+ \frac{1}{4\pi k^{2/3} f_1^{1/3}} \sum_{s=1}^{N-1} \frac{\Psi^+(x_>,\lambda_s)\,\Psi^-(x_<,\lambda_s)}{W_s} F(y,\lambda_s)\, F(y_o,\lambda_s) \frac{d\lambda_s}{ds} \\ &+ \frac{1}{4\pi k^{2/3} f_1^{1/3}} \left(\sum_{s=N}^{\infty} - \int_{s=N}^{\infty} ds\right) \frac{\Psi^+(x_>,\lambda_s)\,\Psi^-(x_<,\lambda_s)}{W_s} F(y,\lambda_s)\, F(y_o,\lambda_s) \frac{d\lambda_s}{ds} \end{aligned} \tag{5-117}$$

The limits on the integral of the form

$$\int_{k^2 f_o}^{\infty} \left[\cdot\cdot\cdot\right] d\lambda$$

are used as a shorthand way of specifying the path of integration. The limit ∞ which appears as the upper limit on this integral is to be interpreted in the sense

$$\int_{k^2 f_o}^{\infty} \left[\cdot\cdot\cdot\right] d\lambda = \lim_{s \to \infty} \int_{k^2 f_o}^{\lambda_s} \left[\cdot\cdot\cdot\right] d\lambda = \int_c \left[\cdot\cdot\cdot\right] d\lambda$$

where c denotes a contour which starts at $\lambda = k^2 f_0$ and goes to infinity along the root locus along which one finds the eigenvalues λ_s. With an $\exp(-i\omega t)$ time dependance this will usually be a contour which starts at a point $\lambda = k^2 f_0$ on the positive real axis and then passes upward into the first quadrant and bends so as to pass into the second quadrant and go off to infinity (with the imaginary part of $\lambda > 0$) along a locus such that the angle between a point on the locus and the negative real λ-axis tends to zero.

The only case of such a locus which has been extensively studied is that of the Hankel functions. Magnus and Kotin (Ref. 12) have shown that for the roots ν_s defined by

$$H^{(1)}_{\nu_s}(z) = H^{(1)}_{\nu_s}(re^{i\phi}) = 0$$

that

$$\nu_s = \alpha_s + i\,\beta_s$$

where

$$\alpha_s = \pi \left(\frac{\pi}{2} - \phi\right) \left(s + \frac{1}{4}\right) \left[\log \frac{(s + 1/4)\,\pi}{er}\right]^{-2}$$

$$\beta_s = \pi\,(s + 1/4) \left[\log \frac{(s + 1/4)\,\pi}{er}\right]^{-1}$$

Let us now make some remarks concerning the integral

$$U_f(x, y; x_0, y_0) = \frac{1}{4\pi k^{2/3} f_1^{1/3}} \int_{k^2 f_0}^{\infty} \frac{\Psi^+(x_>, \lambda)\, \Psi^-(x_<, \lambda)}{W(\lambda)} F(y, \lambda)\, F(y_0, \lambda)\, d\lambda \quad (5\text{-}118)$$

This integral can generally be evaluated by approximating it by a Fresnel integral when

$$\delta = \left\{ \int_{x_<}^{x_>} \frac{du}{\sqrt{g(u) + f_0}} - \left(\int_0^{y} + \int_0^{y_0} \right) \frac{dv}{\sqrt{f(v) - f_0}} \right\} > 0 \quad (5\text{-}119)$$

For $\delta < 0$, we can generally write

$$U_f(x, y; x_0, y_0) = \frac{1}{4\pi k^{2/3} f_1^{1/3}} \int_{-\infty}^{\infty} \left[\cdot \cdot \cdot \right] d\lambda - \frac{1}{4\pi k^{2/3} f_1^{1/3}} \int_{-\infty}^{k^2 f_0} \left[\cdot \cdot \cdot \right] d\lambda \quad (5\text{-}120)$$

where the integral

$$\int_{-\infty}^{\infty} \left[\cdot \cdot \cdot \right] d\lambda$$

can be evaluated by the method of stationary phase and the integral

$$\int_{-\infty}^{k^2 f_0} \left[\cdot \cdot \cdot \right] d\lambda$$

can be approximated by a Fresnel integral.

The stationary phase evaluation of the integral taken between the limits $-\infty$ and $+\infty$ generally yields the "free space field" which is present when there is no reflecting surface at $y = 0$.

Before leaving the subject of these integral representations for the wave function $U(x,y;x_0,y_0)$, we should remark that it is also possible to express this function in the form of a Fourier integral

$$U(x, y; x_0, y_0) = \frac{1}{4\pi k^{2/3} f_1^{1/3}} \int_{-\infty}^{\infty} \frac{\Psi^+(x_>, \lambda)\, \Psi^-(x_<, \lambda)}{W(\lambda)} \Bigg[F_1(y_>, \lambda)\, F_2(y_<, \lambda) - \frac{\dfrac{\partial F_2(0,\lambda)}{\partial y} + i k Z F_2(0,\lambda)}{\dfrac{\partial F_1(0,\lambda)}{\partial y} + i k Z F_1(0,\lambda)} F_1(y, \lambda)\, F_1(y_0, \lambda) \Bigg] d\lambda \qquad (5\text{-}121)$$

If we evaluate this expression in the form of the residue series given in Eq. (5-92) we can readily verify the result given in Eq. (5-102).

5.7 Direct Integration of N_s

In our previous discussions, we arrived at some elegant representations for the normalization integral. Let us now consider an alternative procedure which is more direct and which contains some interesting analysis.

We want to evaluate the integral

$$\int_0^{\infty} \left[G(u, t_s)\right]^2 du = k^{2/3} f_1^{1/3} \int_0^{\infty} \left[F(y, \lambda_s)\right]^2 dy \qquad (5\text{-}122)$$

where $G(u, t_s)$ can be obtained from Eq. 4-148). Thus

$$G(u, t) = w(t - u) - \frac{\beta h_2}{15} \left[(3u + 2t)\, w(t - u) + (3u^2 + 4ut + 8t^2)\, w'(t - u)\right] \qquad (5\text{-}123)$$

Let us define

$$\zeta = t - u \quad , \quad d\zeta = - du$$

and observe that

$$\int_0^{\infty} \left[G(u, t)\right]^2 du = \int_{-\infty}^{t} w^2(\zeta)\, d\zeta - \frac{2\beta h_2}{15} \int_{-\infty}^{t} \Big\{(5t - 3\zeta)\, w^2(\zeta) + (15t^2 - 10\,\zeta t + 3\zeta^2)\, w(\zeta)\, w'(\zeta)\Big\} d\zeta \qquad (5\text{-}124)$$

Since

$$\frac{d}{d\zeta}\left\{\zeta w^2(\zeta) - w'^2(\zeta)\right\} = w^2(\zeta) + 2\zeta w' w - 2\zeta w w' = w^2(\zeta) \tag{5-125a}$$

$$\begin{aligned}\frac{d}{d\zeta}\left\{\frac{1}{3}\zeta^2 w^2 - \frac{1}{3}\zeta w'^2(\zeta) + \frac{1}{3} w w'\right\} &= \frac{2\zeta}{3} w^2 + \frac{2\zeta^2}{3} w w' - \frac{1}{3} w'^2 - \frac{2}{3}\zeta^2 w w' \\ &\quad + \frac{1}{3} w'^2 + \frac{1}{3}\zeta w^2 = \zeta w^2(\zeta)\end{aligned} \tag{5-125b}$$

and

$$\begin{aligned}\frac{d}{d\zeta}\left\{\frac{1}{6}\zeta^2 w^2(\zeta) + \frac{1}{3}\zeta w'^2(\zeta) - \frac{1}{3} w(\zeta) w'(\zeta)\right\} &= \frac{1}{3}\zeta w^2(\zeta) + \frac{1}{3}\zeta^2 w w' + \frac{1}{3} w'^2 \\ &\quad + \frac{2}{3}\zeta^2 w w' - \frac{1}{3} w'^2 - \frac{1}{3}\zeta w^2 \\ &= \zeta^2 w(\zeta) w'(\zeta)\end{aligned} \tag{5-125c}$$

we have the following integrals

$$\int w^2(\zeta)\, d\zeta = \zeta w^2(\zeta) - w'^2(\zeta) \tag{5-126a}$$

$$\int \zeta w^2(\zeta)\, d\zeta = \frac{1}{3}\zeta^2 w^2(\zeta) - \frac{1}{3}\zeta w'^2(\zeta) + \frac{1}{3} w(\zeta) w'(\zeta) \tag{5-126b}$$

$$\int w'(\zeta) w(\zeta)\, d\zeta = \frac{1}{2} w^2(\zeta) \tag{5-126c}$$

$$\int \zeta^2 w(\zeta) w'(\zeta)\, d\zeta = \frac{1}{6}\zeta^2 w^2(\zeta) + \frac{1}{3}\zeta w'^2(\zeta) - \frac{1}{3} w(\zeta) w'(\zeta) \tag{5-126d}$$

Therefore, we find that

$$\begin{aligned}\int_0^\infty \left[G(u, t)\right]^2 du &= t w^2(t) - w'^2(t) - \frac{2\beta h_2}{15}\left\{5t\left(t w^2(t) - w'^2(t)\right) - 3\left[\frac{1}{3} t^2 w^2(t)\right.\right. \\ &\quad \left. - \frac{1}{3} t w'^2(t) + \frac{1}{3} w(t) w'(t)\right] \frac{15}{2} t^2 w^2(t) - 10t \frac{w'^2(t)}{2} \\ &\quad \left. + 3\left[\frac{1}{6} t^2 w^2(t) + \frac{1}{3} t w'^2(t) - \frac{1}{3} w(t) w'(t)\right]\right\} \\ &= w^2(t)\left(t - q^2\right) - \frac{2\beta h_2}{15} w^2(t)\left\{12t^2 - 8tq^2 - 2q\right\} + \cdots\end{aligned} \tag{5-127}$$

since

$$w_1'(t) - q\, w_1(t) = 0$$

We can readily verify this result by using Eq. (5-82) to write

$$\int_0^\infty \left[G(u,\ t)\right]^2 du = \left[\frac{\partial G}{\partial t}\frac{\partial G}{\partial u} - G\,\frac{\partial^2 G}{\partial t \partial u}\right]_{u=0} \tag{5-128}$$

where

$$\left[\frac{\partial G}{\partial t}\ \frac{\partial G}{\partial u}\right]_{u=0} = -w'^2(t) + \frac{\beta h_2}{15}\left[-\,w\,w' + 16t\,w'^2 + 16\,t^3\,w\,w'\right] \tag{5-129a}$$

$$\left[-\,G\,\frac{\partial^2 G}{\partial t \partial u}\right]_{u=0} = t\,w^2(t) + \frac{\beta h_2}{15}\left\{5w\,w' - 24\,t^2 w^2 - 16\,t^3\,w\,w'\right\} \tag{5-129b}$$

It is readily verified that these results confirm Eq. (5-127).

In his paper, Fock (Ref. 10) defined D(t) by means of

$$\int_0^\infty \left[G(u,\ t)\right]^2 du = \left[G(o,\ t)\right]^2 D(t) \tag{5-130}$$

Since

$$\left[G(0,\ t)\right]^2 = \left\{w(t) - \frac{\beta h_2}{15}\left[2t\,w(t) + 8t^2\,w'(t)\right]\right\}^2 = w^2(t)\left\{1 - \frac{\beta h_2}{15}\,(2t + 8t^2 q)\right\} + \cdot \tag{5-131}$$

we find that

$$D(t) = \frac{(t - q^2) - \frac{2\beta h_2}{15}\,(12t^2 - 8t\,q^2 - 2q) + \cdots}{1 - \frac{2\beta h_2}{15}\,(2t + 8t^2\,q) + \cdots}$$

$$= (t - q^2) + \beta h_2\left\{\frac{16}{15}\,t^2 q(t - q^2) - \frac{4}{3}\,t\,(t - q^2) - \frac{8}{15}\,tq^2 + \frac{4}{15}q\right\} \tag{5-132}$$

Since Fock gave D(t) to be [see his Eq. 5.09]

$$D(t) = (t - q^2)\,(1 - 4/3\,\beta h_2 t) + 2/3\,\beta h_2\,q \tag{5-133}$$

we must conclude that his result is in error.

5.8 Eigenvalues Derived from the Langer Approximation

In this section we want to discuss a technique which may be of some value in obtaining more useful numerical techniques for the determination of the eigenvalues. Let us assume that the function $G(u,t)$ which satisfies Eq. (4-77), namely

$$\frac{d^2G}{du^2} + [(-t+u) + \beta\, h_2 u^2 + \beta^2\, h_3 u^3 + \cdots]\, G = 0 \tag{5-134}$$

is to be approximated by an expression of the form

$$G(u,\, t) \sim \left(\frac{d\eta_o}{du}\right)^{-\frac{1}{2}} w_{1,2}\,(-\eta_o) \tag{5-135}$$

where

$$\tfrac{2}{3}\,\eta_o^{\,3/2} = \int_{u_1}^{u} \sqrt{(v-t) + \beta h_2 v^2 + \beta^2 h_3 v^3 + \cdots}\;dv \tag{5-136a}$$

for $u > u_1$, and

$$\tfrac{2}{3}\,(-\eta_o)^{3/2} = \int_{u}^{u_1} \sqrt{(t-v) - \beta h_2 v^2 - \beta h_3 v^3 - \cdots}\;dv \tag{5-136b}$$

for $u < u_1$, where u_1 is the turning point defined by

$$t - u_1 - \beta h_2 u_1^{\,2} - \beta^2 h_3 u_1^{\,3} - \cdots = 0 \tag{5-137}$$

In Section 6 we shall refer to this type of approximation as the "Langer approximation."

Let τ denote the root of

$$w_{1,2}(\tau) = 0 \tag{5-138}$$

Then, in order to solve for the eigenvalue t defined by

$$G\,(0,\, t) \sim \left[\left(\frac{d\eta_o}{du}\right)^{-\frac{1}{2}} w_{1,2}(-\eta_o)\right]_{u=0} \tag{5-139}$$

we must invert the equation

$$\tfrac{2}{3}\,\tau^{3/2} = \int_0^{u_1} \sqrt{t - u - \beta h_2 u^2 - \beta^2 h_3 u^3 - \beta^3 h_4 u^4 - \beta^4 h_5 u^5 - \cdots}\; du \qquad (5\text{-}140)$$

and solve for t in terms of τ. In order to carry out these operations, let us define

$$x = t - u - \beta\, h_2\, u^2 - \beta^2\, h_3\, u^3 + \cdots \qquad (5\text{-}141)$$

and express the integral in the form

$$\tfrac{2}{3}\,\tau^{3/2} = \int_0^{t} \sqrt{x}\, \left(-\frac{du}{dx}\right) dx \qquad (5\text{-}142)$$

Since

$$\begin{aligned} u = {} & (t - x) - \beta\, h_2\, (t - x)^2 - \beta^2\, (h_3 - 2h_2^{\,2})\, (t - x)^3 \\ & - \beta^3\, (h_4 - 5h_2\, h_3 + 5h_2^{\,3})\, (t - x)^4 \\ & - \beta^4\, (h_5 - 6h_2\, h_4 - 3h_3^{\,2} + 21h_2^{\,2}\, h_3 - 14h_2^{\,4})\, (t - x)^5 + \cdots \end{aligned} \qquad (5\text{-}143)$$

we find that

$$\begin{aligned} -\frac{du}{dx} = {} & 1 - 2\beta\, h_2\, (t - x) - 3\beta^2\, (h_3 - 2h_2^{\,2})\, (t - x)^2 \\ & - 4\beta^3\, (h_4 - 5h_2\, h_3 + 5h_2^{\,3})\, (t - x)^3 \\ & - 5\beta^4\, (h_5 - 6h_2\, h_4 - 3h_3^{\,2} + 21h_2^{\,2}\, h_3 - 14h_2^{\,4})\, (t - x)^4 + \cdots \end{aligned} \qquad (5\text{-}144)$$

Therefore, if we use the results

$$\int_0^t \sqrt{x}\, dx = \tfrac{2}{3}\, t^{3/2}$$

$$\int_0^t \sqrt{x}\, (t - x)\, dx = \frac{4}{15}\, t^{5/2}$$

$$\int_0^t \sqrt{x}\, (t - x)^2\, dx = \frac{16}{105}\, t^{7/2}$$

$$\int_0^t \sqrt{x}\, (t - x)^3\, dx = \frac{172}{315}\, t^{9/2}$$

$$\int_0^t \sqrt{x}\, (t - x)^4\, dx = \frac{256}{3465}\, t^{11/2}$$

we arrive at

$$\frac{2}{3}\tau^{3/2} = \frac{2}{3}t^{3/2} - \frac{8}{15}\beta\, h_2\, t^{5/2} - \frac{16}{35}\beta^2 (h_3 - 2h_2^{\,2})\, t^{7/2} - \frac{128}{315}\beta^3 (h_4 - 5h_2 h_3$$
$$+ 5h_2^{\,3})\, t^{9/2} - \frac{256}{693}\beta^4 (h_5 - 6h_2 h_4 - 3h_3^{\,2} + 21h_2^{\,2} h_3$$
$$- 14h_2^{\,4})\, t^{11/2} \qquad (5\text{-}145)$$

In order to invert Eq. (5-145), we assume that t can be represented by a series of the form

$$t = \tau + \beta L_1 + \beta^2 L_2 + \beta^3 L_3 + \beta^4 L_4 + \cdots \qquad (5\text{-}146)$$

we can then use the binomial theorem to form the following expressions:

$$t^{3/2} = \tau^{3/2}\Big\{1 + \beta\left(\frac{3}{2}\frac{L_1}{\tau}\right) + \beta^2\left(\frac{3}{2}\frac{L_2}{\tau} + \frac{3}{8}\frac{L_1^{\,2}}{\tau^2}\right)$$
$$+ \beta^3\left(\frac{3}{2}\frac{L_3}{\tau} + \frac{3}{4}\frac{L_1 L_2}{\tau^2} - \frac{1}{16}\frac{L_1^{\,3}}{\tau^3}\right)$$
$$+ \beta^4\left(\frac{3}{2}\frac{L_4}{\tau} + \frac{3}{4}\frac{L_1 L_3}{\tau^2} + \frac{3}{8}\frac{L_2^{\,2}}{\tau^2} - \frac{3}{16}\frac{L_1^{\,2} L_2}{\tau^3} + \frac{3}{128}\frac{L_1^{\,4}}{\tau^4}\right) + \cdots\Big\} \qquad (5\text{-}147a)$$

$$t^{5/2} = \tau^{5/2}\Big\{1 + \beta\left(\frac{5}{2}\frac{L_1}{\tau}\right) + \beta^2\left(\frac{5}{2}\frac{L_2}{\tau} + \frac{15}{8}\frac{L_1^{\,2}}{\tau^2}\right)$$
$$+ \beta^3\left(\frac{5}{2}\frac{L_3}{\tau} + \frac{15}{4}\frac{L_1 L_2}{\tau^2} + \frac{5}{16}\frac{L_1^{\,3}}{\tau^3}\right) + \cdots\Big\} \qquad (5\text{-}147b)$$

$$t^{7/2} = \tau^{7/2}\Big\{1 + \beta\left(\frac{7}{2}\frac{L_1}{\tau}\right) + \beta^2\left(\frac{7}{2}\frac{L_2}{\tau} + \frac{35}{8}\frac{L_1^{\,2}}{\tau^2}\right) + \cdots\Big\} \qquad (5\text{-}147c)$$

$$t^{9/2} = \tau^{9/2}\Big\{1 + \beta\left(\frac{9}{2}\frac{L_1}{\tau}\right) + \cdots\Big\} \qquad (5\text{-}147d)$$

If we now use Eq. (5-147) in Eq. (5-145), we find that

$$\frac{2}{3}t^{3/2} = \frac{2}{3}\tau^{3/2} + \beta\,(L_1\tau^{1/2} - \frac{8}{15}h_2\tau^{5/2}) + \beta^2\,(L_2\tau^{1/2} + \frac{1}{4}L_1{}^2\tau^{-1/2}$$
$$- \frac{4}{3}h_2L_1\tau^{3/2} - \frac{16}{35}h_3\tau^{7/2} + \frac{32}{35}h_2{}^2\tau^{7/2}) + \beta^3\,(L_3\tau^{1/2}$$
$$+ \frac{1}{2}L_1L_2\tau^{-1/2} - \frac{1}{24}L_1{}^3\tau^{-3/2} - \frac{4}{3}h_2L_2\tau^{3/2} - h_2L_1{}^2\tau^{1/2}$$
$$- \frac{8}{5}h_3L_1\tau^{5/2} + \frac{16}{5}h_2{}^2L_1\tau^{5/2} + \frac{688}{63}h_2h_3\tau^{9/2} - \frac{688}{315}h_4\tau^{9/2}$$
$$- \frac{688}{63}h_2{}^3\,\tau^{9/2}) + \cdots \tag{5-148}$$

Since the coefficients of β^n, $n > 1$, must vanish, we find that

$$L_1 = \frac{8}{15}h_2\,\tau^2 \tag{5-149a}$$

$$L_2 = (\frac{16}{35}h_3 - \frac{48}{175}h_2{}^2)\,\tau^3 \tag{5-149b}$$

$$L_3 = (\frac{128}{315}h_4 - \frac{11968}{17325}h_2\,h_3 + \frac{22912}{70875}h_2{}^3)\,\tau^4 \tag{5-149c}$$

If we compare these value of $L_n(\tau)$ with the values $D_n(\tau)$ for $q = \infty$ in Eq. (5-4) we find that these are precisely the terms which contain the highest power of τ. The same thing is true concerning Eq. (5-7) in which $D_n(\tau)$ are given for $q = 0$. This result suggests a valuable means of extending the computation of λ_s when s (or mod τ) is relatively large. Let T_s denote the solution of Eq. (5-145). Use the expansion

$$T_s = \tau_s + \beta\,L_1(\tau_s) + \beta^2\,L_2(\tau_s) + \beta^3\,L_3(\tau_s) + \cdots \tag{5-150}$$

to compute an approximate value of T_s. Then use this as an approximation in Eq. (5-145) and use numerical methods to fins a more accurate value of T_s. Then compute λ_s by replacing Eq. (5-2) by the equation

$$\lambda_s = k^2 f_o + k^{4/3} f_1{}^{2/3}\left\{T_s + \sum_{n=1}^{\infty}\left[D_n(\tau_s, q) - L_n(\tau_s)\right]\beta^n\right\} \tag{5-151}$$

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Section 6

HEIGHT GAIN FUNCTIONS FOR LARGE HEIGHTS

6.1 The Langer Asymptotic Estimate

The representation

$$F(y, \lambda_s) = \left(\frac{d\eta}{du}\right)^{-\frac{1}{2}} w_1(-\eta) \qquad (6\text{-}1)$$

where η is defined by Eq. (4-118), i.e,

$$\eta = u - t_s + \frac{\beta h_2}{15}\left[3\,u^2 + 4\,ut - 8\,t^2\right] + O\,(\beta^2) \qquad (6\text{-}2)$$

is only useful for small values of u. Since

$$y = \frac{1}{k^{2/3}\,f_1^{1/3}}\,u$$

this requires that we **restrict** the use of this form of representation for small heights. However, for large heights we can still use Eq. (6-1) if we define η by means of Eq. (4-51). Thus, if we let

$$\frac{2}{3}\,\zeta^{3/2} = \int_{y_1}^{y} \sqrt{k^2 f(v) - \lambda}\;dv \qquad (6\text{-}3)$$

we find that

$$\eta = \zeta + \left(\frac{3}{2}\right)^{2/3}\frac{\alpha}{k^{4/3}} + \frac{1}{3}\,\frac{\alpha_0}{k^2}\,\zeta + \frac{1}{5}\left(\frac{2}{3}\right)^{2/3}\frac{\alpha_1}{k^{8/3}}\,\zeta^2 + O\,(k^{-10/3}) \qquad (6\text{-}4)$$

where α_0, α_1, and α_2 are defined in Eqs. (4-37) through (4-39). The approximation

$$\eta \approx \zeta \qquad (6\text{-}5)$$

is often called the Langer asymptotic estimate in recognition of Langer's (Ref. 1) research on problems of this type.

The Harvard University Computation Laboratory tables (Ref. 2) can be used to evaluate $w_{1,2}(-\eta)$ for mod $\eta < 6$. These tables give eight decimal values of

$$h_{1,2}(x + iy) = \frac{\sqrt[6]{12}}{\sqrt{\pi}} \exp\left(\mp i \frac{2}{3}\pi\right) w_{1,2}(-x - iy) \qquad (6\text{-}6)$$

and its derivatives for $\Delta x = \Delta y = 0.1$ and $x^2 + y^2 < 36$. The relation

$$w_{1,2}(x + iy) = \frac{\sqrt{\pi}}{\sqrt[6]{12}} \exp\left(\pm i \frac{2}{3}\pi\right) \left[\text{complex conjugate of } h_{2,1}(-x + iy)\right] \qquad (6\text{-}7)$$

is valuable in connection with the use of these tables since the tabulation of $h_{1,2}(x + iy)$ and $h_{1,2}'(x + iy)$ only for $y > 0$.

For mod $\eta >> 1$, we can compute $w_1(-\eta)$ by using the phase and amplitude functions defined and tabluated by Miller (Ref. 3). Thus, we let

$$w_{1,2}(-\eta) = \sqrt{\pi}\, F(-\eta) \exp\left[\pm i\, \chi(-\eta)\right] \qquad (6\text{-}8a)$$

$$w'_{1,2}(-\eta) = \sqrt{\pi}\, G(-\eta) \exp\left[\pm i\, \psi(-\eta)\right] \qquad (6\text{-}8b)$$

and use for F, G, χ, and ψ the asymptotic expansions given by Miller.

$$\left\{F(-\eta)\right\}^2 \sim \frac{1}{\pi\eta^{1/2}}\left(1 - \frac{1\cdot3\cdot5}{1!\ 96}\frac{1}{\eta^3} + \frac{1\cdot3\cdot5\cdot7\cdot9\cdot11}{2!\ 96^2}\frac{1}{\eta^6} - \frac{1\cdot3\cdot5\cdot7\cdot9\cdot11\cdot13\cdot15\cdot17}{3!\ 96^3\ \eta^9} + \cdots\right) \qquad (6\text{-}9a)$$

$$\left\{G(-\eta)\right\}^2 \sim \frac{1}{\pi}\eta^{1/2}\left(1 + \frac{1\cdot3}{1!\ 96}\frac{7}{\eta^3} - \frac{1\cdot3\cdot5\cdot7\cdot9}{2!\ 96^2}\frac{13}{\eta^6} + \frac{1\cdot3\cdot5\cdot7\cdot9\cdot11\cdot13\cdot15}{3!\ 96^3}\frac{19}{\eta^9} - \cdots\right) \qquad (6\text{-}9b)$$

$$\chi(-\eta) - \frac{1}{4}\pi \sim \frac{2}{3}\eta^{3/2}\left(1 - \frac{5}{32}\frac{1}{\eta^3} + \frac{1105}{6144}\frac{1}{\eta^6} - \frac{82825}{65536}\frac{1}{\eta^9} + \frac{1282031525}{58720256}\frac{1}{\eta^{12}} - \cdots\right) \qquad (6\text{-}9c)$$

$$\psi(-\eta) + \frac{1}{4}\pi \sim \frac{2}{3}\eta^{3/2}\left(1 + \frac{7}{32}\frac{1}{\eta^3} - \frac{1463}{6144}\frac{1}{\eta^6} + \frac{4\,95271}{3\,27680}\frac{1}{\eta^9} - \frac{206530429}{8388608}\frac{1}{\eta^{12}} + \cdots\right) \qquad (6\text{-}9d)$$

The asymptotic expansions given in Eq. (6-9) can be used for complex values of η provided

$$|\arg \eta| < \frac{2\pi}{3}$$

Miller showed that

$$\frac{d\chi(-\eta)}{d\eta} = \frac{1}{\pi F^2(-\eta)} \quad , \quad \frac{d\psi(-\eta)}{d\eta} = \frac{\eta}{\pi G^2(-\eta)} \tag{6-10}$$

We can use these properties to write

$$w_{1,2}(-\eta) = \left(\frac{d\chi}{d\eta}\right)^{-1/2} \exp\left[\pm i\,\chi\right] \tag{6-11}$$

$$w'_{1,2}(-\eta) = \sqrt{\eta}\ \left(\frac{d\psi}{d\eta}\right)^{-1/2} \exp\left[\pm i\psi\right] \tag{6-12}$$

From Eq. (6-1) we deduce the result

$$F_{1,2}(y,\lambda) = \left(\frac{d\chi}{du}\right)^{-1/2} \exp\left[\pm i\chi\right] = k^{1/3} f_1^{1/6} \left(\frac{d\chi}{dy}\right)^{-1/2} \exp\left[\pm i\,\chi\right] \tag{6-13}$$

where $\chi = \chi(-\eta) = \chi\{-\eta[\zeta(y)]\}$ is defined in terms of η by means of Eq. (6-9c) and η is defined in terms of ζ by Eq. (6-4) and ζ is defined in terms of y by Eq. (6-3). Thus, we find that

$$\begin{aligned}\chi\left\{-\eta\ (\zeta)\right\} - \frac{\pi}{4} \sim & \left[\frac{2}{3}\zeta^{3/2} - \frac{5}{48}\frac{1}{\zeta^{3/2}} + \frac{1105}{9216}\frac{1}{\zeta^{9/2}}\right] + \left(\frac{3}{2}\right)^{2/3}\left[\zeta^{1/2} + \frac{5}{32}\zeta^{-5/2}\right.\\ & \left. - \frac{1105}{2048}\zeta^{-11/2}\right]\frac{\alpha}{k^{4/3}} + \left[\frac{1}{3}\zeta^{3/2} + \frac{5}{96}\zeta^{-3/2} - \frac{1105}{6144}\zeta^{-9/2}\right]\frac{\alpha_o}{k^{6/3}}\\ & + \left[\left(\frac{1}{5}\zeta^{5/2} + \frac{1}{32}\zeta^{-1/2} - \frac{221}{2048}\zeta^{-7/2}\right)\alpha_1 + \left(\frac{9}{16}\zeta^{-1/2}\right.\right.\\ & \left.\left. - \frac{225}{512}\zeta^{-7/2}\right)\alpha^2\ \left(\frac{2}{3}\right)^{2/3}\right]\frac{1}{k^{8/3}} + \cdots\end{aligned} \tag{6-14}$$

If we use the first term

$$\chi\left\{-\eta\ (\zeta)\right\} - \frac{\pi}{4} \sim \frac{2}{3}\ \zeta^{3/2} = \int_{y_1}^{y} \sqrt{k^2 f(v) - \lambda}\ dv \tag{6-15}$$

we find that

$$F_{1,2}(y,\lambda) \sim \frac{k^{1/3} f_1^{1/6} \exp(\pm \frac{\pi}{4})}{\sqrt[4]{k^2 f(y) - \lambda}} \exp\left[\pm i \int_{y_1}^{y} \sqrt{k^2 f(v) - \lambda}\, dv \right] \qquad (6\text{-}16)$$

This is the well-known W.K.B. approximation of the physicists or the Jeffrey's approximation of some of the English applied mathematicians.* Therefore, we see that Eq. (6-14) provides an extension of these classical approximations.

6.2 The W.K.B. Approximation

We do not need to solve first for η before arriving at Eq. (6-14) Since $\exp(\pm i\chi)$ is a solution of

$$\frac{d^2 Z}{d\chi^2} + Z = 0 \qquad (6\text{-}17)$$

and $F(y,\lambda)$ is a solution of

$$\frac{d^2 F}{dy^2} + \left[-\lambda + k^2 f(y) \right] F = 0 \qquad (6\text{-}18)$$

we can use Eq. (4-51) to show that $\chi(y)$ is a solution of

$$(\dot{\chi})^2 - \frac{3}{4}\left(\frac{\ddot{\chi}}{\dot{\chi}}\right)^2 + \frac{1}{2}\frac{\dddot{\chi}}{\dot{\chi}} = k^2 f(y) - \lambda \qquad (6\text{-}19)$$

where the dots denote differentiation with respect to y. From Eq. (6-19) we find that

$$(\dot{\chi})^2 = \left[k^2 f(y) - \lambda\right] - \frac{1}{4}\frac{k^2 \ddot{f}(y)}{\left[k^2 f(y) - \lambda\right]} + \frac{5}{16}\left[\frac{k^2 \dot{f}(y)}{k^2 f(y) - \lambda}\right]^2 + O(k^{-2}) \qquad (6\text{-}20a)$$

or

$$\dot{\chi} = \frac{d\chi}{dy} = \sqrt{k^2 f(y) - \lambda} - \frac{1}{8}\frac{k^2 \ddot{f}(y)}{\left[k^2 f(y) - \lambda\right]^{3/2}} + \frac{5}{32}\frac{k^4 \left[\dot{f}(y)\right]^2}{\left[k^2 f(y) - \lambda\right]^{5/2}} + O(k^{-3}) \qquad (6\text{-}20b)$$

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*For one of the most recent publications which deals with the W.K.B. approximation, see N. Fröman and P. O. Fröman, <u>JWKB Approximation: Contributions to the Theory</u>, Amsterdam, North-Holland Publishing Co., 1965.

This equation can be integrated by writing

$$\chi(y) = \int_{y_0}^{y} \dot{\chi}(z)\,dz + \chi(y_0) \tag{6-21}$$

where y_0 is arbitrary. However, for comparison with Eq. (6-14), we should take $y_0 = y_1$, where y_1 is the turning point.

If we reconsider the result

$$F_{1,2}(y,\lambda) = (d\eta/du)^{-1/2} w_{1,2}(-\eta) = (d\chi/du)^{-1/2} \exp(\pm i\chi) \tag{6-22}$$

we see that Eq. (6-9c) allows us to calculate χ if we know η. Therefore, let us now show how to determine η when we know χ. We do this by inverting Eq. (6-9c). Let us first define γ by letting

$$\chi = \frac{\pi}{4} + \gamma \tag{6-23}$$

and then write

$$F_{1,2}(y,\lambda) = \underbrace{\left(\frac{d\gamma}{du}\right)^{-1/2} \exp\left(\pm i\frac{\pi}{4}\right) \exp(\pm i\gamma)}_{\substack{\text{extended}\\ \text{Jeffreys form}}} = \underbrace{\left(\frac{d\eta}{du}\right)^{-1/2} w_{1,2}(-\eta)}_{\substack{\text{extended}\\ \text{Langer form}}} \tag{6-24}$$

From Eq. (6-9c), we find that

$$\gamma = \frac{2}{3}\eta^{3/2}\left\{1 - \frac{5}{32}\frac{1}{\eta^3} + \frac{1105}{6144}\frac{1}{\eta^6} - \frac{82825}{65536}\frac{1}{\eta^9} + \frac{1282031525}{58720256}\frac{1}{\eta^{12}} + \cdots\right\} \tag{6-25}$$

If we let

$$\frac{2}{3}\eta^{3/2} = \gamma + \frac{A_1}{\gamma} + \frac{A_2}{\gamma^3} + \cdots$$

we can show that

$$\frac{2}{3}\eta^{3/2} \sim \gamma + \frac{5}{72}\frac{1}{\gamma} - \frac{1255}{31104}\frac{1}{\gamma^3} + \frac{272075}{2239488}\frac{1}{\gamma^5} + \cdots \tag{6-26a}$$

or

$$\eta = \left(\frac{3}{2}\gamma\right)^{2/3} + \frac{5}{48}\left(\frac{3}{2}\gamma\right)^{-4/3} - \frac{295}{2304}\left(\frac{3}{2}\gamma\right)^{-10/3} + \cdots \qquad (6\text{-}26b)$$

This important result permits us to change from the extended Jeffreys form to the extended Langer form. On the other hand, Eq. (6-25) permits us to change from the extended Langer form to the extended Jeffreys form. These results are elegant statements of a procedure which is suggested in a remark made by Cherry (Ref. 4, p. 250, footnote) and which was used (in a different form) by Olver (Ref. 5, p. 341).

In Eq. (4-29) we showed that η is a solution of

$$\eta\,(\dot{\eta})^2 - \frac{3}{4}\left(\frac{\ddot{\eta}}{\dot{\eta}}\right)^2 + \frac{1}{2}\frac{\dddot{\eta}}{\dot{\eta}} = k^2 f(y) - \lambda \qquad (6\text{-}27)$$

It may be convenient in solving this equation to let

$$\xi = \frac{2}{3}\eta^{3/2}$$

and observe that $\xi(y)$ is a solution of

$$(\dot{\xi})^2 - \frac{3}{4}\left(\frac{\ddot{\xi}}{\dot{\xi}}\right)^2 + \frac{1}{2}\frac{\dddot{\xi}}{\dot{\xi}} + \frac{5}{36}\left(\frac{\dot{\xi}}{\xi}\right)^2 = k^2 f(y) - \lambda \qquad (6\text{-}28)$$

where the dots denote differentiation with respect to y. Eq. (6-28) is remarkably similar to Eq. (6-19), the only difference being the term containing the square of the logarithmic derivative of ξ.

We should also observe that Imai (Ref. 6, Eq. (19)) has given a result which can be interpreted as an extended Jeffreys' approximation. The result is of the form

$$F_{1,2}(y,\lambda) \sim \frac{k^{1/3} f_1^{1/6}}{\sqrt[4]{k^2 f(y) - \lambda}} \exp\left(\pm i\frac{\pi}{4}\right) \exp\left[\pm ikz \pm \frac{1}{2ik} Q_1(z) + \frac{1}{4k^2} Q(z)\right] \qquad (6\text{-}29)$$

where

$$z = \int_{y_1}^{y} \sqrt{f^2(v) - \lambda k^{-2}}\; dv \qquad (6\text{-}30a)$$

$$Q_1(z) = \int_0^z \left[Q(v) + \frac{5}{36}\frac{1}{v^2}\right] dv + \frac{5}{36}\frac{1}{z} \qquad (6\text{-}30b)$$

$$Q(z) = -\left(\frac{5}{36}\right)\frac{1}{z^2} + \alpha\, z^{-2/3} + \alpha_0 + \alpha_1\, z^{2/3} + \cdots) \tag{6-30c}$$

and α, α_o, and α_1 are defined by Eq. (4-37), Eq. (4-38), and Eq. (4-39), respectively. If we let

$$\gamma \sim kz - \frac{1}{2k} Q_1(z) \tag{6-31}$$

and observe that

$$\frac{d\gamma}{d(kz)} \sim 1 - \frac{1}{2k^2} Q(z) \quad , \quad \frac{d(kz)}{dy} = \sqrt{k^2 f(y) - \lambda}$$

we can show that

$$\left(\frac{d\gamma}{dy}\right)^{-1/2} = \frac{1}{\sqrt[4]{k^2 f(y) - \lambda}} \left[1 - \frac{1}{2k^2} Q(z)\right]^{-1/2} \sim \frac{1}{\sqrt[4]{k^2 f(y) - \lambda}} \left[1 + \frac{1}{4k^2} Q(z)\right]$$

$$\sim \frac{1}{\sqrt[4]{k^2 f(y) - \lambda}} \exp\left[\frac{1}{4k^2} Q(z)\right] \tag{6-32}$$

Therefore, we can rewrite Imai's result in the concise form

$$F_{1,2}(y,\lambda) \sim k^{1/3} f_1^{1/6} \left(\frac{d\gamma}{dy}\right)^{-1/2} \exp\left(\pm i \frac{\pi}{4}\right) \exp(\pm i\, \gamma) \tag{6-33}$$

if we let γ be defined by Eq. (6-31). In order to use these results, we must require that

$$\left|\arg \gamma\right| < \pi$$

6.3 Some Further Forms for $F_{1,2}(y,\lambda)$

A study of a classic memoir by Lorenz (Ref. 7) has led the present author to see that the methods which this 19th century author used to obtain asymptotic estimates for the Bessel functions provides a suggestion for an alternative approach to the computation of $F_{1,2}(y,\lambda)$. Let

$$F_{1,2}(y,\lambda) = \sqrt{\pm i A(y)} \exp(\pm i \gamma) \qquad (6\text{-}34)$$

If we let primes denote differentiation with respect to y, we can show that the function A(y) has the properties

$$A(y) = F_1(y,\lambda) F_2(y,\lambda) = F_1 F_2$$

$$A'(y) = F_1' F_2 + F_1 F_2'$$

$$A''(y) = F_1'' F_2 + F_1 F_2'' + 2F_1' F_2' = 2\left[k^2 f(y) - \lambda\right] A(y) + 2F_1' F_2'$$

$$A'''(y) = 2\left[k^2 f(y) - \lambda\right] A'(y) + 2k^2 f'(y) A(y) + 2F_1' F_2'' + 2F_1'' F_2'$$

$$= 2\left[k^2 f(y) - \lambda\right] A'(y) + 2k^2 f'(y) A(y) + 2\left[k^2 f(y) - \lambda\right] A'(y)$$

Therefore, we see that A(y) is a solution of the third-order differential equation

$$\frac{d^3A}{dy^3} + 4\left[\lambda - k^2 f(y)\right] \frac{dA}{dy} - 2k^2 f'(y) A(y) = 0 \qquad (6\text{-}35)$$

Differentiation of the relation

$$\exp(i\, 2\, \gamma) = \frac{F_1(y,\lambda)}{F_2(y,\lambda)}$$

and use of the Wronskian relation $W(F_1, F_2)$ defined by Eq. (5-99), namely

$$W(F_1, F_2) = F_1(y,\lambda) \frac{dF_2(y,\lambda)}{dy} - F_2(y,\lambda) \frac{dF_1(y,\lambda)}{dy} = -2i\, k^{2/3} f_1^{1/3}$$

leads to a first-order differential equation for $\gamma(y)$ which is of the form

$$\frac{d\gamma}{dy} = \frac{\frac{dF_1}{dy} F_2 - \frac{dF_2}{dy} F_1}{2i\, F_1 F_2} = \frac{k^{2/3} f_1^{1/3}}{A(y)} \qquad (6\text{-}36)$$

We can find an interesting special case of these relations in the introduction of Miller's (Ref. 3) table of the Airy functions.

Miller showed that if a,b, and c were arbitrary constants, that the function

$$Z(x) = a\,Ai^2(x) + b\,Ai(x)\,Bi(x) + c\,Bi^2(x) \tag{6-37}$$

was a solution of the third-order differential equation

$$\frac{d^3Z}{dx^2} - 4x\frac{dZ}{dx} - 2Z = 0 \tag{6-38}$$

We have already cited some results taken from Miller's work which is related to these ideas employed by Lorenz. For example, since the Airy functions are solutions of

$$\frac{d^2w_{1,2}(t)}{dt^2} = t\,w_{1,2}(t)$$

we can use the definitions

$$w_{1,2}(t) = \sqrt{\pi}\,[Bi(t) \pm iAi(t)] \tag{6-39}$$

to show that

$$Z(t) = w_1(t)w_2(t) = \pi\{[Bi(t)]^2 + [Ai(t)]^2\} = \pi F^2(t)$$

is a solution of Eq. (6-37) provided we merely change the indepedent variable from x to t. We readily see that Eq. (6-10) provides an example of Eq. (6-36) if we compare Eq. (6-39) with Eq. (6-11). If we compare the Lorenz form in Eq. (6-34) with the extended Jeffreys form in Eq. (6-34) we see that they differ only in the manner in which the factor multiplying $\exp(\pm i\Upsilon)$ are defined; however, Eq. (6-36) even provides a relationship between these factors. The basic difference in the Lorenz formulation is that the third-order differential equation in Eq. (6-35) is to be solved, and then the "phase factor" determined by integrating the first-order equation given in Eq. (6-36). From Eq. (6-23) we can show that $\Upsilon(y)$ is a solution of the non-linear, third-order differential equation

$$(\dot{\Upsilon})^2 - \frac{3}{4}\left(\frac{\ddot{\Upsilon}}{\dot{\Upsilon}}\right)^2 + \frac{1}{2}\left(\frac{\dddot{\Upsilon}}{\dot{\Upsilon}}\right) = k^2 f(y) - \lambda \tag{6-40}$$

From the experience of the present author, it appears that if one is interested in deriving asymptotic expansions that the results are arrived at with the greatest ease by solving Eq. (6-40) by assuming a form for the solution (such as we have done in Eq. (4-115) in order to solve Eq. (4-114) and using the non-linear, third-order differential equation to determine the unknown coefficients. The amplitude (which we should put in quotation marks as "amplitude" since the function will generally be a complex-valued function) can then be determined by computing the derivative $d\Upsilon/du$ as shown in the extended Jeffreys form of Eq. (6-24). However, from the standpoint of obtaining the solutions for $F_{1,2}(y,\lambda)$ by means of numerical methods there are advantages associated with using the form given in Eq. (6-34) and in starting the calculations by solving for $A(y)$ from the linear third-order differential equation of Eq. (6-35). The "phase" function $\Upsilon(y)$ is then to be determined by integrating the first-order differential equation given as Eq. (6-36).

Miller (Ref. 8) has made extensive use of Eq. (6-35) in his studies of the parabolic cylinder functions which satisfy the differential equation

$$\frac{d^2 f}{dx^2} + \left(-a + \frac{1}{4}x^2\right) f = 0 \tag{6-41}$$

The use of Eq. (6-35) and Eq. (6-36) is most attractive when λ is real and positive, and $k^2 f(y)$ is real, and $\lambda > k^2 f(y)$. Under these conditions, both $A(y)$ and $\Upsilon(y)$ are real.

Lorenz (Ref. 7) also gave a method (for the study of the Bessel functions) which suggests a method which can be useful when λ is real and $[-\lambda + k^2 f(y)] < 0$ for the values of y of interest. In this region we can show that for y not near the turning point y_1 that $F_{1,2}(y,\lambda)$ has the asymptotic behavior

$$F_{1,2}(y,\lambda) \sim \frac{k^{1/3} f_1^{\,1/6}}{\sqrt[4]{\lambda - k^2 f(y)}} \exp\left[\int_{y_1}^{y} \sqrt{\lambda - k^2 f(v)}\, dv\right] \tag{6-42}$$

Since this result does not permit us to distinguish between F_1 and F_2, we shall illustrate the idea set forth by Lorenz by considering the analogous situation with the Airy functions as a model of the method. Let us define two functions $U(y,\lambda)$ and $V(y,\lambda)$ by means of the relation

$$2F_{1,2}(y,\lambda) = U(y,\lambda) \pm i\,V(y,\lambda) \tag{6-43}$$

so that

$$U(y,\lambda) = \frac{F_1(y,\lambda) - F_2(y,\lambda)}{2i} \tag{6-44a}$$

$$V(y,\lambda) = \frac{F_1(y,\lambda) + F_2(y,\lambda)}{2} \tag{6-44b}$$

The corresponding Airy function formulae are

$$2\,w_{1,2}(z) = u(z) \pm i\,v(z) \tag{6-45}$$

$$v(z) = \frac{w_1(z) - w_2(z)}{2i} = \sqrt{\pi}\ \mathrm{Ai}(z) \tag{6-46a}$$

$$u(z) = \frac{w_1(z) + w_2(z)}{2} = \sqrt{\pi}\ \mathrm{Bi}(z) \tag{6-46b}$$

For y near the turning point y_1, we use the function η defined by Eq. (4-118) and write

$$U(y,\lambda) = k^{1/3} f_1^{1/6} \left(\frac{d\eta}{dy}\right)^{-1/2} v(-\eta) \tag{6-47a}$$

$$V(y,\lambda) = k^{1/3} f_1^{1/6} \left(\frac{d\eta}{dy}\right)^{-1/2} u(-\eta) \tag{6-47b}$$

where η is negative and real when $[-\lambda + k^2 f(y)]$ is real and negative.

Miller (Ref. 3) showed that for $\arg x < \frac{2}{3}\pi$,

$$v(x) = \sqrt{\pi}\ \mathrm{Ai}(z) \sim \frac{1}{2}\, x^{-1/4} e^{-\xi} \left(1 - \frac{3\cdot 5}{1!\ 216}\,\frac{1}{\xi} + \frac{5\cdot 7\cdot 9\cdot 11}{2!\,(216)^2}\,\frac{1}{\xi^2} - \frac{7\cdot 9\cdot 11\cdot 13\cdot 15\cdot 17}{3!\ (216)^3}\,\frac{1}{\xi^3} + \cdots\right) \tag{6-48a}$$

$$u(x) = \sqrt{\pi}\ \mathrm{Bi}(z) \sim x^{-1/4}\, e^{\xi} \left(1 + \frac{3\cdot 5}{1!\ 216}\ \frac{1}{\xi} + \frac{5\cdot 7\cdot 9\cdot 11}{2!\ (216)^2}\ \frac{1}{\xi^2} + \frac{7\cdot 9\cdot 11\cdot 13\cdot 15\cdot 17}{3!\ (216)^3}\ \frac{1}{\xi^3} + \cdots\right) \tag{6-48b}$$

where

$$\xi = \frac{2}{3} x^{3/2}$$

However, the form of asymptotic expansion which is suggested by a study of Lorenz's memoir (Ref. 7) is dependent upon the introduction of the auxiliary functions $r(x)$ and $\mu(x)$ by means of

$$2v(x) = 2\sqrt{\pi}\ \mathrm{Ai}(x) = \sqrt{r(x)}\ \exp\left[-\mu(x)\right] = \left(\frac{d\mu}{dx}\right)^{-1/2} \exp\left[-\mu(x)\right]$$

$$u(x) = \sqrt{\pi}\ \mathrm{Bi}(x) = \sqrt{r(x)}\ \exp\left[\mu(x)\right] = \left(\frac{d\mu}{dx}\right)^{-1/2} \exp\left[\mu(x)\right]$$

Since

$$r(x) = 2v(x)u(x) = 2\pi \mathrm{Ai}(x)\mathrm{Bi}(x)$$

we see that $r(x)$ is a solution of Eq. (6-38) and that

$$\frac{d\mu(x)}{dx} = \frac{1}{r(x)} \tag{6-50}$$

or

$$(\dot{\mu})^2 + \frac{3}{4}\left(\frac{\ddot{\mu}}{\dot{\mu}}\right)^2 - \frac{1}{2}\ \frac{\dddot{\mu}}{\dot{\mu}} = x \tag{6-51}$$

where the dots denote differentiation with respect to x. We can show that $r(x)$ and $\mu(x)$ possess asymptotic expansions of the

$$r(x) \sim \frac{1}{\sqrt{x}} \left\{1 + \frac{1\cdot 3\cdot 5}{1!\,96}\ \frac{1}{x^3} + \frac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11}{2!\ (96)^2}\ \frac{1}{x^6} + \frac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13\cdot 15\cdot 17}{3!\ (96)^3}\ \frac{1}{x^9} + \cdots\right\} \tag{6-52a}$$

$$\mu(x) \sim \frac{2}{3}\, x^{3/2} \left[1 + \frac{5}{32}\,\frac{1}{x^3} + \frac{1105}{6144}\ \frac{1}{x^6} + \frac{82825}{65536}\ \frac{1}{x^9} + \cdots\right] \tag{6-52b}$$

In a similar way, we can show that

$$2v'(x) = 2\sqrt{\pi}\ Ai'(x) = -\sqrt{p(x)}\ \exp\left[-\lambda(x)\right]$$

$$u'(x) = \sqrt{\pi}\ Bi'(x) = \sqrt{p(x)}\ \exp\left[\lambda(x)\right]$$

where

$$\frac{d\lambda}{dx} = \frac{x}{p(x)} \tag{6-53}$$

and $p(x)$ is a solution of the third-order differential equation

$$x^2 \frac{d^3p}{dx^3} - 3x \frac{d^2p}{dx^2} + 3 \frac{dp}{dx} - 4x^3 \frac{dp}{dx} + 2x^2 p = 0 \tag{6-54}$$

We can show that $p(x)$ and $\lambda(x)$ possess asymptotic expansions of the form

$$p(x) \sim x^{1/2}\left\{1 - \frac{1\cdot 3}{1!\,96}\frac{7}{x^3}\ \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2!\ 96^2}\frac{13}{x^6} - \frac{1\cdot 3\cdot 5\cdot 9\cdot 11\cdot 13\cdot 15}{3!\ 96^3}\frac{1}{x^9}\cdots\right\} \tag{6-55a}$$

$$\lambda(x) \sim \frac{2}{3}x^{3/2}\left\{1 - \frac{7}{32}\frac{1}{x^3} - \frac{1463}{6144}\frac{1}{x^6} - \frac{495291}{327680}\frac{1}{x^9} - \frac{206530429}{8388608}\frac{1}{x^{12}} + \cdots\right\} \tag{6-55b}$$

Let us now turn to Eq. (6-44) and obtain generalizations of Eq. (6-48) by making the definitions

$$2V(y,\lambda) = \sqrt{R(y)}\ \exp[-\Gamma(y)] \tag{6-56a}$$

$$U(y,\lambda) = \sqrt{R(y)}\ \exp[+\Gamma(y)] \tag{6-56b}$$

The function $R(y)$ will be a solution of Eq. (6-35) and $\Gamma(y)$ will be a solution of

$$\dot{\Gamma} = \frac{d\Gamma}{dy} = \frac{k^{2/3}\, f_1^{\,1/3}}{R(y)} \tag{6-57}$$

or

$$(\dot{\Gamma})^2 + \frac{3}{4}\left(\frac{\ddot{\Gamma}}{\dot{\Gamma}}\right)^2 - \frac{1}{2}\frac{\dddot{\Gamma}}{\dot{\Gamma}} = k^2 f(y) - \lambda \tag{6-58}$$

The procedures which we have derived from the methods employed by Lorenz were used extensively by Nicholson (Ref. 9). The casual

manner in which Lorenz used these procedures suggests that they may have been well-known methods for analysts of the late nineteenth century. One of the most extensive uses of these methods in recent years has been in the study of the parabolic cylinder functions by Miller (Ref. 8).

Watson (Ref. 10, p. 224) says of the method used in Lorenz's memoir: "It is not easy to estimate exactly the magnitude or the sign of the remainder after any number of terms in these asymptotic expansions when this method is used." This is still a very valid criticism. Nevertheless, these methods permit one to readily "grind out" successive terms in the asymptotic expansions. These series are often found to be useful for computational purposes. The value of these expansions can be greatly enhanced if one follows the suggestions of Miller (Ref. 11) and seeks to determine converging factors. Slater (Ref. 12) has had some outstanding success with Miller's converging factors in the study of the confluent hypergeometric functions which are solutions of Kummer's equation

$$x\frac{d^2y}{dx^2} + (b - x)\frac{dy}{dx} - a\,y = 0$$

If we define z(x) by means of

$$y(x) = x^{-b/2}\exp(x/2)\;z(x)$$

we see that these studies involve Whittaker's equation

$$\frac{d^2w}{dx^2} + \left[-\frac{1}{4} + \frac{b - 2a}{2x} - \frac{b(b - 2)}{4x^2}\right] z = 0$$

Since Whittaker's equation is of the same form as the height gain function, some of the methods and results which have been discussed above can be used in the study of this function. In quantum physics, this equation leads to the Coulomb wave functions for which there is an extensive literature.

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Section 7

ASYMPTOTIC ESTIMATES FOR $U(x,y;x_0,y_0)$

7.1 The Diffraction Functions

In Section 1 we have discussed a mathematical model for the diffraction of waves by a convex surface which involved a function $D(\xi,\zeta,\zeta_0,q)$ which could be represented by a series of the form

$$D(\xi, \zeta, \zeta_0, q) = i \sum_{s=1}^{\infty} \frac{\exp(i\xi\tau_s)\, w_1(\tau_s-\zeta)\, w_1(\tau_s-\zeta_0)}{\tau_s\,[w_1(\tau_s)]^2 - [w_1'(\tau_s)]^2} \tag{7-1}$$

in which the functions $w_1(z)$ and $w_1'(z)$ are the Airy function

$$w_1(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty e^{i\frac{2}{3}\pi}} \exp(\tfrac{1}{3}\omega^3 - \omega z)\,d\omega \tag{7-2}$$

and its derivative

$$w_1'(z) = \frac{dw_1(z)}{dz}$$

and the quantitites τ_s are the roots of the equation

$$w_1'(\tau_s) - q\,w_1(\tau_s) = 0 \tag{7-3}$$

We can refer to Eq. (7-1) as the van der Pol-Bremmer formula since this function was first extensively used in their classic papers (Refs. 1 and 2) which appeared during the late 1930's. Pryce (Ref. 3) and Fock (Ref. 4) also expressed $D(\xi,\zeta,\zeta_0,q)$ in the form of a Fourier integral

$$D(\xi,\zeta,\zeta_0,q) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(i\xi t) F(t,\zeta,\zeta_0,q)\,dt \tag{7-4a}$$

where

$$F(t,\zeta,\zeta_0,q) = w_1(t-\zeta_>)v(t-\zeta_<) - \frac{v'(t)-qv(t)}{w_1'(t)-qw_1(t)} w_1(t-\zeta)w_1(t-\zeta_0)$$

$$= \tfrac{1}{2}w_1(t-\zeta_0)\left[w_2(t-\zeta) - \frac{w_2'(t)-qw_2(t)}{w_1'(t)-qw_1(t)} w_1(t-\zeta)\right] \qquad (7\text{-}4b)$$

where $v(t)$ is the Airy integral

$$v(t) = \frac{1}{\sqrt{\pi}} \int_0^\infty \cos(\tfrac{1}{3}x^3 - xt)dx = \frac{w_1(t) + w_2(t)}{2i} \qquad (7\text{-}5)$$

We have discussed three limiting forms of Eq.(7-1) and three families of two-parameter functions have been introduced and have been referred to as "diffraction functions."

The <u>attenuation function</u> was defined by

$$V_0(\xi,q) = \frac{1}{2}\sqrt{\frac{-i\xi}{\pi}} \int_{-\infty}^{\infty} \exp(i\xi t)\frac{w_1(t)}{w_1'(t)-qw_1(t)} dt \qquad (7\text{-}6a)$$

$$= \sqrt{i\pi\xi} \sum_{s=1}^{\infty} \frac{\exp(i\xi\tau_s)}{\tau_s - q^2} \qquad (7\text{-}6b)$$

The <u>current distribution function</u> was defined by

$$V_1(\xi,q) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\exp(i\xi t)}{w_1'(t)-qw_1(t)} dt \qquad (7\text{-}7a)$$

$$= i2\sqrt{\pi} \sum_{s=1}^{\infty} \frac{\exp(i\xi\tau_s)}{(\tau_s - q^2)w_1(\tau_s)} \qquad (7\text{-}7b)$$

The <u>reflection coefficient function</u> was defined by

$$V_{11}(\xi,q) = -\sqrt{i/\pi}\int_{-\infty}^{\infty} \exp(i\xi t)\frac{v'(t)-qv(t)}{w_1'(t)-qw_1(t)}\, dt \tag{7-8a}$$

$$= -2\sqrt{-i\pi}\sum_{s=1}^{\infty} \frac{\exp(i\xi\tau_s)}{(\tau_s - q^2)[w_1(\tau_s)]^2} \tag{7-8b}$$

$$= V_2(\xi,q) + \frac{\sqrt{i}}{2\sqrt{\pi}\,\xi} \tag{7-8c}$$

These functions can serve as "models" for the defining of special functions which can be derived from the Green's function which was defined in Eq. (4-13), namely

$$U(x, y; x_0, y_0) = \sum_{s=1}^{\infty} \frac{\Psi^+(x_>,\lambda_s)\,\Psi^-(x_<,\lambda_s)}{W_s}\;\frac{F(y,\lambda_s)\;F(y_0,\lambda_s)}{N_s} \tag{7-9}$$

However, we shall more often be interested in actually employing the functions $D(\xi,\zeta,\zeta_0,q)$, $V_0(\xi,q)$, $V_1(\xi,q)$, $V_{11}(\xi,q)$, and $V_2(\xi,q)$ as asymptotic approximations to $U(x,y,;x_0,y_0)$.

7.2 The Reflection Formula

An exception to our aim of actually employing the functions defined in Eqs. (7-1) through (7-8) to approximate Eq. (7-9) is the case of the obtaining of a representation for the reflected wave which corresponds to the result obtained from geometrical optics. In this case, we shall use the theory associated with the function $D(\xi,\zeta,\zeta_0,q)$ as a "model" for how to proceed with the more general function $U(x,y;x_0,y_0)$.

According to Eq. (1-42), we have

$$D_0(\xi,\zeta,\zeta_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(i\xi t)\, w_1(t-\zeta_>)\, v(t-\zeta_<) = \frac{\sqrt{i}}{2\sqrt{\pi\xi}} \exp(-i\Omega) \tag{7-10a}$$

where

$$\Omega = \Omega(\xi,\zeta,\zeta_0) = \frac{\xi^3}{12} - \frac{\xi}{2}(\zeta + \zeta_0) - \frac{(\zeta - \zeta_0)^2}{4\xi} \tag{7-10b}$$

If we decompose $D(\xi,\zeta,\zeta_0,q)$ into the components

$$D(\xi,\zeta,\zeta_0,q) = D_0(\xi,\zeta,\zeta_0) + R(\xi,\zeta,\zeta_0,q) \tag{7-11a}$$

we can use the fact that $D_0(\xi,\zeta,\zeta_0)$ is the field which exists when the "reflecting surface" does not exist to refer to the portion $R(\xi,\zeta,\zeta_0,q)$ as the "reflected wave." We can use the first of the equations in Eq. (7-4b) to arrive at the explicit form for $R(\xi,\zeta,\zeta_0,q)$ which is given in Eq. (1-43). However, we are interested at this point in our discussion in making a different decomposition of $D(\xi,\zeta,\zeta_0,q)$ by deforming the path of integration and by using the second of the equations in Eq. (7-4b). Thus, we define

$$D(\xi,\zeta,\zeta_0,q) = D_i(\xi,\zeta,\zeta_0) + D_r(\xi,\zeta,\zeta_0,q) \tag{7-11b}$$

where

$$D_i(\xi,\zeta,\zeta_0) = \frac{i}{4\pi}\int_C \exp(i\xi t) w_1(t-\zeta_>) w_2(t-\zeta_<)\,dt \tag{7-12a}$$

$$D_r(\xi,\zeta,\zeta_0,q) = \frac{-i}{4\pi}\int_C \exp(i\xi t) R(t,q) w_1(t-\zeta) w_1(t-\zeta_0)\,dt \tag{7-12b}$$

$$R(t,q) = \frac{w_2'(t) - q w_2(t)}{w_1'(t) - q w_1(t)} \tag{7-12c}$$

The evaluation of $D_i(\xi,\zeta,\zeta_0)$ and $D_r(\xi,\zeta,\zeta_0,q)$ by means of the method of stationary phase has been discussed in considerable detail by Fock (Ref. 4, pp. 241-246, under the heading "The Reflection Formula."). The contour C associated with the integrals is to be chosen so as to permit the application of the method of stationary phase to the evaluation of the integrals.

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Fock's analysis leads to the results

$$D_i(\xi,\zeta,\zeta_0) \approx D_0(\xi,\zeta,\zeta_0) = \frac{\sqrt{i}}{2\sqrt{\pi\xi}} \exp(-i\Omega) \tag{7-13a}$$

where Ω is defined in Eq. (7-10b), and

$$D_r(\alpha+\alpha_0, \alpha^2+2\alpha\beta, \alpha_0{}^2+2\alpha_0\beta_0, q) \xrightarrow[\beta>>1]{}$$

$$- \frac{\sqrt{i}}{2\sqrt{\pi}} \frac{q - i\beta}{q + i\beta} \frac{\sqrt{\beta}}{\sqrt{2\alpha\alpha_0+\beta(\alpha+\alpha_0)}} \exp(i\omega^*) \tag{7-13b}$$

where

$$\omega^* = \frac{2}{3}(\alpha^3+\alpha_0{}^3) + 2\beta(\alpha^2+\alpha_0{}^2) + \beta^2(\alpha + \alpha_0) \tag{7-13c}$$

Eq. (7-13c) is merely a restatement of the result which we have already given as Eq. (1-44).

With the background which we have sketched above, we can now turn to the case of the Green's function $U(x,y;x_0,y_0)$. If we consider the Fourier integral representation given in Eq. (5-121) we see that the counterpart of $D_r(\xi,\zeta,\zeta_0,q)$ which is defined in Eq. (7-12b) is the integral

$$U_r(x, y; x_0, y_0) = - \frac{1}{4\pi k^{2/3} f_1^{1/3}} \int_P \frac{\Psi^+(x_>, \lambda)\, \Psi^-(x_<, \lambda)}{W(\lambda)}$$

$$\frac{F_2'(0, \lambda) + i k Z F_2(0, \lambda)}{F_1'(0, \lambda) + i k Z F_1(0, \lambda)} F_1(y, \lambda) F_1(y_0, \lambda)\, d\lambda \tag{7-14}$$

We define

$$\lambda = k^2\mu$$

and let y_1 denote the "turning point" at which

$$f(y_1) = \mu$$

If we recall the Wronskian property defined by Eq. (4-19)

$$W_s = W_s(\lambda) = \psi^+ \frac{d\psi^-}{dx} - \psi^- \frac{d\psi^+}{dx}$$

and use $\lambda = k^2\mu$, we see that the asymptotic estimate (WKB approximation) of Eq. (4-16), namely

$$\Psi^{\pm}(x, \lambda) \xrightarrow[k\to\infty]{} \frac{\text{constant}}{\sqrt[4]{k^2 g(x) + \lambda}} \exp\left[\pm i \int_0^x \sqrt{k^2 g(u) + \lambda}\ du\right]$$

leads to the asymptotic result

$$\frac{\Psi^+(x_>, \lambda)\Psi^-(x_<, \lambda)}{W(\lambda)} \xrightarrow[k\to\infty]{} \frac{\exp\left[i k \int_{x_<}^{x_>} \sqrt{g(u) + \mu}\ du\right]}{-2i k \sqrt[4]{g(x) + \mu}\ \sqrt[4]{g(x_o) + \mu}} \tag{7-15}$$

If we use the leading term of Eq. (6-29) for $F_{1,2}(y,\lambda)$, namely

$$F_{1,2}(y,\lambda) \approx \frac{k^{\frac{1}{3}}\sqrt{\pm i f_1^{\frac{1}{3}}}}{\sqrt{\sqrt{k^2 f(y) - \lambda}}} \exp\left(\pm i \int_{y_1}^{y} \sqrt{k^2 f(v) - \lambda}\ dv\right) \tag{7-16}$$

and use $\lambda = k^2\mu$, we find that

$$\frac{F_2'(0, \lambda) + i k Z F_2(0, \lambda)}{F_1'(0, \lambda) + i k Z F_1(0, \lambda)} \xrightarrow[k\to\infty]{} -i \frac{Z - \sqrt{f_o - \mu}}{Z + \sqrt{f_o - \mu}} \exp\left[-i\, 2k \int_{y_1}^{0} \sqrt{f(v) - \mu}\ dv\right] \tag{7-17}$$

and

$$F_1(y, \lambda) F_1(y_o, \lambda) \xrightarrow[k\to\infty]{} i \frac{k^{-1/3} f_1^{1/3}}{\sqrt{\sqrt{f(y) - \mu}\ \sqrt{f(y_o) - \mu}}} \exp\left[i k \left(\int_{y_1}^{y_o} + \int_{y_1}^{y}\right) \sqrt{f(v) - \mu}\ dv\right] \tag{7-18}$$

where we recall that $f_0 = f(0)$ and $f_1 = f'(0)$. We can now use Eqs. (7-15) through (7-18) to show that these asymptotic results permit us to express Eq. (7-14) in the form

$$U_r(x, y; x_o, y_o) \xrightarrow[k\to\infty]{} -\frac{i}{8\pi} \int_P A(\mu) \exp\left[i k \phi(\mu)\right] d\mu \tag{7-19}$$

where

$$A(\mu) = \frac{Z - \sqrt{f_o - \mu}}{Z + \sqrt{f_o + \mu}} \frac{1}{\sqrt[4]{g(x)+\mu}\,\sqrt[4]{g(x_o)+\mu}\,\sqrt[4]{f(y)-\mu}\,\sqrt[4]{f(y_o)-\mu}} \tag{7-20}$$

$$\phi(\mu) = \int_{x_<}^{x_>} \sqrt{g(u)+\mu}\,du + \int_0^{y} \sqrt{f(v)-\mu} + \int_0^{y_o} \sqrt{f(v)-\mu}\,dv \tag{7-21}$$

and the symbol P on the integral sign denotes the fact that the path of the integral is to pass through the point of stationary phase. According to the theory of the method of stationary phase,

$$\int g(z) \exp[i\,k\,f(z)]\,dz \sim \sqrt{\frac{2\pi}{-k\,f''(z_o)}}\; g(z_o) \exp[i\,k\,f(z_o) - i\frac{\pi}{4}] \tag{7-22}$$

where z_0 is defined by the condition

$$f'_1(z_0) = 0$$

and in the form given in Eq. (7-22) we assume that $f''(z_0) < 0$. Therefore, we find that

$$U_r(x,y;x_o,y_o) \xrightarrow[k\to\infty]{} - \frac{1}{\sqrt{-32\,\pi\,k\,\phi''(\mu_o)}}\; A(\mu_o) \exp[i\,k\,\phi(\mu_o) + i\frac{\pi}{4}] \tag{7-23}$$

where the point of stationary phase μ_0 is defined by

$$\phi'(\mu_o) = \frac{1}{2}\int_{x_<}^{x_>} \frac{1}{\sqrt{g(u)+\mu_o}}\,du - \int_0^{y} \frac{1}{\sqrt{f(v)-\mu_o}}\,dv - \int_0^{y_o} \frac{1}{\sqrt{f(v)-\mu_o}}\,dv = 0 \tag{7-24}$$

The second derivative of $\phi(\mu)$ is of the form

$$\phi''(\mu) = -\frac{1}{4}\left[\int_{x_<}^{x_>} [g(u)+\mu]^{-3/2}\,du + \int_0^{y} [f(v)-\mu]^{-3/2}dv + \int_0^{y_o} [f(v)-\mu]^{-3/2}dv\right]$$

For the case $\mu = \mu_0$ we can express $\phi''(\mu)$ in the form

$$\phi''(\mu_0) =$$

$$-\frac{1}{4\mu_o}\left[\int_0^{y} \frac{f(v)}{[f(v)-\mu_o]^{3/2}}\,dv + \int_0^{y_o} \frac{f(v)}{[f(v)-\mu_o]^{3/2}}\,dv - \int_{x_<}^{x_>} \frac{g(u)}{[g(u)+\mu_o]^{3/2}}\,du\right] \tag{7-25}$$

If we collect these results, we see that Eq.(7-23) informs us that the evaluation by means of the method of stationary phase has led us to the approximation

$$U_r(x,y;x_0,y_0) \xrightarrow[k \to \infty]{} \frac{\sqrt{f_0-\mu_0} - Z}{\sqrt{f_0-\mu_0} + Z} F(x,y,;x_0,y_0)\exp[iP(x,y,x_0,y_0)] \quad (7\text{-}26)$$

where

$$[F(x,y,;x_0,y_0)]^2 = \frac{\left\{\frac{8\pi k}{\mu_0}\left[\left(\int_0^y + \int_0^{y_0}\right)\frac{f(v)}{[f(v)-\mu_0]^{3/2}}dv - \int_{x_<}^{x_>}\frac{g(u)}{[g(u)+\mu_0]^{3/2}}du\right]\right\}^{-1}}{\sqrt{[g(x)+\mu_0][g(x_0)+\mu_0][f(y)-\mu_0][f(y_0)-\mu_0]}}$$

and

$$P(x,y,;x_0,y_0) = \left(\int_{x_<}^{x_>}\sqrt{g(u)+\mu_0}\,du + \int_0^y \sqrt{f(v)-\mu_0} + \int_0^{y_0}\sqrt{f(v)-\mu_0}\,dv\right) + \frac{\pi}{4}$$

Further discussions of the result described by Eq. (7-26) can be found in Refs. 5 through 11. In the above discussion we have merely attempted to sketch the manner in which the WKB approximations are employed to obtain this reflection formula for the field in the lighted region and well away from the "horizon." In this report we are primarily concerned with the applications of the diffraction functions which make it possible to describe the transition across the "horizon".

However, before returning to the applications of the diffraction functions, we will pause to make some remarks concerning the use of the WKB approximation in the case of inhomogeneous media. The author is of the opinion that for many of the practical problems which confront the microwave engineer that the changes in the index

of refraction are often too rapid to permit one to obtain a result of the desired accuracy by means of the WKB approximation. Since the extensions of the WKB approximation to obtain higher-order corrections generally involves the use of asymptotic series (such as the series we gave as Eq. (4-42)), the theory is difficult to "translate" into practical results and the use of these classical results has been rather limited.

The future work on the height gain function will probably be very much influenced by the techniques which are developed for the treatment of the problem of propagation of waves through inhomogeneous layers. Rydbeck (Ref. 12) has laid the foundation for a method which is attractive from the point of view of numerical analysis. Tischer (Ref. 13) has presented a brief discussion of this method in his monograph on space communications. There is already a sizeable literature upon the theory behind these methods. Schelkunoff (Ref. 14) and Bremmer (Ref. 15) have written papers that indicate how one can begin to tackle the problem of a medium in which the rate of change of the index of refraction plays an important role in the obtaining of numerical results. Bremmer (Ref. 15) showed that the WKB approximation could be interpreted as the first term in a series in which the successive terms are classified according to the number of "reflections" which they have undergone. This series has become known as the "Bremmer series" and the reader is cautioned to not confuse the "Bremmer series" discussed by Atkinson (Ref. 16) and Kay (Ref 17) with the series described by Eq. (7-1) which the present author sometimes refers to as the "van der Pol-Bremmer formula."

The basic idea which is contained in Refs. 12 through 17 is that we should seek to determine the "total" solution of height gain differential equation instead of trying to find the two solutions $F_{1,2}(y,\lambda)$ and seek representations for each of these solutions.

To illustrate the difference in approach which may be expected to play an important role in future work, let us observe that for $y_0 > y$ that Eq. (5-121) contains a function

$$F(y,\lambda) = F_2(y,\lambda) - \frac{\frac{\partial F_2(0,\lambda)}{\partial y} + ikZ\, F_2(0,\lambda)}{\frac{\partial F_1(0,\lambda)}{\partial y} + ikZ\, F_1(0,\lambda)} F_1(y,\lambda) \qquad (7\text{-}27)$$

which is a solution of the height gain differential equation

$$\frac{d^2F(y,\lambda)}{dy^2} + [-\lambda + k^2 f(y)]\, F(y,\lambda) = 0 \qquad (7\text{-}28a)$$

which satisfies the boundary condition

$$\frac{\partial F(0,\lambda)}{\partial y} + ikZ\, F(0,\lambda) = 0 \qquad (7\text{-}28b)$$

If we use the approximations of Eq. (7-17) and Eq. (7-18) to obtain an asymptotic result for the quantity

$$H(y,y_0,\lambda,k) = F(y,\lambda)F_1(y_0,\lambda) \qquad (7\text{-}29)$$

we arrive at the result (with $\lambda = k^2\mu$)

$$H(y,y_0,\mu,k) \approx H_0(y,y_0,\mu,k) + H_1(y,y_0,\mu,k) \qquad (7\text{-}30a)$$

where

$$H_0(y,y_0,\mu,k) = \frac{k^{-\frac{1}{3}} f_1^{\frac{1}{3}}}{\sqrt{\sqrt{f(y)-\mu}\,\sqrt{f(y_0)-\mu}}} \exp\left(ik\int_y^{y_0} \sqrt{f(v)-\mu}\, dv\right) \qquad (7\text{-}30b)$$

$$H_1(y,y_0,\mu,k) = \frac{Z - \sqrt{f_0-\mu}}{Z + \sqrt{f_0-\mu}} \; \frac{k^{-\frac{1}{3}} f_1^{\frac{1}{3}}}{\sqrt{\sqrt{f(y)-\mu}\,\sqrt{f(y_0)-\mu}}} \exp(ik\Lambda) \qquad (7\text{-}30c)$$

and

$$\Lambda = \Lambda(y,y_0,\mu) = \left(\int_0^y + \int_0^{y_0}\right)\sqrt{f(v) - \mu}\, dv \qquad (7\text{-}30d)$$

We can interpret Eq. (7-30c) as a wave which has been reflected from the surface $y = 0$ with a reflection coefficient

$$R(Z,f_0,\mu) = \frac{Z - \sqrt{f_0 - \mu}}{Z + \sqrt{f_0 - \mu}} \, ,$$

a "diverging factor"

$$S(y,y_0,\mu,k) = \frac{k^{-\frac{1}{3}} f_1^{\frac{1}{3}}}{\sqrt{\sqrt{f(y)-\mu}\,\sqrt{f(y_0)-\mu}}}$$

and a phase shift which is represented by the factor $\exp(ik\Lambda)$. These considerations can be shown to permit one to interpret these results on the basis of a "geometrical optics" of inhomogeneous media. However, if the function $f(y)$ varies "rapidly" we should expect that the effect of traversing the inhomogeneous medium should involve some "internal reflections" in the medium as well as the "phase shift." If these effects are to be included in Eq. (7-27), one may be forced to obtain some extremely accurate solutions of $F_1(y,\lambda)$ and $F_2(y,\lambda)$. The author's experience with this problem has led him to favor seeking to solve for an approximation to $F(y,\lambda)$, as defined in Eq. (7-28), in the case of media in which these "internal reflections" are important.

In order to illustrate the techniques which can be learned from a study of Refs. 12 through 17, let us consider the problem of finding a solution of the differential equation

$$\frac{d^2U}{dx^2} + p^2(x)U = 0 \tag{7-31}$$

under the condition that $p(x)$ is continuous function of the variable x. We assume that $p(x) = p_0$ = constant when $x < 0$. Then, if the time dependence $\exp(-i\omega t)$ is suppressed, an incident plane wave is given by $\exp(ip_0x)$ in the region $x < 0$. For the sake of

simplicity, we shall assume that the medium to the right of $x = 0$ extends to infinity. The theory can be readily extended to the case in which

$$\frac{dU}{dx} + iZU = 0$$

at $x = x_0 > 0$. However, this complication "clutters up" the theory and our aim at this point is merely to point the way for further work in this type of treatment of Eq. (7-31).

Let us assume that in the region $x > 0$ the total field $U(x)$ can be represented as the sum of two terms

$$U(x) = V(x) + W(x) \tag{7-32}$$

where $V(x)$ is the contribution of all waves moving foward which have arisen from points to the left of x, and $W(x)$ represents the total of all backward waves starting from points to the right of x. From the theory of the WKB approximation, we know that the function

$$Y(x) = \frac{1}{\sqrt{p(x)}} \exp\left(i\int_0^x p(s)ds\right) \tag{7-33a}$$

represents a transmitted wave propagating in the foward direction, and

$$Z(x) = \frac{1}{\sqrt{p(x)}} \exp\left(-i\int_0^x p(s)ds\right) \tag{7-33b}$$

represents a wave propagating in the backward direction. We make the observation that $Y(x)$ and $Z(x)$ satisfy the first-order differential equations

$$Y' = \left(+ip - \frac{p'}{2p}\right)Y \tag{7-34a}$$

$$Z' = \left(-ip - \frac{p'}{2p}\right)Z \tag{7-34b}$$

The "trick" which is found in Refs. 12 through 17 is that of

observing that the functions V(x) and W(x) are solutions of the <u>coupled</u> first-order differential equations

$$V' = (+ip - \frac{p'}{2p})V + \frac{p'}{2p} W \qquad (7\text{-}35a)$$

$$W' = (-ip - \frac{p'}{2p})W + \frac{p'}{2p} V \qquad (7\text{-}35b)$$

We can readily verify that this pair of equations is equivalent to Eq. (7-31) by observing that

$$U' = V' + W_1' = ip(V - W)$$

and

$$U'' = ip'(V - W) + ip(V' - W') = - p^2U$$

If we define $\Psi(x)$ to be the phase factor

$$\Psi(x) = \int_0^x p(s)ds \qquad (7\text{-}36)$$

we can replace the pair of coupled differential equations given above as Eq. (7-35) by the coupled integral equations

$$W(x) = - \frac{\exp[-i\Psi(x)]}{\sqrt{p(x)}} \int_x^{\infty} \exp[+i\Psi(t)] \frac{p'(t)}{2\sqrt{p(t)}} V(t)dt \qquad (7\text{-}37a)$$

$$V(x) = \frac{\exp[+i\Psi(x)]}{\sqrt{p(x)}} \left[A + \int_0^x \exp[-i\Psi(t)] \frac{p'(t)}{2\sqrt{p(t)}} W(t)dt \right] \qquad (7\text{-}37b)$$

The constant <u>A</u> which appears in Eq. (7-37b) is to be determined by the boundary conditions at the interface $x = 0$. If we assume that the field for $x < 0$ is of the form

$$U(x) = \exp(+ip_0x) + R \exp(-ip_0x) \qquad (7\text{-}38)$$

If we require that both U(x) and its derivative be continuous at x = 0, we readily see that

$$A = \sqrt{p(0)} = \sqrt{p_0} \tag{7-39a}$$

$$R = -\frac{1}{\sqrt{p_0}}\int_0^{\infty} \exp[+i\Psi(t)]\,\frac{p'(t)}{2\sqrt{p(t)}}\,V(t)dt \tag{7-39b}$$

The author has not been able to find any literature which deals with the numerical integration of Eq. (7-37) on an electronic computer. Atkinson (Ref. 16) and Kay (Ref. 17) have discussed certain iteration processes which lead to solutions. One of the chief applications of this theory appears to be that of introducing the concept of a "reflection coefficient" which is defined as the ratio of W(x) to V(x), namely

$$R(x) = \frac{W(x)}{V(x)}$$

It is customary to approximate V(x) by neglecting the integral in Eq. (7-37b). This leads to the approximation

$$R(x) \approx -\exp[-i2\Psi(x)]\int_x^{\infty} \exp[+i2\Psi(t)]\,\frac{p'(t)}{2p(t)}\,dt \tag{7-40}$$

Our discussion of the replacement of the second-order equation of the form of Eq. (7-31) by the coupled first-order differential equations has taken us quite far astray from our main purpose which was that of finding approximations for the reflected wave which was defined by Eq. (7 -14). However, the practical importance of problems involving inhomogeneous media surrounding convex surfaces (such as the ionized layers surrounding a re-entry body) cannot be overemphasized. The WKB approximation has only limited applicability in some of these problems and in order to enlarge the scope of the applicability of our diffraction functions we need to

be aware of new methods which may offer the prospect of extending the theory to a wider range of problems.

The author would like to direct attention to the fact that the presence of the reciprocal of p(x) in Eq. (7-35) indicates that the type of decomposition of U(x) which is given in Eq. (7-32) can only be used "far away" from the turning points at which p(x) vanishes.

7.3 The Diffraction Formula

Let us now turn to the problem of expressing the Green's function $U(x,y;x_0,y_0)$ as defined in Eq. (7-9) in terms of the diffraction function $D(\xi,\zeta,\zeta_0,q)$ as defined in Eq. (7-1). The first step will be that of using the WKB estimates in Eq. (4-16) and the Wronskian definition in Eq. (4-19) to show that

$$\frac{\Psi^+(x_>,\lambda_s)\Psi^-(x_<,\lambda_s)}{W_s} \xrightarrow[k\to\infty]{} \frac{\exp\left[i\int_{x_<}^{x_>}\sqrt{[k^2g(u)+\lambda_s]}\,du\right]}{-i2\sqrt{\sqrt{[k^2g(x)+\lambda_s]}\,\sqrt{[k^2g(x_0)+\lambda_s]}}} \tag{7-41}$$

We then recall Eq. (5-2)

$$\lambda_s = k^2f_0 + (k^{\frac{2}{3}}f_1^{\frac{1}{3}})^2t_s$$

and observe that

$$\begin{aligned}\sqrt{[k^2g(x)+\lambda_s]} &= k\sqrt{[g(x)+f_0+(f_1/k)^{\frac{2}{3}}t_s]}\\ &\approx k\sqrt{[g(x)+f_0]} + \frac{1}{2}\frac{k^{\frac{1}{3}}f_1^{\frac{2}{3}}t_s}{\sqrt{[g(x)+f_0]}} + \cdots\end{aligned} \tag{7-42}$$

Let us now use the leading term of Eq. (7-42) in the denominator of Eq. (7-41) and retain the first two terms in the phase factor. We then have the approximation

$$\frac{\Psi^{+}(x_{>}, \lambda_s)\, \Psi^{-}(x_{<}, \lambda_s)}{W_s} \xrightarrow[k \to \infty]{}$$

$$\frac{\exp\left\{ ik \int_{x_<}^{x_>} \sqrt{g(u) + f_o}\, du + i \frac{1}{2} t_s\, k^{1/3} f_1^{\,2/3} \int_{x_<}^{x_>} \frac{1}{\sqrt{g(u) + f_o}}\, du \right\}}{- i\, 2\, k \sqrt[4]{g(x) + f_o}\ \ \sqrt[4]{g(x) + f_o}} \qquad (7\text{-}43)$$

We now recall from Eq. (5-2) that

$$t_s = \tau_s + \sum_{n=1}^{\infty} D_n(\tau_s, q)\, \beta^n \quad , \quad \beta = k^{-\frac{2}{3}} f_1^{\,-\frac{1}{3}} \qquad (7\text{-}44)$$

and from Eq. (5-112) that

$$\frac{1}{N_s} = \frac{k^{\frac{2}{3}} f_1^{\,\frac{1}{3}}}{[w_1'(\tau_s)]^2 - \tau_s [w_1(\tau_s)]^2}\ \frac{dt_s}{d\tau_s} \qquad (7\text{-}45)$$

Let us now introduce several definitions which will simplify the writing out of the formulae which are to follow. Let

$$\Upsilon = \Upsilon(x, x_0) = \int_{x_<}^{x_>} \sqrt{g(u) + f_o} \qquad (7\text{-}46)$$

$$\xi = \xi(x, x_0) = \frac{1}{2} k^{\frac{1}{3}} f_1^{\,\frac{2}{3}} \int_{x_<}^{x_>} \frac{1}{\sqrt{g(u) + f_o}}\, du \qquad (7\text{-}47)$$

$$U(x, y; x_0, y_0) \xrightarrow[k \to \infty]{} \frac{i}{2} \frac{(f_1/k)^{\frac{1}{3}} \exp(ik\Upsilon)}{\sqrt[4]{g(x) + f_o}\ \sqrt[4]{g(x_0) + f_o}} \sum_{s=1}^{\infty} \left\{ \exp(i\xi t_s) \right.$$

$$\left. \frac{F(y, \lambda_s)\, F(y_0, \lambda_s)}{\tau_s [w_1(\tau_s)]^2 - [w_1'(\tau_s)]^2} \left[1 + \sum_{n=1}^{\infty} \frac{\partial D_n(\tau_s)}{\partial \tau_s} \beta^n \right] \right\} \qquad (7\text{-}48)$$

From the first several terms of Eq. (4-148) we find that we can

express the height gain function $F(y,\lambda)$ for $y \to 0$ and $k \to \infty$ in the form

$$F(y,\lambda) \sim w(t-u) - \frac{\beta h_2}{15}\left[(3u+2t)\,w(t-u) + (3u^2+4ut+8t^2)\,w'(t-u)\right] + \cdots \quad (7\text{-}49)$$

where

$$u = k^{2/3} f_1^{\,1/3}\, y \ , \quad \lambda = k^2 f_o + k^{4/3} f_1^{\,2/3}\, t \ , \quad \beta = k^{-2/3} f_1^{\,-1/3}$$

$$t = \tau + D_1(\tau)\,\beta + D_2(\tau)\,\beta^2 + \cdots$$

and the Airy function $w(z)$ is understood to be $w_1(z)$. For sufficiently large value of k, we have the parameter $\beta \to 0$ and we can introduce the further approximations

$$t_s \approx \tau_s \quad (7\text{-}50)$$

$$F(y_0,\lambda)F(y,\lambda) \approx w_1(\tau_s - \mu^2)\,w_1(\tau_s - \mu_0^{\,2}) \quad (7\text{-}51)$$

where

$$\mu^2 = k^{\frac{2}{3}} f_1^{\,\frac{1}{3}}\, y \quad (7\text{-}52a)$$

$$\mu_0^{\,2} = k^{\frac{2}{3}} f_1^{\,\frac{1}{3}}\, y_0 \quad (7\text{-}52b)$$

If we now use Eq. (7-50) and Eq. (7-51) in Eq. (7-48), and recall that we have $\beta \to 0$, we find that

$$U(x,y;x_0,y_0) \xrightarrow[\substack{\beta \to 0 \\ y \to 0 \\ y_0 \to 0}]{} \frac{(f_1/k)^{\frac{1}{3}} \exp(ik\Upsilon)}{2\sqrt[4]{g(x)+f_o}\ \sqrt[4]{g(x_o)+f_o}}\, D(\xi,\mu^2,\mu_0^{\,2},q) \quad (7\text{-}53)$$

This result should only be used in the thin boundary layer for which both μ^2 and $\mu_0^{\,2}$ are small (where "small" probably means that we should require that these quantities be less than unity).

For larger heights, we should use the WKB approximation given in Eq. (6-16), namely

$$F_1(y,\lambda) \sim \frac{k^{1/3} f_1^{\ 1/6}}{\sqrt[4]{k^2 f(y) - \lambda}} \exp\left(i\frac{\pi}{4}\right) \exp\left[i \int_{y_1}^{y} \sqrt{k^2 f(v) - \lambda}\ dv \right] \tag{7-54}$$

where

$$\lambda = k^2 f_o + k^{4/3} f_1^{\ 2/3}\, t$$

From Eqs. (5-42) and (5-45) we see that

$$y_1 \xrightarrow[\beta \to 0]{} \beta^2 \tau_s$$

and therefore we will replace the lower limit in Eq. (7-54) by 0 when we consider the case $\beta \to 0$. We also take

$$\lambda_s \approx k^2 f_0 + (k^2 f_1)^{\frac{2}{3}} \tau_s$$

and observe that the radicals can be approximated by

$$\sqrt{k^2 f(y) - \lambda_s} \approx k\sqrt{f(y) - f_0} + \frac{1}{2} \frac{k^{\frac{1}{3}} f_1^{\ \frac{2}{3}} \tau_s}{\sqrt{f(y) - f_0}} + \cdots$$

We can then approximate Eq. (7-54) by

$$F_1(y,\lambda_s) \approx \left(\frac{f_1}{k}\right)^{1/6} \frac{\exp(i\,\pi/4)}{\sqrt[4]{f(y) - f_o}} \exp\left\{ ik \int_0^y \sqrt{f(v) - f_o}\ dv \right.$$

$$\left. -i\frac{1}{2}\,\tau_s k^{\frac{1}{3}} f_1^{\ \frac{2}{3}} \int_0^y \frac{dv}{\sqrt{f(v) - f_o}} \right\} \tag{7-55}$$

We can then express $U(x,y;x_0,y_0)$ in the form

$$U(x,y,x_0,y_0) \xrightarrow[\substack{\beta \to 0 \\ y_0 \to \infty}]{} \left\{ \frac{i(f_1/k)^{\frac{1}{3}} \exp(ik\Upsilon + ik\Lambda_0 + i\frac{\pi}{4})}{\sqrt[4]{g(x) + f_o}\ \sqrt[4]{g(x_o) + f_o}\ \sqrt[4]{f(y_o) - f_o}} \right.$$

$$\left. \sum_{s=1}^{\infty} \frac{\exp(i\,\zeta\,\tau_s)\ w_1(\tau_s - y)}{\tau_s [w_1(\tau_s)]^2 - [w_1'(\tau_s)]} \right\} \tag{7-56}$$

where Λ_0 and ζ are defined by

$$\Lambda_0 = \Lambda(y_0) = \int_0^{y_0} \sqrt{f(v) - f_0}\, dv \qquad (7\text{-}57a)$$

$$\zeta = \frac{1}{2} k^{\frac{1}{3}} f_1^{\frac{2}{3}} \left[\int_{x_<}^{x_>} \frac{1}{\sqrt{g(u) + f_0}}\, du - \int_0^{y_0} \frac{1}{\sqrt{f(v) - f_0}}\, dv \right] \qquad (7\text{-}57b)$$

Since $w_1'(\tau_s) = q w_1(\tau_s)$, we observe that

$$\tau_s [w_1(\tau_s)]^2 - [w_1'(\tau_s)]^2 = (\tau_s - q^2)[w_1(\tau_s)]^2 \qquad (7\text{-}58)$$

and, therefore, $V_1(\zeta, q)$ as defined by Eq. (7-7b) can also be expressed in the form

$$V_1(\zeta, q) = i\, 2\sqrt{\pi} \sum_{s=1}^{\infty} \frac{\exp(i \zeta \tau_s)}{(\tau_s - q^2)\, w_1(\tau_s)} \qquad (7\text{-}59)$$

Therefore, if we let $y = 0$ in Eq. (7-56), we obtain

$$U(x,0;x_0,y_0) \xrightarrow[\substack{\beta \to 0 \\ y_0 \to \infty}]{} \left[\frac{(f_1/k)^{\frac{1}{3}} \exp(ik\Upsilon + ik\Lambda_0 + i\frac{\pi}{4})}{2\sqrt{\pi}\, \sqrt[4]{g(x) + f_0}\, \sqrt[4]{g(x_0) + f_0}\, \sqrt[4]{f(y_0) - f_0}} \right] V_1(\zeta, q) \qquad (7\text{-}60)$$

When both y and y_0 are large, we use Eq. (7-55) for both $F(y_0, \lambda_s)$ and $F(y, \lambda_s)$ and arrive at

$$U(x,y;x_0,y_0) \xrightarrow[\substack{\beta \to 0 \\ y_0 \to \infty \\ y \to \infty}]{} -\left[\frac{(f_1/k)^{\frac{1}{3}} \exp(ik\sigma)}{\sqrt{[\sqrt{g(x) + f_0}\, \sqrt{g(x_0) + f_0}\, \sqrt{f(y) - f_0}\, \sqrt{f(y_0) - f_0}\,]}} \right.$$

$$\left. \sum_{s=1}^{\infty} \frac{\exp(i \eta \tau_s)}{\tau_s [w_1(\tau_s)]^2 - [w_1'(\tau_s)]^2} \right]$$

$$= \frac{(f_1/k)^{\frac{1}{3}} \exp(ik\sigma + \frac{\pi}{4})}{2\sqrt{\pi [\sqrt{g(x) + f_0}\, \sqrt{g(x_0) + f_0}\, \sqrt{f(y) - f_0}\, \sqrt{f(y_0) - f_0}\,]}} V_{11}(\eta, q) \qquad (7\text{-}61)$$

where Eq. (7-58) has been used in Eq. (7-8b) in order to show that the reflection function can be written in the form

$$V_{11}(\eta,q) = i2\sqrt{i\pi} \sum_{s=1}^{\infty} \frac{\exp(i\eta\tau_s)}{\tau_s[w_1(\tau_s)]^2 - [w_1'(\tau_s)]^2} \tag{7-62}$$

and where σ is defined by

$$\sigma = \int_{x_<}^{x_>} \sqrt{g(u)+f_o}\,du + \int_0^y \sqrt{f(v)-f_o}\,dv + \int_0^{y_o} \sqrt{f(v)-f_o}\,dv \tag{7-63}$$

and the argument η is defined in terms of a quantity δ by means of

$$\eta = \frac{1}{2} k^{1/3} f_1^{2/3} \int_{x_<}^{x_>} \frac{du}{\sqrt{g(u)+f_o}} - \frac{1}{2} k^{1/3} f_1^{2/3} \left(\int_0^y + \int_0^{y_o} \right) \frac{dv}{\sqrt{f(v)-f_o}}$$

$$= \frac{1}{2} k^{1/3} f_1^{2/3} \delta \tag{7-64}$$

and δ is defined by

$$\delta = \int_{x_<}^{x_>} \frac{du}{\sqrt{g(u)+f_o}} - \int_0^y \frac{dv}{\sqrt{f(v)-f_o}} - \int_0^{y_o} \frac{dv}{\sqrt{f(v)-f_o}} \tag{7-65}$$

The physical significance of the parameters σ and δ has been discussed by Friedlander (Ref. 6). The path of the diffracted rays is depicted in Fig. 7-1. Friedlander showed that η is the geodesic distance of (x,y) from (x_0,y_0) measured along the descending branch (PA) of the glancing ray, the segment (AB) along the boundary, and the diffracted ray (BQ). The variable δ is the geodesic distance from A to B, i.e., the distance the wave travels along the surface. Friedlander showed that it was possible to also express δ in the form

$$\delta = \int_{x_1}^{x_2} \frac{du}{\sqrt{g(u)+f_o}} \tag{7-66}$$

where x_1 and x_2 are defined in Fig. 7-1. Similar discussions have been given by Seckler and Keller (Ref. 7) and by Jones (Refs. 8 and 9).

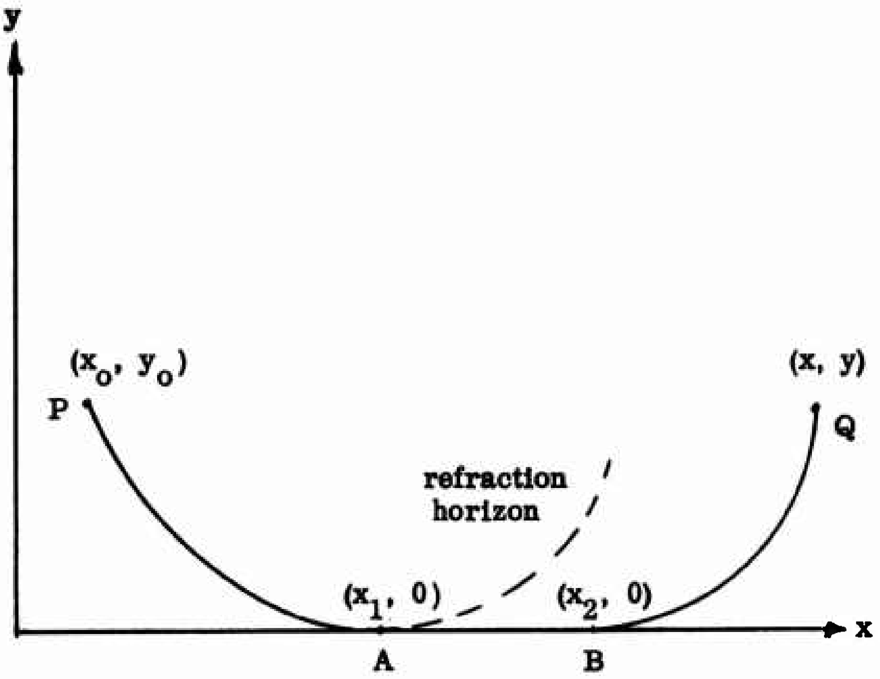

Fig. 7-1 Ray Paths in Inhomogeneous Medium

These geometrical interpretations for the parameters appearing in the asymptotic estimate for $U(x,y;x_0,y_0)$ are made possible by the use of the asymptotic estimate for the height gain function which is given as Eq. (7-55). This estimate was used in the above form by Friedlander (Ref. 6). However, an analogous result had already been used by Fock (Ref. 4, Eqs. (6.10) and (6.11, p. 274) in his classic paper of 1948.

Since δ as defined by Eq. (7-66) is the same as γ defined by Eq. (7-46), we observe that for $y = y_0 = 0$ that we can use the asymptotic estimate

$$U(x,0;x_0,0) \xrightarrow[\beta \to \infty]{} \frac{i(f_1/k)^{\frac{1}{3}} \exp(ik\sigma)}{\sqrt[4]{g(x)+f_0}\ \sqrt[4]{g(x_0)+f_0}} \sum_{s=1}^{\infty} \frac{\exp(i\eta\tau_s)[w_1(\tau_s)]^2}{\tau_s[w_1(\tau_s)]^2 - [w_1'(\tau_s)]^2} \quad (7\text{-}67)$$

along with the property in Eq. (7-58) to show that

$$U(x,0;x_0,0) \xrightarrow[\beta \to \infty]{} \frac{(f_1/k)^{\frac{1}{3}} \exp(ik\sigma + \frac{\pi}{4})}{\sqrt{\pi\eta[\sqrt{g(x)+f_0}\sqrt{g(x_0)+f_0}]}} V_0(\eta,q) \qquad (7\text{-}68)$$

where $V_0(\eta,q)$ is the attenuation function defined in Eq. (7-6), and Eq. (7-63) reduces to

$$\sigma = \int_{x_1}^{x_2} \sqrt{g(u) + f_0}\, du$$

For the case $y = 0$, we observe that

$$\sigma = \Upsilon + \Lambda_0$$

where Υ and Λ_0 are defined by Eqs. (7-46) and (7-57a), respectively. For this case, we also observe that ζ as defined by Eq. (7-57b) is equivalent to η as defined by Eq. (7-64). Therefore, we can express Eq. (7-60) in the alternative form

$$U(x,0;x_0,y_0) \xrightarrow[\substack{\beta \to 0 \\ y_0 \to \infty}]{} \frac{(f_1/k)^{\frac{1}{3}} \exp(ik\sigma + \frac{\pi}{4})}{2\sqrt{\pi\sqrt{g(x)+f_0}\sqrt{g(x_0)+f_0}\sqrt{f(y_0)-f_0}}} V_1(\eta,q) \quad (7\text{-}69)$$

The forms in which we have put Eqs. (7-61), (7-68) and (7-69) reveal that in each case where we have used the two-parameter functions $V_0(\eta,q)$, $V_1(\eta,q)$, and $V_{11}(\eta,q)$ that the parameter η is related to the distance that the wave has traveled along the interface at $y = 0$, whereas the parameter σ which has appeared in the exponential factor that multiplies these special functions is related to the geodesic distance that the wave has traveled along the ray path between the source at (x_0,y_0) and the receiver at (x,y).

Although we have obtained some useful special results in the course of the above discussion, we do not have a formula which is valid for arbitrary values of y and y_0 as $\beta \to 0$. In Eq. (7-53) we have obtained a result for the boundary layer in which both y and y_0

are assumed to be small. In Eq. (7-69) we required that $y = 0$ and $y_0 \to \infty$, whereas in Eq. (7-61) we required that <u>both</u> $y \to \infty$ and $y_0 \to \infty$. Let us now show how to obtain a result which contains all of the results given above as special cases.

7.4 A Digression on Hankel Functions

Before introducing the type of result which we want to employ in place of Eq. (7-55) or Eq. (7-51), let us consider the special case of the Hankel function for which Eq. (7-55) yields the asymptotic estimate

$$H_\nu^{(1)}(kr) \sim \sqrt{\frac{-2i}{\pi k\sqrt{r^2 - a^2}}}\ \exp\left[ik\left(\sqrt{r^2 - a^2} - a\tan^{-1}\frac{\sqrt{r^2 - a^2}}{a}\right) - i\,t\left(\frac{ka}{2}\right)^{1/3}\tan^{-1}\frac{\sqrt{r^2 - a^2}}{a}\right] \tag{7-70}$$

where

$$\nu^2 = (ka)^2 + 4\left(\frac{ka}{2}\right)^{4/3} t \tag{7-71}$$

The corresponding special form taken by Eq. (7-49) is

$$H_\nu^{(1)}(kr) = -\frac{i}{\sqrt{\pi}}\left(\frac{2}{ka}\right)^{1/3}\left\{w_1(t-u) - \frac{1}{30}\left(\frac{2}{ka}\right)^{2/3}\left[(3u + 2t)\,w_1(t-u) + (3u^2 + 4ut + 8t^2)\,w_1'(t-u)\right] + O\left[\left(\frac{2}{ka}\right)^{4/3}\right]\right\} \tag{7-72}$$

where

$$u = \left(\frac{2}{ka}\right)^{1/3} ka \log\frac{r}{a} = \left(\frac{ka}{2}\right)^{2/3}\log\left(\frac{r}{a}\right)^2 \tag{7-73}$$

The results in Eq. (7-72) are new since in the work of van der Pol, Bremmer, Fock, and other writers it has been customary to replace the Hankel functions $H_\nu^{(1)}(kr)$, $H_\nu^{(1)\prime}(kr)$, where

$$\nu = (ka) + (ka/2)^{1/3}\,\tau \tag{7-74}$$

by the asymptotic approximations

$$H_\nu^{(1)}(kr) \sim -\frac{i}{\sqrt{\pi}} \left(\frac{2}{ka}\right)^{1/3} w_1(\tau - y) \tag{7-75}$$

$$H_\nu^{(1)'}(kr) \sim \frac{i}{\sqrt{\pi}} \left(\frac{2}{ka}\right)^{2/3} w_1'(\tau - y) \tag{7-76}$$

where

$$y = \left(\frac{2}{ka}\right)^{1/3} k(r - a) \tag{7-77}$$

Since

$$u = \left(\frac{2}{ka}\right)^{1/3} ka \log\frac{r}{a} = \left(\frac{2}{ka}\right)^{1/3} ka \left[\frac{r-a}{a} - \frac{1}{2}\,\frac{(r-a)^2}{a^2} + \ldots\right] \tag{7-78}$$

we can express Eq. (7-72) in the alternative form

$$H_\nu^{(1)}(kr) = -\frac{i}{\sqrt{\pi}} \left(\frac{2}{ka}\right)^{1/3} \left\{ w_1(t - y) - \frac{1}{60}\left(\frac{2}{ka}\right)^{2/3} [(6y + 4t)\, w_1(t - y) + (9y^2 + 8y\,t + 16\,t^2)\, w_1'(t - y)] + O\,[(2/ka)^{4/3}] \right\} \tag{7-79}$$

A comparison of Eq.(7-71) and Eq. (7-74) leads to

$$t = \tau + \frac{1}{4}\left(\frac{2}{ka}\right)^{2/3} \tau^2 \tag{7-80}$$

Therefore, Eq. (7-79) can be replaced by

$$H_\nu^{(1)}(kr) = -\frac{i}{\sqrt{\pi}} \left(\frac{2}{ka}\right)^{1/3} \left\{ w_1(\tau - y) - \frac{1}{60}\left(\frac{2}{ka}\right)^{2/3} [(6y + 4\tau)\, w_1(\tau - y) + (9y^2 + 8y\tau + \tau^2)\, w_1'(\tau - y)] + O\,[(2/ka)^{4/3}] \right\} \tag{7-81}$$

Although Eq. (7-81) is an extension of the classical approximation described by Eq. (7-75), the result is of limited usefulness unless ka is very large and (r - a) is small, or, more specifically,

$$r = a \left\{1 + O[(2/ka)^{2/3}]\right\} \tag{7-82}$$

The results in Eqs. (7-49) and (7-72) are also useful only in the "boundary layer" defined by Eq. (7-82).

If we use the property

$$w_1(t-y) \xrightarrow[y \gg t]{} \frac{1}{\sqrt[4]{y-t}} \exp\left[i\frac{2}{3}(y-t)^{3/2} + i\frac{\pi}{4}\right] \tag{7-83a}$$

we obtain

$$w_1(t-y) \xrightarrow[y\to\infty]{} \frac{1}{\sqrt[4]{y}} \exp\left[i\left(\frac{2}{3}y^{3/2} - y^{1/2}\,t + \frac{\pi}{4}\right)\right] \tag{7-83b}$$

We then find that according to the "boundary layer" approximation of Eq. (7-75) that

$$H_\nu^{(1)}(kr) \sim -\frac{i}{\sqrt{\pi}}\left(\frac{2}{ka}\right)^{1/3} w_1(\tau - y) \xrightarrow[y\to\infty]{} \frac{1}{\sqrt{\pi}\,\sqrt[4]{y}}\left(\frac{2}{ka}\right)^{1/3} \exp\left[i\frac{2}{3}y^{3/2} - i\,y^{1/2}\,\tau - i\frac{\pi}{4}\right] \tag{7-84}$$

or

$$H_\nu^{(1)}(kr) \sim \sqrt{\frac{-2i}{\pi k\sqrt{2a(r-a)}}} \exp\left\{i\frac{1}{3}ka\left[\frac{2(r-a)}{a}\right]^{3/2} - i\tau\left(\frac{ka}{2}\right)^{1/3}\sqrt{\frac{2(r-a)}{a}}\right\} \tag{7-85}$$

This result does not agree with Eq. (7-70). However, if in Eq. (7-70) we use the approximations

$$\sqrt{r^2 - a^2} = \sqrt{(r + a(r-a)} \sim \sqrt{2a(r-a)}$$

$$k\left(\sqrt{r^2-a^2} - a\tan^{-1}\frac{\sqrt{r^2-a^2}}{a}\right) \sim \frac{k}{3}\frac{(r^2-a^2)^{3/2}}{a^2} \sim \frac{ka}{3}\left[\frac{2(r-a)}{a}\right]^{3/2}$$

$$\tan^{-1}\frac{\sqrt{r^2-a^2}}{a} \sim \frac{\sqrt{r^2-a^2}}{a} \sim \sqrt{2\frac{r-a}{a}}$$

we see that Eq. (7-70) "joins" onto the result derived from Eq. (7-75). However, these results serve to emphasize the fact that Eq. (7-75) is only useful in the "boundary layer" defined by Eq. (7-82).

A study of these results reveals that if we merely replace the variable y by a new parameter

$$\zeta^2 = \left(\frac{ka}{2}\right)^{2/3}\frac{r^2-a^2}{a^2} \tag{7-86}$$

we obtain a modified approximation which has the property

$$H_\nu^{(1)}(kr) \sim -\frac{i}{\sqrt{\pi}} \left(\frac{2}{ka}\right)^{1/3} w_1(\tau - \zeta^2)$$

$$\xrightarrow[\zeta \to \infty]{} \sqrt{\frac{-2i}{\pi k\sqrt{r^2 - a^2}}} \exp\left[i \frac{ka}{3} \left(\frac{r^2 - a^2}{a^2}\right)^{3/2} - i \left(\frac{ka}{2}\right)^{1/3} \tau \left(\frac{r^2 - a^2}{a^2}\right)^{1/2} \right] \qquad (7\text{-}87)$$

This result has the advantage of leading to the same amplitude as given by Eq. (7-70). However, the phase is only an approximation to the phase in the exponent of Eq. (7-70). The phase can be readily corrected by observing that for given (finite) values of ζ and τ,

$$\exp\left[ik \left(\sqrt{r^2-a^2} - a \tan^{-1} \frac{\sqrt{r^2-a^2}}{a}\right) - i \left(\frac{ka}{2}\right)^{1/3} \tau \tan^{-1}\left(\frac{\sqrt{r^2-a^2}}{a}\right) - i\frac{2}{3}\zeta^3 + i\zeta\tau \right]$$

$$= \exp\left\{ i\left[\left(\frac{2}{ka}\right)^{2/3} \left(-\frac{2}{5}\zeta^5 + \frac{1}{3}\tau\zeta^3\right) + \left(\frac{2}{ka}\right)^{4/3} \left(\frac{2}{7}\zeta^7 - \frac{1}{5}\tau\zeta^5\right) + \cdots \right] \right\}$$

$$= 1 + O\left[\left(\frac{2}{ka}\right)^{2/3}\right] \qquad (7\text{-}88)$$

Therefore, the modified asymptotic estimate

$$H_\nu^{(1)}(kr) \sim \frac{i}{\sqrt{\pi}} \left(\frac{2}{ka}\right)^{1/3} \exp(i\Phi)\, w_1(\tau - \zeta^2) \qquad (7\text{-}89)$$

where

$$\Phi = k\sqrt{r^2-a^2} - ka \tan^{-1} \frac{\sqrt{r^2-a^2}}{a} - \left(\frac{ka}{2}\right)^{1/3} \tau \tan^{-1}\left(\frac{\sqrt{r^2-a^2}}{a}\right) - \frac{2}{3}\zeta^3 + \zeta\tau \qquad (7\text{-}90)$$

leads to Eq. (7-70) for $\zeta \to \infty$ and to Eq. (7-75) for $\zeta \to 0$.

If we use the property

$$\zeta^2 = y + \frac{1}{4}\left(\frac{2}{ka}\right)^{2/3} y^2 \qquad (7\text{-}91)$$

we can use Eq. (7-81) to obtain the result

$$H_\nu^{(1)} \sim -\frac{i}{\sqrt{\pi}} \left(\frac{2}{ka}\right)^{1/3} \left\{ w_1(\tau - \zeta^2) + \left(\frac{2}{ka}\right)^{2/3} \left[\left(-\frac{1}{15}\tau - \frac{1}{10}\zeta^2\right) w_1(\tau - \zeta^2) + \left(-\frac{\tau^2}{60} - \frac{2}{15}\tau\zeta^2 + \frac{2}{5}\zeta^4\right) w_1'(\tau - \zeta^2) \right] + O\left[\left(\frac{2}{ka}\right)^{4/3}\right] \right\} \qquad (7\text{-}92)$$

$$w'_1(\tau - \zeta^2) \xrightarrow[\zeta\to\infty]{} -i\sqrt{\zeta^2-\tau}\left[1 + i\frac{1}{4(\zeta^2-\tau)^{3/2}} - \frac{5}{32(\zeta^2-\tau)^3} + \cdots\right] w_1(\tau-\zeta^2)$$

$$= -i\left[\zeta - \frac{\tau}{2\zeta} + i\frac{1}{4\zeta^2} - \frac{1}{8}\frac{\tau^2}{\zeta^3} + i\frac{\tau}{4\zeta^4} - \frac{1}{16}\frac{\tau^3}{\zeta^5} - \frac{5}{32}\frac{1}{\zeta^5} + \cdots\right] w_1(\tau - \zeta^2) \tag{7-93}$$

we can show that

$$H_\nu^{(1)}(kr) \xrightarrow[\zeta\to\infty]{} -\frac{i}{\sqrt{\pi}}\left(\frac{2}{ka}\right)^{1/3}\left\{1 + \left(\frac{2}{ka}\right)^{2/3}\left[-i\frac{2}{5}\zeta^5 + i\frac{1}{3}\tau\zeta^3 + i\frac{1}{16}\frac{1}{\zeta} + \cdots\right] + O\left[\left(\frac{2}{ka}\right)^{4/3}\right]\right\} w_1(\tau - \zeta^2) \tag{7-94}$$

The presence of the positive powers of ζ in the coefficient of $(ka/2)^{-\frac{2}{3}}$ reveals that this form of representation for $H_\nu^{(1)}(kr)$ is only useful for moderate values of ζ. However, if we write Eq. (7-90) in the form

$$\Phi = 2\left(\frac{ka}{2}\right)^{2/3}\zeta - 2\left(\frac{ka}{2}\right)\tan^{-1}\left[\left(\frac{2}{ka}\right)^{1/3}\zeta\right] - \tau\left(\frac{ka}{2}\right)^{1/3}\tan^{-1}\left[\left(\frac{2}{ka}\right)^{1/3}\zeta\right] - \frac{2}{3}\zeta^3 + \zeta\tau$$

$$= \left(\frac{2}{ka}\right)^{2/3}\left(-\frac{2}{5}\zeta^5 + \frac{1}{3}\tau\zeta^3\right) + \left(\frac{2}{ka}\right)^{4/3}\left(\frac{2}{7}\zeta^7 - \frac{1}{5}\tau\zeta^5\right) + \cdots \tag{7-95}$$

we find that

$$\exp(i\Phi) = 1 + \left(\frac{2}{ka}\right)^{2/3}\left[-i\frac{2}{5}\zeta^5 + i\frac{1}{3}\tau\zeta^3\right] + O\left[\left(\frac{2}{ka}\right)^{4/3}\right] \tag{7-96}$$

Therefore, if we replace Eq. (7-94) by

$$H_\nu^{(1)}(ka) \sim -\frac{i}{\sqrt{\pi}}\left(\frac{2}{ka}\right)^{1/3}\exp(i\Phi)\left\{1 + \left(\frac{2}{ka}\right)^{2/3}\left(i\frac{1}{16}\frac{1}{\zeta} + \cdots\right) + O\left[\left(\frac{2}{ka}\right)^{4/3}\right]\right\} w_1(\tau - \zeta^2) \tag{7-97}$$

we find that

$$H_\nu^{(1)}(ka) \xrightarrow[\zeta\to\infty]{} -\frac{1}{\sqrt{\pi}}\left(\frac{2}{ka}\right)^{1/3}\exp(i\Phi)\left\{1 + O\left[\left(\frac{2}{ka}\right)^{4/3}\right]\right\} w_1(\tau - \zeta^2) \tag{7-98}$$

This result shows that not only do we get the correct phase for the leading term by using Eq. (7-89), but we also obtain a result

for $\zeta \to \infty$ in which the corrections of order $(ka/2)^{-\frac{2}{3}}$ vanish.

Eq. (7-97) cannot be used for small values of ζ because of the presence of the inverse powers of ζ. However, the alternative form

$$H_\nu^{(1)}(kr) \sim -\frac{i}{\sqrt{\pi}}\left(\frac{2}{ka}\right)^{1/3} \exp[i\,\Phi] \left\{ w_1(\tau-\zeta^2) + \left(\frac{2}{ka}\right)^{2/3} \left[\left(i\frac{2}{5}\zeta^5 - i\frac{1}{3}\tau\zeta^3 - \frac{1}{15}\tau - \frac{1}{10}\zeta^2\right) w_1(\tau-\zeta^2) + \left(-\frac{\tau^2}{60} - \frac{2}{15}\tau\zeta^2 + \frac{2}{5}\zeta^4\right) w_1'(\tau-\zeta^2)\right] + \cdots \right\} \qquad (7\text{-}99)$$

can be used for both large and small values of ζ. For large values of ζ, the form in Eq. (7-97) may be unsatisfactory because it may be more convenient to replace $w_1(\tau - \zeta^2)$ by an asymptotic expansion. We can obtain such a result from Eq. (7-99) by using the expansions

$$w_1(\tau-\zeta^2) \xrightarrow[\zeta \gg \tau]{} \frac{1}{\sqrt[4]{\zeta^2-\tau}} \exp\left\{i\frac{2}{3}(\zeta^2-\tau)^{3/2} + i\frac{\pi}{4}\right\} \left\{1 - \frac{1\,3\cdot5}{1!216}\left(\frac{3}{2}\right)\frac{1}{(\zeta^2-\tau)^{3/2}} - \frac{5\cdot7\cdot9\cdot11}{2!(216)^2}\left(\frac{3}{2}\right)^2 \frac{1}{(\zeta^2-\tau)^{3/2}} + \cdots\right\}$$

$$\xrightarrow[\zeta \to \infty]{} \frac{\exp\left(i\frac{2}{3}\zeta^3 - i\tau\zeta + i\frac{\pi}{4}\right)}{\sqrt{\zeta}} \left\{1 + \frac{i\tau^2}{4\zeta} + \frac{1}{\zeta^2}\left(\frac{\tau}{4} - \frac{\tau^4}{32}\right) + \frac{1}{\zeta^3}\left(-\frac{\tau^6}{384} + \frac{5}{48}\tau^3 - \frac{5}{48}\right) + \frac{1}{\zeta^4}\left(\frac{\tau^8}{6144} - \frac{7\tau^5}{384} + \frac{35}{192}\tau^2\right) + \frac{1}{\zeta^5}\left(\frac{\tau^{10}}{122880} - \frac{\tau^7}{512} + \frac{35}{512}\tau^4 - \frac{35}{192}\tau\right) + O\left(\frac{1}{\zeta^6}\right)\right\} \qquad (7\text{-}100)$$

and

$$w_1'(\tau-\zeta^2) \xrightarrow[\substack{\zeta \gg \tau \\ \zeta \to \infty}]{} \sqrt{\zeta}\exp\left(i\frac{2}{3}\zeta^3 - i\tau\zeta - i\frac{\pi}{4}\right)\left\{1 + \frac{i\tau^2}{4\zeta} - \frac{1}{\zeta^2}\left(\frac{\tau^4}{32} + \frac{\tau}{4}\right) + \frac{1}{\zeta^3}\left(-\frac{\tau^6}{384} - \frac{\tau^3}{48} + \frac{7}{48}\right) + \frac{1}{\zeta^4}\left(\frac{\tau^8}{6144} - \frac{\tau^5}{384} \cdot \frac{25}{192}\tau^2\right) + \frac{1}{\zeta^5}\left(\frac{\tau^{10}}{122880} - \frac{\tau^7}{1536} - \frac{35}{1536}\tau^4 + \frac{35}{192}\tau\right)\right.$$

(Eq. concluded on next page.)

$$+\frac{1}{\zeta^6}\left(-\frac{\tau^{12}}{2949120}+\frac{5}{73728}\tau^9+\frac{7}{6144}\tau^6-\frac{245}{2304}\tau^3+\frac{455}{4608}\right)$$

$$+O\left(\frac{1}{\zeta^7}\right)\Bigg\} \qquad (7\text{-}101)$$

In the discussion between Eq. (7-75) and Eq. (7-99), we have used τ as defined by Eq. (7-74) in order to conform with the usage of Fock, Wait, and other authors. However, for the sake of completeness, we will also set down the Hankel function results which involve t as defined in Eq. (7-71). In order to do this, we use Eq. (7-80) to write

$$\tau = 2\left(\frac{ka}{2}\right)^{2/3}\left[\sqrt{1+\left(\frac{2}{ka}\right)^{2/3} t} - 1\right] = t - \frac{1}{4}\left(\frac{2}{ka}\right)^{2/3} t^2 + O\left[\left(\frac{2}{ka}\right)^{4/3}\right] \qquad (7\text{-}102)$$

We then obtain from Eq. (7-72) the result

$$H_\nu^{(1)}(kr) \sim -\frac{i}{\sqrt{\pi}}\left(\frac{2}{ka}\right)^{1/3}\Bigg\{w_1(t-\zeta^2) - \frac{1}{30}\left(\frac{2}{ka}\right)^{2/3}\left[(3\zeta^2+2t)\,w_1(t-\zeta^2)\right.$$
$$\left. + (-12\zeta^4+4\zeta^2 t+8t^2)\,w_1'(t-\zeta^2)\right] + O\left[\left(\frac{2}{ka}\right)^{4/3}\right]\Bigg\} \qquad (7\text{-}103)$$

We can also obtain this result from Eq. (7-72) by using the property

$$u = \left(\frac{ka}{2}\right)^{2/3}\log\left(\frac{r}{a}\right)^2 = \zeta^2 - \frac{1}{2}\left(\frac{2}{ka}\right)^{2/3}\zeta^4 + \cdots \qquad (7\text{-}104)$$

In order to improve Eq. (7-103) for the case when ζ is very large, we define

$$\Psi = k\sqrt{r^2-a^2} - ka\tan^{-1}\frac{\sqrt{r^2-a^2}}{a} - t\left(\frac{ka}{2}\right)^{1/3}\tan^{-1}\frac{\sqrt{r^2-a^2}}{a} - \frac{2}{3}\zeta^3 + \zeta t$$
$$= \left(\frac{2}{ka}\right)^{2/3}\left(-\frac{2}{5}\zeta^5 + \frac{1}{3}t\zeta^3\right) + O\left[\left(\frac{2}{ka}\right)^{4/3}\right] \qquad (7\text{-}105)$$

and obtain the modified result

$$H_\nu^{(1)}(kr) \sim -\frac{i}{\sqrt{\pi}}\left(\frac{2}{ka}\right)^{1/3} \exp(i\Psi)\left\{ w_1(t-\zeta^2) + \left(\frac{2}{ka}\right)^{2/3} \left[\left(i\frac{2}{5}\zeta^5 - i\frac{1}{3}t\zeta^3 - \frac{1}{15}t - \frac{1}{10}\zeta^2\right) w_1(t-\zeta^2) + \left(-\frac{4}{15}t^2 - \frac{2}{15}t\zeta^2 + \frac{2}{5}\zeta^4\right) w_1'(t-\zeta^2)\right] + O\left[\left(\frac{2}{ka}\right)^{4/3}\right]\right\} \qquad (7\text{-}106)$$

These results which we have given for the Hankel function have proven to be very useful in the case of the diffraction of waves by spheres and circular cylinders. We want to now use these concepts to obtain a more useful form for the more general height gain function $F(y,\lambda)$.

7.5 The Extended Height Gain Approximation

The form taken by Eq. (7-99) can serve as a model for an extended result for $F_1(y,\lambda)$ which behaves like Eq. (7-49) for small values of y and which behaves like Eq. (7-55) for large values of y. The first step is to start from the asymptotic estimate of Eq. (7-49) and observe that

$$F_1(y,\lambda) \sim w_1(t-u) = w_1(t - k^{2/3} f_1^{1/3}) \xrightarrow[y\to\infty]{} \left(\frac{f_1}{k}\right)^{1/6} \frac{\exp(i\frac{\pi}{4})}{\sqrt[4]{f_1 y}} \exp\left\{ i\frac{2}{3} k f_1^{1/2} y^{3/2} - i\,t\,k^{1/3} f_1^{1/6} y^{1/2}\right\} \qquad (7\text{-}107)$$

Since the factor multiplying the exponent in Eq. (7-55) differs from that in Eq. (7-107), we take a hint from Eq. (7-86) and define

$$\zeta^2 = \left(\frac{k}{f_1}\right)^{2/3} [f(y) - f_0] = u + \beta h_2 u^2 + \beta^2 h_3 u^3 + O(\beta^3) \qquad (7\text{-}108a)$$

or

$$u = \zeta^2 - \beta h_2 \zeta^4 + O(\beta^2) \qquad (7\text{-}108b)$$

Next we take a hint from Eq. (7-105) and observe that the exponent in Eq. (7-55) is of the form

$$k\int_0^y \sqrt{f(v)-f_o}\,dv - \frac{1}{2}k^{1/3}f_1^{2/3}\,t\int_0^y \frac{dv}{\sqrt{f(v)-f_o}} = \frac{2}{3}\zeta + \beta\,(-\frac{4}{5}h_2\zeta^5 + \frac{2}{3}h_2\zeta^3 t) + O(\beta^2) \tag{7-109}$$

Therefore, by defining

$$\Psi = k\int_0^y \sqrt{f(v)-f_o}\,dv - \frac{1}{2}k^{1/3}f_1^{2/3}\,t\int_0^y \frac{dv}{\sqrt{f(v)-f_o}} - \frac{2}{3}\zeta^3 + t\zeta \tag{7-110a}$$

$$= 2\beta h_2\,(-\frac{2}{5}\zeta^5 + \frac{1}{3}t\,\zeta^3) + O(\beta^2) \tag{7-110b}$$

and observing that

$$w_1(t-u) = w_1(t-\zeta^2) + \beta h_2\zeta^4\,w_1'(t-\zeta^2) + O(\beta^2) \tag{7-111}$$

we can modify Eq. (7-49) to obtain the representation

$$F_1(y,\lambda) = \exp(i\Psi)\Big[w_1(t-\zeta^2) + 2\beta h_2\,[(i\frac{2}{5}\zeta^5 - i\frac{1}{3}t\zeta^3 - \frac{1}{15}t - \frac{1}{10}\zeta^2)\,w_1(t-\zeta^2) + (-\frac{4}{15}t^2 - \frac{2}{15}t\zeta^2 + \frac{2}{5}\zeta^4)\,w_1'(t-\zeta^2)] + O(\beta^2)\Big] \tag{7-112}$$

This extended form of the asymptotic expansion for $F_1(y,\lambda)$ is very useful in the problem of finding representation for the Green's function $U(x,y;x_0,y_0)$ which involve the van der Pol-Bremmer formula $D(\xi,\zeta,\zeta_0,q)$. Thus, we can now replace Eq. (7-53) by

$$U(x,y;x_o,y_o) \sim \left(\frac{f_1}{k}\right)^{1/3} \frac{\exp(i k \sigma)}{2\sqrt[4]{g(x)+f_o}\,\sqrt[4]{g(x_o)+f_o}}\,E(\eta,\gamma^2,\gamma_o^2,q) \tag{7-113}$$

where

$$E(\eta,\gamma^2,\gamma_o^2,q) = \exp[-i\frac{2}{3}(\gamma^3+\gamma_o^3)]\,D(\xi,\gamma^2,\gamma_o^2,q) \tag{7-114}$$

and

$$\xi = \eta + \gamma + \gamma_o \tag{7-115a}$$

$$\gamma^2 = \left(\frac{k}{f_1}\right)^{2/3} [f(y) - f_0] \tag{7-115b}$$

$$\gamma_0^2 = \left(\frac{k}{f_1}\right)^{2/3} [f(y) - f_0] \tag{7-115c}$$

and σ and η are defined in Eqs. (7-63) and (7-64), respectively.

The result contained in Eq. (7-113) is the desired extension of the previous results. For example, the limiting form

$$E(\eta, \gamma^2, \gamma_0^2, q) \xrightarrow[\substack{\gamma \to \infty \\ \gamma_0 \to \infty}]{} \frac{1}{2} \sqrt{\frac{1}{\pi \gamma \gamma_0}} \; V_{11}(\eta, q) \tag{7-116}$$

leads to Eq. (7-61). The limiting form

$$E(\eta, 0, \gamma_0^2, q) \xrightarrow[\gamma_0 \to \infty]{} \frac{1}{2} \sqrt{\frac{1}{\pi \gamma_0}} \; V_1(\eta, q) \tag{7-117}$$

leads to Eq. (7-69). If we use the approximation

$$f(y) \approx f_0 + f_1 y$$

in Eqs. (7-115b) and (7-115c) we see that

$$\gamma^2 \approx \mu^2$$

$$\gamma_0^2 \approx \mu_0^2$$

the boundary layer approximation of Eq. (7-53) follows from the more general result in Eq. (7-113).

7.6 Extension of Bremmer's Results

The approximation which we have given above as Eq. (7-113) is one of the key results of our investigation. This result permits us have an asymptotic estimate which is valid for arbitrary values of the height parameters provided the parameter k is large enough, i.e., $\beta = k^{-\frac{2}{3}} f_1^{-\frac{1}{3}} \to 0$. Eq. (7-113) is only the leading term in

an asymptotic expansion of $U(x,y;x_0,y_0)$ in powers of β. We have not attempted to include above the further terms in the expansion but it is readily seen that we have laid out the necessary formulae from which such an expansion can be obtained. The further terms in the expansion will involve various derivatives (and integrals) of the basic function $D(\xi,\zeta,\zeta_0,q)$ which was defined in Eq. (7-1). In a later phase of the present research work we will give the explicit forms for these coresctions along with numerical work related to finding the deriva tives and the integrals of $D(\xi,\zeta,\zeta_0,q)$ which are required.

We would like to conclude this discussion by indicating how one can modify the theory as given by Bremmer (Ref. 18) for the propagation of waves through a concentrically stratified troposphere. The results that we wish to give could be obtained directly from Eq. (7-113). However, we have chosen to employ the notation used by Bremmer and to assume that the reader who wishes to follow the discussion which we now undertake will have a copy of Bremmer's paper before him.

We start by making a few changes in the notation employed by Bremmer. We define

$$n_o = n_{eff}(a) = n(a) \tag{7-118a}$$

$$a_o = a_{eff} = a\,\frac{n(a)}{n(a) + a\,n'(a)} \tag{7-118b}$$

$$n(r) = n_{eff}(r) \tag{7-118c}$$

for convenience in writing out certain equations for which Bremmer employs "eff" as subscripts. We then replace Eq. (58) of Bremmer's paper with the asymptotic estimate

$$u_\ell(r) \sim \frac{2}{3}\eta_\infty^{3/2} \tag{7-119a}$$

where

$$\frac{2}{3}\eta_\infty^{3/2} = k\int_a^r \frac{1}{\rho}\sqrt{\rho^2 n^2(\rho) - a^2 n^2(a)}\, d\rho - \left(\frac{k n_o a^3}{2 a_o^2}\right)^{\frac{1}{3}} \theta(r)\, t_s \qquad (7\text{-}119b)$$

$$\theta(r) = n_o a \int_a^r \frac{1}{\rho} \frac{1}{\sqrt{\rho^2 n^2(\rho) - a^2 n^2(a)}}\, d\rho \qquad (7\text{-}119c)$$

The results which we have given in Eq. (7-119) are directly related to the form of the exponent which is used in the modified WKB approximation which we gave above as Eq. (7-55). In fact, we now replace Bremmer's WKB approximation for his height gain function by the result

$$f_\ell(r) \sim \frac{1}{r\sqrt[4]{n^2(r) - \frac{a^2}{r^2}\, n^2(a)}} \exp\left(i\,\frac{2}{3}\eta_\infty^{3/2}\right) \qquad (7\text{-}120)$$

which is essentially our Eq. (7-55). This result is to be used when

$$k\int_a^r \frac{1}{\rho}\sqrt{\rho^2 n^2(\rho) - a^2 n^2(a)}\, d\rho \gg 1 \qquad (7\text{-}121)$$

which requires $r > a$. For $r \to a$, we use the asymptotic approximation which involves the Airy function, namely

$$f_\ell(r) \sim \sqrt{\frac{i}{ar}} \left(\frac{k a_o}{2 n_o^2}\right)^{1/6} w_1\left(t_s - \zeta^2\right) \qquad (7\text{-}122)$$

where

$$\zeta^2 = \left(\frac{k n_o a_o}{2}\right)^{2/3} \frac{r^2 n^2(r) - a^2 n^2(a)}{a^2 n^2(a)} \qquad (7\text{-}123)$$

Since

$$w_1\left(t_s - \zeta^2\right) \xrightarrow[\zeta \to \infty]{} \frac{1}{\sqrt{\zeta}} \exp\left(i\,\frac{2}{3}\zeta^3 - i\zeta t_s - i\,\frac{\pi}{4}\right) \qquad (7\text{-}124)$$

we obtain the result

$$f_\ell(r) \xrightarrow[\zeta \to \infty]{} \frac{1}{r\sqrt[4]{n^2(r) - \frac{a^2}{r^2}\, n^2(a)}} \exp\left(i\frac{2}{3}\zeta^3 - i\zeta t_s\right) \tag{7-125}$$

Since this result does not agree with that given as Eq. (7-120), it is proposed that Eq. (7-122) be replaced by the modified form

$$f_\ell(r) \sim \sqrt{\frac{1}{ar}}\left(\frac{ka_o}{2n_o^2}\right)^{1/6} \exp\left[i\frac{2}{3}\eta_\infty^{3/2} - i\frac{2}{3}\zeta^3 + i\zeta t_s\right] w_1\left(t_s - \zeta^2\right) \tag{7-126}$$

which contains both the properties of Eq. (7-120) and Eq. (7-122).

We observe that

$$f_\ell(a) \sim \frac{\sqrt{1}}{a}\left(\frac{ka_o}{2n_o^2}\right)^{1/6} w_1\left(t_s\right) \tag{7-127}$$

and therefore the height gain ratio given as Eq. (59) by Bremmer is to be replaced by

$$\frac{f_\ell(r)}{f_\ell(a)} \sim \frac{a}{r} \exp\left[i\frac{2}{3}\eta_\infty^{3/2} - i\frac{2}{3}\zeta^3 + i\zeta t_s\right] \frac{w_1\left(t_s - \zeta^2\right)}{w_1\left(t_s\right)} \tag{7-128}$$

Let us now turn to the problem of evaluating the expression referred to in Bremmer's paper as Eq. (38). The quantity to be evaluated is

$$\frac{\partial}{\partial \ell}\left\{\frac{\frac{\partial}{\partial r}\left(r f_\ell\right)}{r f_\ell}\right\} = -\frac{2kan_o}{a^2 f_\ell^2(a)} \int_a^\infty f_\ell^2(r)\, dr \tag{7-129}$$

and we will use the approximations

$$f_\ell(r) \sim \frac{\sqrt{1}}{a}\left(\frac{ka_o}{2n_o^2}\right)^{1/6} w_1\left(t_s - \zeta_o^2\right) \tag{7-130a}$$

$$\zeta_o^2 = \left(\frac{2}{kn_oa_o}\right)^{1/3} kn_o y = \left(\frac{2k^2n_o^2}{a_o}\right)^{1/3} y \quad , \; y = r - a \tag{7-130b}$$

which are valid for the situation $r \to a$. From the integral which we designated as Eq. (5-126a) we find that

$$\int_0^\infty w_1^2(t_s - \alpha y)\, dy = \frac{1}{\alpha}\left[t_s w_1^2(t_s) - w_1'^2(t_s)\right] \qquad (7\text{-}131)$$

Therefore, we can use the approximate result

$$\int_a^\infty f_\ell^2(r)\, dr \sim \left[\frac{\sqrt{I}}{a}\left(\frac{ka_o}{2n_o^2}\right)^{1/6}\right]^2 \int_0^\infty w_1^2\left[t_s - \left(\frac{2k^2 n_o^2}{a_o}\right)^{1/3} y\right] dy$$

$$= \left(\frac{a_o}{2k^2 n_o^2}\right)^{1/3} \left[\frac{\sqrt{I}}{a}\left(\frac{ka_o}{2n_o^2}\right)^{1/6}\right]^2 \left\{t_s\left[w_1(t_s)\right]^2 - \left[w_1'(t_s)\right]^2\right\} \qquad (7\text{-}132)$$

to show that

$$\frac{\partial}{\partial \ell}\left\{\frac{\frac{\partial}{\partial r}(rf_\ell)}{rf_\ell}\right\} \sim -\frac{2kn_o}{a}\left(\frac{a_o}{2k^2 n_o^2}\right)^{1/3} \frac{t_s\left[w_1(t_s)\right]^2 - \left[w_1'(t_s)\right]^2}{\left[w_1(t_s)\right]^2}$$

$$= \frac{2^{2/3} k^{1/3} n_o^{1/3} a_o^{1/3}}{a}\left[-t_s + q^2\right] \qquad (7\text{-}133)$$

where t_s is defined by the familiar equation.*

$$w_1'(t_s) - q\, w_1(t_s) = 0$$

where we must let q be related to Bremmer's notation by means of the relation

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*The present author suspects that Bremmer's Eq. (38) and all results based upon it are in error. We need to observe that in Bremmer's notation

$$-t_s = \left[\frac{3}{2}\, u_\ell(a)\right]^{2/3}$$

and therefore a comparison of our Eq. (7-133) with Bremmer's Eq. (38) suggests that $n_{eff}(a)$ should be replaced by $[n_{eff}(a)]^{\frac{1}{3}} = n_0^{\frac{1}{3}}$.

$$q = \frac{\Gamma}{a}\left\{\frac{a_o}{2k^2 n_o^2}\right\}^{1/3} = \frac{\Gamma}{kn_o a}\left\{\frac{kn_o a_o}{2}\right\}^{1/3} \tag{7-134}$$

Let us observe that Bremmer's Eq. (41) appears as

$$\ell \sim kn_o a + \left(\frac{kn_o a^3}{2a_o^2}\right)^{1/3} t_s \tag{7-135}$$

in our notation. We can then use the results which we have given above to rewrite Bremmer's Eq. (42) in the extended form

$$\Pi(r,\theta) \sim \left(\frac{B}{Cb}\right)\sqrt{\frac{i8\pi kn_o a}{r\,b\,\sin\theta}}\;\frac{1}{2}\left(\frac{2}{kn_o a_o}\right)^{1/3} \exp\left[ikL - i\omega t\right] E\left(\xi, \zeta^2(r), \zeta^2(b), q\right) \tag{7-136}$$

where $E(\xi,\zeta^2(r),\zeta^2(b),q)$ is the function defined above as Eq. (7-114) and L is the path length based upon the geometrical optics of inhomogeneous media

$$L = n_o a\theta + \left[\left(\int_a^r + \int_a^b\right)\frac{1}{\rho}\sqrt{\rho^2 n^2(\rho) - a^2 n^2(a)}\; d\rho\right] \tag{7-137}$$

and the parameters ξ and ζ are defined by

$$\xi = \left(\frac{kn_o a^3}{2a_o^2}\right)^{1/3}\left[\theta - an_o\left[\int_a^r + \int_a^b\right]\frac{d\rho}{\rho\sqrt{\rho^2 n^2(\rho) - a^2 n^2(a)}}\right] \tag{7-138}$$

$$\zeta^2(r) = \left(\frac{kn_o a_o}{2}\right)^{2/3} \frac{r^2 n^2(r) - a^2 n^2(a)}{a^2 n^2(a)} \tag{7-139}$$

In Bremmer's paper the concept of a "critical altitude" is introduced and certain expressions are derived which are to be used in place of his Eq. (42) when the source or the receiver lie above

this critical height. A comparison of Bremmer's discussion with the discussion which we have given above will reveal that Bremmer's Eq. (60) is the analog of our Eq. (7-69) and his Eq. (61) is the analog of our Eq. (7-61). Just as we have caused Eqs. (7-61) and (7-69) to follow as special cases of the more general result given in Eq. (7-113), so also the results of Bremmer's Eqs. (60) and (61) follow as special cases of the more general result which we have given above as Eq. (7-136). All of these results which we have based upon the discussion in Bremmer's paper correspond to results which are valid in the vicinity of, and below, the horizon. This is also true of our Eq. (7-136)

Further research is needed to obtain an extension of the results which involve the function $D(\xi,\zeta,\zeta_0,q)$ which will be valid within the lighted region and which will pass over smoothly to agree with the results of the method of stationary phase as outlined in Section 7.2. The results which we have discussed in Section 3 will provide some "checks" upon the extended theory when the research is completed.

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Section 8
CHEBYSHEV POLYNOMIAL REPRESENTATIONS

8.1 Introduction

The time-span of the present contract was not adequate for the author to be able to develop the representations in terms of Chebyshev polynomials which he envisioned at the start of this program. The author's increasing awareness of the necessity to deal with functions defined in the complex-plane has dampened a little his orginal enthusiasm for the utility of the Chebyshev polynomials. However, the conviction that the Chebyshev polynomials will play an important role in future work has not been undermined in spite of the growing awareness of the fact that the use of these polynomials does not provide a panecea for all of the representation problems that confront the numerical analyst in the problems that arise in diffraction theory.

The past decade has witnessed a rapid growth of interest in a branch of applied mathematics which will probably come to be known as the constructive theory of functions.* Until recently, there appeared to be much more interest in this subject displayed by the papers appearing in Russian mathematical journals than in the papers being presented for publication in journals whose authors were predominately from the United States. This fact can probably be traced to the fact that the Russian students of Chebyshev and Bernstein found their motives for the study of this class of problems through inspiration from their teachers who had

*An excellent introduction to the subject is to be found in the monograph: J. Todd, Introduction to the Constructive Theory of Functions, Basel, Birkhauser, 1963

found these problems to be fascinating. The recent upsurge of interest in the United States would seem to be directly related to the fact that the digital computer has given a "new life" to this "more-or-less classical" branch of mathematics.

The basic problem is that of constructing approximate representations of functions in terms of simpler functions. To the rapidly expanding literature in English, there have recently been added a number of English translations of standard works which originally appeared in Russian.*

In this section, we are interested in only a very narrow spectrum of the topics in the theory of the construction of approximate representations for mathematical functions. We want to consider the advantages of expanding a function defined over a given range of a real variable in terms of certain types of Fourier series in which one employs the so-called Chebyshev polynomials $T_n(x)$ and $T_n^*(x)$ which are defined by means of the relations

$$T_n(x) = \cos[n \cos^{-1}(x)] \quad , -1 < x < 1 \qquad (8\text{-}1a)$$

$$T_n^*(x) = \cos[n \cos^{-1}(2x - 1)] \, , \quad 0 < x < 1 \qquad (8\text{-}1b)$$

In view of the fact that there exists such a voluminous literature upon the so-called Chebyshev approximation problem, it is almost regretable that the nomenclature "Chebyshev polynomials" has come to be associated with the method of trigonometrical interpoltation of empirical and analytical functions which has been developed since the publication in 1938 of a classic paper by

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*I.P. Natanson, Constructive Theory of Functions, AEC Trans. TR-4503, 1961

E. Remez, General Computation Methods for Chebyshev Approximation, AEC Trans. TR-4491, 1960

N. I. Achieser, Theory of Approximation, New York, Ungar, 1956

Lanczos (Ref. 1). The "Chebyshev approximation problem" deals with the developing of a representation for a function which possesses the minimum maximum error. The reader who is not familiar with these so-called "best fit" procedures may find it particularly instructive to "thumb through" the small handbook prepared by Hastings (Ref. 2) and to read the chapter on these methods which appeared in a recent book by Ralston (Ref. 37).

The author wishes the reader to grasp the essential point that the Chebyshev polynomials arise only in those problems for which the form chosen for the approximation to the function is that of a polynomial. Since the Chebyshev approximation theory often deals with representations in the form of continued fractions, rational fractions, series of exponentials, etc., there is often no relevance between the papers on Chebyshev "best fit" and the class of polynomials problems which the author proposes as a means of representing certain functions which occur in diffraction theory. Even when the mode of representation which is chosen is that of a polynomial, the representation advocated by the author is that of a truncated Chebyshev series expansion. Although these truncated series may be closely related to the "best fit" polynomial, the amount of labour necessary to compute the "Chebyshev approximation" is far greater than that of the method of trigonometric interpolation which is employed when the truncated Chebyshev series expansion is used to represent the function. In a recent paper, Clenshaw (Ref. 3) has discussed this problem and his references will lead the reader to other sources in which the Chebyshev series expansions are compared with the "best fit" polynomials. The technical report by Murnaghan and Wrench (Ref. 4), which is summarized by the recent journal article (Ref. 5), affords an excellent example of the labour which is required to arrive at the "best fit" polynomial. For most practical problems, the obtaining of the "best fit" polynomial appears to the present author to be primarily of academic interest if one has

at his disposal the means of obtaining readily a truncated Chebyshev polynomial expansion.

8.2 Polynomial Expansions

Unless the reader has had experience with numerical analysis, he will probably be inclined to think of representations of functions in terms of Taylor series, asymptotic expansions in inverse powers of the variable, or in terms of Fourier series. The Taylor series and the Fourier series are the most attractive from the standpoint of our knowledge of what types of functions can be represented and of the rules for the determination of the coefficients in the expansions. Each of these types of expansions involve well-defined infinite processes. Within the range of convergence of the series, we can achieve an arbitrary degree of accuracy by considering a sufficient number of terms and by carrying each term to a sufficient number of significant figures. The asymptotic series, on the other hand, are far more difficult to deal with since the terms in the series may not display a behavior which permits one to evaluate the function for the values of the variable which are of interest.

We can work with the Taylor series with the knowledge that the series will converge within any circle about the expansion point which does not intersect a point at which the function possesses a singularity. Lanzcos (Ref. 1) has written a very interesting paper in which he demonstrates that the Taylor series is an <u>extrapolating series</u> which takes all its information about the function at a given point and then employs this set of information to represent the function in a finite region which surrounds the given point. The demands placed upon such a function are quite severe and we describe these conditions by saying that the function must be analytie in the neighborhood of the expansion point. The function must possess derivatives

of all orders at the point, and the value of the derivative must be independent of the direction in which we approach the point from a neighboring point in the complex plane.

Lanczos (Ref. 1) emphasizes the fact that the Fourier series is an interpolating series. If we consider the relations

$$f(x) = \sum_{n=-\infty}^{\infty} A_n(a,b)\exp\{i2\pi n[(x-a)/(b-a)]\} \tag{8-2}$$

$$A_n(a,b) = \int_a^b f(x)\exp\{-i2\pi n[(x-a)/(b-a)]\}[dx/(b-a)] \tag{8-3}$$

we see that $f(x)$ is represented on $a < x < b$ in terms of the values of the function over this range. The function $f(x)$ need not be defined anywhere except on the range $a < x < b$, and, in particular, it need not be analytic in the neighborhood of this range since the only requirement is that the integral be integrable. Therefore, the Fourier series permits us to work with a much broader class of functions than the exremely restricted class of functions which possess Taylor series.

Research in the diffraction of waves by obstacles which are larger than a wavelength requires the investigator to possess a certain amount of familiarity with asymptotic series. Since many of these series have the property that they converge in the beginning but that the terms increase in magnitude after a certain minimum, the practical use of such series is sufficiently difficult that even the subject of such series is avoided in many textbooks. However, the modern reader now has access to two very clear introductions in the form of the little monographs by Copson (Ref. 6) and Erdélyi (Ref. 7). Lanzcos (Ref. 1) discusses these types of series from the standpoint of interpolation and extrapolation and shows that they resemble the Taylor series in that they are extrapolating

series which involve information about the function at a single point (usually the point at infinity since the series are usually in inverse powers of the variable). Furthermore, Lanzcos remarks that, unlike the Taylor series, the value of the derivative need not be the same when infinity is approached along different rays in the complex plane. This phenomena greatly complicates the practical use of asymptotic expansions of functions for complex values of the variable since it is often found that different asymptotic expansions are required for different sectors in the complex plane as the variable is allowed to tend to infinity. With a number of problems that occur in diffraction theory, this phenomenon is known as the "Stokes' phenomenon" and it has been discussed in a number of texts which have appeared in recent years. The author particularly recommends to the reader who is unfamiliar with this phenomenon that he consult the monographs by Jeffreys (Ref. 8) and by Heading (Ref. 9).

The Taylor series and the asymptotic series have something in common in the sense that when they are employed to evaluate a function that only a finite number of terms are employed. The numerical problem then becomes that of evaluating a polynomial. Until the usage of electronic computers became widespread, very little thought was given to the fact that the classical Taylor series and the asymptotic series are very inefficient except for the cases in which the value of the function is obtained by use of only a few terms. Not only does the electronic computer afford us a motivation for seeking a more efficient polynomial representation, but the ease of evaluating polynomials on a computer has given a "new life" to the classic theorem of Weierstrass which asserts that: For any continuous function f(x) defined upon the range $a < x < b$, there exists a sequence of algebraic polynomials of the form

$$P_n(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots + c_nx^n$$

which converges uniformly to f(x) on $a \leq x \leq b$. This theorem informs us that there exist polynomial representations for functions which may not possess a Taylor series over the range of interest.

The finite word-length of the electronic computer has changed some of the basic philosophy regarding what constitutes a "good" algorithm for the evaluation of a function. If we are to work with floating point numbers which consist of 8-digits (which is generally the case in FORTRAN compilers when one works in single-precision), we readily see that any term in the polynomial which is less than 10^{-8} times the sum of the polynomial will not be used in forming the sum. Also, since we seldom can find a single algorithm which is useful for an arbitrary value of the variable, we will find ourselves frequently writing a computer program which is to be used only for a certain range of the variables. When the values of the variable are taken over a real range, and the value of the variable which we will use in the program is likely to lie anywhere in the range with almost equal probability, we arrive at a situation where the conventional power series and asymptotic expansions become relatively inefficient as compared with a Fourier series for the function developed over this range.

The Chebyshev polynomials are nothing more than a specific type of Fourier series which are particularly useful when the function is to be computed for arbitrary values of x lying in the range $a \leq x \leq b$ and the computer to be employed has a finite word-size and hence the accuracy to which the function can be computed is limited by the computer.

Since the transformation

$$x = (2y - b - a)/(b - a) \qquad (8\text{-}4)$$

transforms the segment $a \leq y \leq b$ into the segment $-1 \leq x \leq 1$, we can simplify the discussion by restricting the range to the

x-axis between -1 and +1 when discussing the applications of the Chebyshev polynomials $T_n(x)$. In a similar manner, the transformation

$$x = (y - a)/(b - a) \tag{8-5}$$

transforms the segment $a < y < b$ into the segment $0 < x < 1$ which is the range over which the "T-star" polynomials $T_n^*(x)$ have been defined. However, the $T_n^*(x)$ polynomials will generally be most useful when the function is to be represented over the infinite range $a \leq y < \infty$. In this important class of problems we can define the variable x to be

$$x = y/a \tag{8-6}$$

and then we see that $x = 0$ corresponds to $y \to \infty$ and $x = 1$ corresponds to $y = a$.

If we now assume that a function f(x) which is defined on (-1,1) has been expanded in a series of Chebyshev polynomials with coefficients A_n

$$f(x) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n T_n(x) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \cos(n \cos^{-1} x) \tag{8-7}$$

we can intuitively make some predictions about the behavior of the A_n if the function f(x) behaves in a sufficiently smooth manner. Since Eq. (8-7) is essentially a resolution of the function f(x) into its "spectral components," we can expect that after the first several values of A_n that, as n increases, the magnitude of A_n will diminish rather rapidly because they represent "small ripples" of successive "corrections." Since no A_n which is less than 10^{-8} times the largest of the A_n will contribute to the sum (when employing conventional FORTRAN compilers), we can discard all such "spectral terms" and arrive thereby at an efficient means of evaluating the function f(x) for values of x in the range (-1,1). From the conventional theory of Fourier series, we recognize the fact that we can use Eq. (8-3) in the form

$$A_n = \frac{2}{\pi}\int_0^\pi f(\cos\theta)\cos(n\theta)d\theta = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_n(x)}{\sqrt{1-x^2}}\,dx \qquad (8\text{-}8)$$

However, we shall show below that it is seldom necessary to resort to Eq. (8-8) to determine the A_n.

8.3 Properties of Chebyshev Polynomials

The reader who undertakes to employ the Chebyshev polynomials should arrange to have access to Refs. (10) and (11) where a rather complete summary of the properties of the Chebyshev polynomials will be found.

Explicit expressions for the first several Chebyshev polynomials are

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_2(x) = 2x^2 - 1$$
$$T_3(x) = 4x^3 - 3x$$
$$T_4(x) = 8x^4 - 8x^2 + 1$$
$$T_5(x) = 16x^5 - 20x^3 + 5x$$

Further polynomials are readily obtained from the recursion formula

$$T_{n+1}(x) - 2x\,T_n(x) + T_{n-1}(x) = 0 \qquad (8\text{-}9)$$

The $T_n^*(x)$ polynomials are related to the $T_n(x)$ polynomials through the relationships

$$T_n^*(x^2) = T_n(2x^2 - 1) = T_{2n}(x) \qquad (8\text{-}10)$$

and therefore it is merely a matter of notational convenience to introduce the "T-star" polynomials. The first few $T_n^*(x)$ are

$$T_0^*(x) = 1$$
$$T_1^*(x) = 2x - 1$$
$$T_2^*(x) = 8x^2 - 8x + 1$$
$$T_3^*(x) = 32x^3 - 48x^2 + 18x - 1$$
$$T_4^*(x) = 128x^4 - 256x^3 + 160x^2 - 32x + 1$$
$$T_5^*(x) = 512x^5 - 1280x^4 + 1120x^3 - 400x^2 + 50x - 1$$

and the recursion formula is

$$T_{n+1}^*(x) = 2(2x - 1)T_n^*(x) - T_{n-1}^*(x) \tag{8-11}$$

These relations can be employed when one wishes to convert a representation in terms of Chebyshev polynomials into an algebraic polynomial, i.e., when carrying out the operations indicated by the following formulae:

$$\begin{aligned} f(x) &= \tfrac{1}{2}A_0 + A_1T_1(x) + A_2T_2(x) + A_3T_3(x) + \cdots + A_NT_N(x) \\ &= B_0 + B_1x^1 + B_2x^2 + B_3x^3 + B_4x^4 + \cdots + B_Nx^N \end{aligned} \tag{8-12}$$

$$\begin{aligned} f(x) &= \tfrac{1}{2}C_0 + C_1T_1^*(x) + C_2T_2^*(x) + C_3T_3^*(x) + \cdots + C_NT_N^*(x) \\ &= D_0 + D_1x^1 + D_2x^2 + D_3x^3 + D_4x^4 + \cdots + D_Nx^N \end{aligned} \tag{8-13}$$

If one wishes to start with the coefficients B_n in Eq. (8-12) and arrive at the coefficients A_n, the following inverse relations will be found useful

$$x^0 = T_0(x)$$
$$x^1 = T_1(x)$$
$$2x^2 = T_0(x) + T_2(x)$$
$$4x^3 = 3T_1(x) + T_3(x)$$
$$8x^4 = 3T_0(x) + 4T_2(x) + T_4(x)$$

If one wishes to start with the coefficients D_n in Eq. (8-13) and arrive at the coefficients C_n, the following inverse relations will be found useful

$$\begin{aligned}
x^0 &= T_0^*(x) \\
2x^1 &= T_0^*(x) + T_1^*(x) \\
8x^2 &= 3T_0^*(x) + 4T_1^*(x) + T_2^*(x) \\
32x^3 &= 10T_0^*(x) + 15T_1^*(x) + 6T_2^*(x) + T_3^*(x) \\
128x^4 &= 35T_0^*(x) + 56T_1^*(x) + 28T_2^*(x) + 8T_3^*(x) + T_4^*(x)
\end{aligned}$$

Lanczos (Ref. 10) has listed both types of inverse relations for all terms up and including x^{12}. However, in actual practice one will seldom need these explicit representations because the necessary coefficients are readily generated by algorithms which will be discussed below.

8.4 Summing Series of Chebyshev Polynomials

Extensive tables of the Chebyshev polynomials are contained in a publication (Ref. 10) in the NBS Applied Mathematics Series. However, in practice one seldom has a need for these tables since Clenshaw (Ref. 11) has published an algorithm which permits one to sum a series of Chebyshev polynomials without actually computing the values of $T_n(x)$ or $T_n^*(x)$ for the particular value of x and the range of values of n. Before we present the algorithm, let us consider what might be necessary if the algorithm were not known.

Let us introduce a set of FORTRAN variables which are related to the mathematical variables as follows:

$$\begin{aligned}
X &= x \\
SUM &= f(x) \\
A0 &= A_0 \\
A(I) &= A_n
\end{aligned}$$

where $I = n$, for $n = 1,2,3,\cdots,N$. In order to evaluate the sum in Eq. (8-12) we could employ the FORTRAN program designated as Program 8-1.

Program 8-1

```
      SUBRØUTINE CHEBYS(X,SUM,A0,A,N)
      DIMENSIØN A(N),T(N)
      IF(ABS(X).GT.(1.0)) GØ TØ 10
      T(1) = 1.0
      T(2) = X
      DØ 1  I = 3,N
      T(I) = 2.0*X*T(I-1) - T(I-2)
    1 CØNTINUE
      SUM = 0.5*A0
      DØ 2 I = 2,N
      SUM = SUM + A(I)*T(I)
    2 CØNTINUE
      RETURN
   10 WRITE(6,11) X
      CALL EXIT
   11 FØRMAT(37H1SUBRØUTINE CHEBYS CANNØT HANDLE X = ,E15.8)
      END
```

In the above program we have first set up an array of numbers denoted by T(N) which correspond to the values of the Chebyshev polynomials $T_n(x)$. This has been done by programming the recursion formula in Eq. (8-9). Then, in the second DØ-loop we have formed the sum which appears in Eq. (8-12).

Clenshaw's algorithm permits us to sum the series in Eq. (8-12) by letting

$$b_{N+2} = b_{N+1} = 0$$

and then recurring backwards with the recurrence formula

$$b_r = 2xb_{r+1} - b_{r+2} + A_r \qquad (8\text{-}14)$$

until we have b_2, b_1, and b_0. We then determine $f(x)$ by means of the relation

$$f(x) = \frac{1}{2}(b_0 - b_2)$$
$$= xb_1 - b_2 + \frac{1}{2}A_0 \qquad (8\text{-}15)$$

We observe that the only difference between the recurrence relation in Eq. (8-14) and that in Eq. (8-9) is the presence of the term A_r in Eq. (8-14).

We can carry out the operations indicated in Eqs. (8-14) and (8-15) by means of the FORTRAN program which is designated as Program 8-2.

Program 8-2

```
      SUBRØUTINE CLENSW(X,SUM,AO,A,N)
      DIMENSIØN A(N)
      IF(ABS(X).GT.(1.0)) GØ TØ 10
      B2 = 0.0
      B1 = 0.0
      DØ 1 I = 1,N
      M = N + 1 - I
      BO = 2.0*X*B1 - B2 + A(M)
      B2 = B1
      B1 = BO
    1 CØNTINUE
      SUM = X*B1 - B2 + 0.5*AO
      RETURN
   10 WRITE(6,11) X
      CALL EXIT
   11 FØRMAT(37H1SUBRØUTINE CLENSW CANNØT HANDLE X = ,E15.8)
      END
```

A comparison of Program 8-1 with Program 8-2 reveals that we now have only one DØ-loop, and that it is no longer necessary to set up the array T(I), I = 1 to N.

Since $T_n^*(x) = T_n(2x - 1)$, we observe that the algorithm in Eq. (8-14) can be employed to sum the series in Eq. (8-13) if we merely replace the recurrence formula of Eq. (8-14) by

$$b_r = 2(2x - 1)b_{r+1} - b_{r+2} + C_r \qquad (8\text{-}16)$$

Therefore, we can sum a series of "T-star" polynomials as given in Eq. (8-13) by means of Program 8-3. We observe that

$$CO = C_0$$

$$C(I) = C_n$$

for I = n, with n ranging from 1 to N.

Program 8-3

```
      SUBRØUTINE TNSTAR(X,SUM,CO,C,N)
      DIMENSIØN C(N)
      IF((X.LT.(0.0)).ØR.(X.GT.(1.0))) GØ TØ 10
      Z = 2.0*X - 1.0
      CALL CLENSW(Z,SUM,CO,C,N)
      RETURN
   10 WRITE(6,11) X
      CALL EXIT
   11 FØRMAT(37H1SUBRØUTINE TNSTAR CANNØT HANDLE X = ,E15.8)
      END
```

The property of the Chebyshev polynomials

$$T_{2n}(x) = T_n(2x^2 - 1)$$

permits us to sum a series of even polynomials of the form

$$f(x) = \tfrac{1}{2}E_0 + E_1T_2(x) + E_2T_4(x) + E_3T_6(x) + \cdots + E_NT_{2N}(x) \quad (8\text{-}17)$$

by replacing the recurrence formula of Eq. (8-14) by

$$b_r = 2(2x^2 - 1)b_{r+1} - b_{r+2} + E_r \quad (8\text{-}18)$$

If we then let

$$EO = E_0$$

$$E(I) = E_n$$

for I = n, and n ranging from 1 to N, we can evaluate the sum in Eq. (8-17) by means of Program 8-4.

Program 8-4

```
      SUBRØUTINE TNEVEN(X,SUM,EO,E,N)
      DIMENSION E(N)
      IF(ABS(X).GT.(1.0)) GØ TØ 10
      Z = 2.0*X*X - 1.0
      CALL CLENSW(Z,SUM,EO,E,N)
      RETURN
   10 WRITE(6,11) X
      CALL EXIT
   11 FØRMAT(37H1SUBRØUTINE TNEVEN CANNØT HANDLE X = ,E15.8)
      END
```

If $f(x)$ is expressed in the form of a odd series of the form

$$f(x) = F_1T_1(x) + F_2T_3(x) + F_3T_5(x) + \cdots + F_NT_{2N-1}(x) \tag{8-19}$$

it will often be convenient to use the property described by Eq. (8-9) to express $f(x)$ in the form

$$f(x) = x[\tfrac{1}{2}E_0 + E_1T_2(x) + E_2T_4(x) + \cdots + E_{N-1}T_{2N-2}(x)] \tag{8-20}$$

where

$$2F_1 = E_0\ ,\ 2F_2 = E_1 + E_2,\ 2F_3 = E_2 + E_3,\ \ldots,\ 2F_N = E_{N-1}$$

Clenshaw (Ref. 11) has pointed out that the form in Eq. (8-20) has the desirable property of yielding the correct behavior of $f(x)/x$ near $x = 0$. However, if one finds an occasion where the form in Eq. (8-19) is preferred, the recurrence relation in Eq. (8-14) has to be replaced by

$$b_r = 2(2x^2 - 1)b_{r+1} - b_{r+2} + F_r \tag{8-21}$$

and the value of $f(x)$ is given by

$$f(x) = x(b_0 - b_1) \tag{8-22}$$

The form of Eq. (8-22) reveals that the Clenshaw algorithm yields the correct behavior of $f(x)/x$ for x near zero. Since many practical applications lead to indeterminate forms (such as the

familiar form sin(αx)/x which occurs in many diffraction problems) in which the ratio f(x)/x occurs, it will be convenient to replace Eq. (8-22) by the alternative form

$$f(x) = x\, g(x) = x(b_0 - b_1) \tag{8-23}$$

and prepare a FORTRAN program in which

SUM1 = f(x)

SUM2 = g(x)

$F(I) = F_n$

where I = n, and n ranges through the values 1,2,3,...,N. The series given in Eq. (8-19) can be summed by Program 8-5.

Program 8-5

```
      SUBRØUTINE TNØDDS(X,SUM1,SUM2,F0,F,N)
      DIMENSIØN F(N)
      IF(ABS(X).GT.(1.0)) GØ TØ 10
      Z = 2.0*(2.0*X*X - 1.0)
      B2 = 0.0
      B1 = 0.0
      DØ 1 I = 1,N
      M = N + 1 - I
      B0 = Z*B1 - B2 + F(M)
      B2 = B1
      B1 = B0
    1 CØNTINUE
      SUM2 = B1 - B2
      SUM1 = X*SUM2
      RETURN
   10 WRITE(6,11) X
      CALL EXIT
   11 FORMAT(37H1SUBRØUTINE TNØDDS CANNØT HANDLE X = ,E15.8)
      END
```

The reader will observe that each of the programs given above has been provided with a CALL EXIT which will cause the computer program which employs these subroutines to exit from the electronic computer when the variable x lies outside the allowed range. However, in each case, the computer will write a message to indicate

that the appropriate subroutine could not handle the value of x which caused the program to be terminated by the CALL EXIT statement.

8.5 Further Remarks on the Clenshaw Algorithm

The technique which we have employed in the discussion above to sum the Chebyshev series belongs to a wider class of algorithms which was introduced by Clenshaw (Ref. 12). Let f(x) be represented in the form

$$f(x) = \sum_{n=0}^{N} A_n p_n(x) \tag{8-24}$$

where the functions denoted by $p_n(x)$ satisfy a linear recurrence relation of the form

$$p_{n+1}(x) + \alpha_n p_n(x) + \beta_n p_{n-1}(x) = 0 \tag{8-25}$$

where α_n and β_n may be functions of x as well as of n. The Clenshaw algorithm asserts that if one defines

$$b_{N+2} = b_{N+1} = 0$$

and then constructs the sequence $b_N, b_{N-1}, b_{N-2}, \ldots, b_1, b_0$ according to the rule

$$b_r + \alpha_r b_{r+1} + \beta_{r+1} b_{r+2} = A_r \ , \quad r = N, N-1, \ldots, b_1, b_0 \tag{8-26}$$

that the sum of the series in Eq. (8-24) is given by

$$f(x) = b_0 p_0(x) + b_1 [p_1(x) + \alpha_0 p_0(x)] \tag{8-27}$$

A well-known example of this algorithm is the case in which

$$p_n(x) = x^n$$

for which Eq. (8-25) takes the simple form

$$p_{n+1}(x) - x p_n(x) = 0$$

Therefore, we see that for this case

$$\alpha_n = -x \quad , \quad \beta_n = 0$$

and Eq. (8-27) reduces to the simple form

$$f(x) = b_0$$

and the recurrence relation reduces to

$$b_r = xb_{r+1} + A_r$$

If we consider a simple case, such as

$$f(x) = A_0 + A_1x + A_2x^2 + A_3x^3$$

we see that the algorithm yields the successive results

$$\begin{aligned} b_3 &= A_3 \\ b_2 &= xb_3 + A_2 = A_2 + A_3x \\ b_1 &= xb_2 + A_1 = A_1 + A_2x + A_3x^2 \\ b_0 &= xb_1 + A_0 = A_0 + A_1x + A_2x^2 + A_3x^3 \end{aligned}$$

The reader who is experienced in the methods of numerical analysis will probably recognize this algorithm as being merely the so-called "nested-multiplication" method of evaluating a polynomial, i.e.,

$$f(x) = A_0 + A_1x + A_2x^2 + A_3x^3 = \{A_0 + x[A_1 + x(A_2 + xA_3)]\}$$

The Clenshaw algorithm provides an extension of the "nested multiplication" technique which enables one to evaluate the function f(x) and its derivatives when it is expressed in the form of a Taylor series

$$f(x) = \sum_{n=0}^{N} A_n \frac{x^n}{n!} \tag{8-28a}$$

and

$$f^{(m)}(x) = \frac{d^m f(x)}{dx^m} = \sum_{n=m}^{N} A_n\, [n(n-1)\cdots(n-m+1)]\frac{x^n}{n!} \tag{8-28b}$$

$$= \sum_{k=0}^{N-m} A_{m+k}\frac{x^k}{k!} \tag{8-28c}$$

If we observe that the $p_n(x)$ in Eq. (8-28) are $x^n/x!$ we see that the recurrence relation is

$$p_{n+1}(x) = \frac{x}{n+1}\, p_n(x)$$

and hence α_n and β_n are defined by

$$\alpha_n = -\frac{x}{n+1} \quad , \quad \beta_n = 0$$

Therefore, if we let

$$b_{N-m+1} = 0$$

$$b_r = \frac{x}{r+1} b_{r+1} + A_{r+m} \quad , \; r = N-m,\; N-m-1, \ldots, 2, 1, 0$$

we will find that

$$f^{(m)}(x) = b_0$$

The summation of Fourier series of the form

$$f(\theta) = \sum_{n=0}^{N} A_n \cos(n\theta) \tag{8-29a}$$

$$g(\theta) = \sum_{n=1}^{N} A_n \sin(n\theta) \tag{8-29b}$$

can be accomplished readily by using the algorithm in the form

$$b_{N+2} = b_{N+1} = 0$$

$$b_r = 2\cos\theta\, b_{r+1} - b_{r+2} + A_r \ , \quad r = N, N-1, N-2, \ldots, 2, 1, 0$$

The algorithm asserts that

$$f(\theta) = b_0 - b_1 \cos\theta \tag{8-30a}$$

$$g(\theta) = b_1 \sin\theta \tag{8-30b}$$

Since many diffraction problems involving a circular cylinder can be solved by series of the form in Eq. (8-29), the use of this algorithm can be extremely helpful in making more efficient computer programs for the calculation of the scattering patterns, current distributions, and other quantities which can be expressed in a Fourier series of the form of Eq. (8-29).

One can also obtain more efficient programs for the summing of certain series of spherical harmonics such as occur in the theory of the diffraction of waves by a sphere. In the case of a series of Legendre functions $P_n(\cos\theta)$ such as

$$f(\theta) = \sum_{n=0}^{N} A_n P_n(\cos\theta) \tag{8-31}$$

we consider the recurrence formula for the polynomials, namely,

$$(n + 1)P_{n+1}(\cos\theta) = (2n + 1)\cos\theta\, P_n(\cos\theta) - nP_{n-1}(\cos\theta)$$

and find that the α_n and β_n of Eq. (8-25) are

$$\alpha_n = -\frac{2n+1}{n+1}\cos\theta \ , \qquad \beta_n = \frac{n}{n+1}$$

Therefore, if we set

$$b_{N+2} = b_{N+1} = 0$$

and recur backwards with the formula

$$b_r = \frac{2n+1}{n+1}\cos\theta\, b_{r+1} - \frac{r+1}{r+2}\, b_{r+2} + A_r \tag{8-32}$$

we will find* that

$$f(\theta) = b_0 \tag{8-33}$$

The result contained in Eq. (8-33) can be used as the basis for summing the many series of the form of Eq. (8-31) which occur in boundary value problems involving a sphere.

There are many other problems in applied mathematics which can be summed by means of the Clenshaw algorithm. For example, in his original paper (Ref. 12), Clenshaw observed that the finite Neumann series of Bessel functions

$$f(x) = \sum_{n=0}^{N} A_n J_{n+\nu}(x)$$

could be evaluated in this manner.

The author has encountered several cases in which the coefficients in series of the form described by Eq. (8-28) behaved in such a manner that the backward recursion led to incorrect results because of some unfavorable cancellation which took place during the recurrence process. Therefore, until such time as the theory of the propagation of errors in such processes is better understood, the author would like to advise the reader that it may be advisable to test the algorithm by comparing its results with the results obtained by computing the magnitude of the quantities $A_n p_n(x)$ in Eq. (8-24) and listing these for visual inspection and

---

*The author suggests that "skeptics" join him in observing that Eq. (8-33) is correct by considering the case of the simple function $f(\theta) = P_0(\cos\theta) + P_1(\cos\theta) + 2P_2(\cos\theta) = \cos\theta + 3\cos^2\theta$. We start with $b_4 = b_3 = 0$, and then compute $b_2 = 2$; $b_1 = (3/2)\cos\theta\,(2) + 1 = 1 + 3\cos\theta$; and $b_0 = \cos\theta(1 + 3\cos\theta) - (1/2)(2) + 1 = \cos\theta + 3\cos^2\theta$. Therefore, we do find that $f(\theta) = b_0$ as implied by Eq. (8-33).

comparing the sum of this "column" of figures with the results obtained from the algorithm.

8.6 Determination of Coefficients

We now want to take up the problem of the determination of the coefficients A_n in the truncated series of Chebyshev polynomials which is employed to represent a function f(x).

$$f(x) = \frac{1}{2} A_0 + A_1T_1(x) + A_2T_2(x) + A_3T_3(x) + \cdots + A_NT_N(x)$$

$$= B_0 + B_1x^1 + B_2x^2 + B_3x^3 + B_4x^4 + \cdots + B_Nx^N \tag{8-34}$$

Let us first of all dispose of a related problem. Suppose we already have the A_n for $n = 0,1,2,\ldots,N$ and we would like to convert from the series of Chebyshev polynomials to the conventional algebraic polynomial, i.e., we know the A_n in Eq. (8-34) and we would like to know the B_n. This was one of the first problems which the author undertook under the present contract, but during the course of the work there have appeared several programs which will accomplish this task. Golden (Ref. 47, pp. 120-124) has presented a good treatment of this problem and actual FORTRAN IV programs are constructed. Prager (Ref. 79) restricts his discussion to the "shifted" Chebyshev polynomials (which we have referred to as the "T star" polynomials) and he has given FORTRAN IV programs that permit one to convert from the C_n of Eq. (8-13) to the D_n, or vice versa. Arden has discussed this problem when it is in the form of Eq. (8-34) (which is the same as Eq.(8-12)), but his treatment (Ref. 48) seems (to the present author) to fail to correctly yield the B_n from the A_n if N is greater than 5. Since it would be convenient to have the ability to carry out these operations as a subroutine, we give in Program 8-6 a subroutine which will take a set of $A(I) = A_n$, for $I = n$ (and $n < 97$) and convert them to a set of $B(I) = B_n$.

The algorithm which has been programmed in Program 8-6 has two features which may require modification if the reader wishes to compile this FORTRAN IV subroutine on the electronic computer which is available to him. The UNIVAC 1107 compiler permits one to use positive and negative integers, as well as zero, as a subscript. This compiler also permits one to write DØ-loops which proceed in "negative" increments. Both of these features have been employed in Program 8-6 (and in Program 8-7). The reader will observe that in order for the program to have access to

Program 8-6

```
      SUBRØUTINE CBS2PS(N,A,B)
      DIMENSIØN A(N),B(N)
      DIMENSIØN AB1(98),AAB1(99),AB2(98),AAB2(99)
      EQUIVALENCE (AB1(1),AAB1(2)),(AB2(1),AAB2(2))
      IF(N.GT.0) GØ TØ 10
      B(0) = A(0)
      RETURN
   10 IF(N.LE.97) GØ TØ 20
      WRITE(6,11) N
      CALL EXIT
   11 FØRMAT(37H1SUBRØUTINE CBS2PS CANNØT HANDLE N = ,I6)
   20 DØ 9 I=0,N
      AB1(I) = A(I)
    9 CØNTINUE
      NM1 = N-1
      DØ 8 I=NM1,1,-1
      AB2(I+2) = AB2(I+1)
      DØ 7 J=I,1,-1
      AB2(J) = 2.0*AB1(J+1) - AB2(J+2)
    7 CØNTINUE
      AB2(0) = AB1(1) - 0.5*AB2(2)
      B(N-I-1) = AB1(0) - 0.5*AB2(1)
      DØ 6 J=0,I
      AB1(J) = AB2(J)
    6 CØNTINUE
    8 CØNTINUE
      B(N-1) = AB1(0)
      B(N)   = AB1(1)
      RETURN
      END
```

the array AB1(0), AB1(1), AB1(2),... that it has been necessary to dimension AB1(98) and then introduce a second array AAB1(99). The EQUIVALENCE statement is then used in such a manner that a reference to AB1(0) is actually a reference to AAB1(1) and hence the compiler is able to work with zero as a subscript. Many compilers for FORTRAN IV will not permit a DØ-loop of the form

```
DØ 8 I = NM1,1,-1
```

since the allowed form is generally

```
DØ 8 I = N1,N2,N3
```

where N1,N2, and N3 are <u>unsigned</u> (i.e., positive) integers. Both of these features of the UNIVAC 1107 compiler (and certain other FORTRAN IV compilers) are very convenient because they permit us to write an algorithm which is closer to the mathematical language.

Since the Program 8-6 expects to find a value for A(0) and will give back a result B(0), the calling program must contain a "gimmick" such as:

```
DIMENSIØN A(N),B(N),AA(2),BB(2)
EQUIVALENCE (A(1),AA(2)),(B(1),BB(2))
```

Program 8-6 may be found useful in the case of a function which is to be evaluated many times because there are fewer steps involved in evaluating the series when expressed in the form of an algebraic polynomial. However, Clenshaw (Ref. 11, p. 13) has shown that the Chebyshev coefficients of ascending order usually decrease more rapidly than the coefficients of ascending powers. Therefore, the evaluation of the algebraic polynomial may be faster than that of the equivalent Chebyshev expansion, but the Chebyshev expansion may be subject to smaller errors. Clenshaw's example shows definitely some advantages to leaving the function represented as a Chebyshev series. However, Beasley (Ref. 27) has shown that a different rearrangement of the series

used in Clenshaw's example can lead to an algebraic polynomial which does not suffer the disadvantages of the algebraic polynomial which was considered by Clenshaw.

Let us turn now to the problem of obtaining the coefficients A_n in Eq. (8-34) when the coefficients B_n are assumed to be known. A program which will accomplish this is given as Program 8-7. Many of the remarks made in connection with the discussion of Program 8-6 apply equally well to this program since we have also employed the zero subscripts in A(0), B(0), etc., and backward steps are made in DØ-loops such as "DØ 7 J = I,1,-1."

Program 8-7

```
      SUBRØUTINE PS2CBS(N,A,B)
      DIMENSIØN A(N),B(N)
      DIMENSIØN AB1(98),AAB1(99),AB2(98),AAB2(99)
      EQUIVALENCE (AB1(1),AAB1(2)),(AB2(1),AAB2(99))
      IF(N.GT.0) GØ TØ 10
      A(0) = B(0)
      RETURN
   10 IF(N.LE.97) GØ TØ 20
      WRITE(6,11) N
      CALL EXIT
   11 FØRMAT(37H1SUBRØUTINE PS2CBS CANNØT HANDLE N = ,I6)
   20 DØ 2 I=0,N
      AB1(I) = B(I)
    2 CØNTINUE
      NMS1 = N-1
      DØ 3 I=NMS1,1,-1
      AB2(I+2) = 0.0
      AB2(I+1) = 0.0
      DØ 4 J=I,1,-1
      AB2(J) = 2.0*AB1(J+1) - AB2(J+2)
    4 CØNTINUE
      AB2(0) = AB1(1) - 0.5*AB2(2)
      A(N-I-1) = AB1(0) - 0.5*AB2(1)
      DØ 5 J=1,I
      AB1(J) = AB2(J)
    5 CØNTINUE
    3 CØNTINUE
      A(N-1) = AB1(0)
      A(N)   = AB1(1)
      RETURN
      END
```

Although Program 8-7 is quite limited in its usefulness (since one needs to already have a polynomial represenation for the function), most of the literature on the Chebyshev series has evolved around the concept of using the Chebyshev polynomials to "telescope" a power series. This concept has been quite clearly discussed by Prager (Ref. 79), Golden (Ref. 47), Arden (Ref. 48), Lanzcos (Ref. 1 and Ref. 36), Minnick (Ref. 44), Spielberg (Ref. 42), and other authors whose papers are listed in the references for this Section. Often times the polynomial which one starts with is the Taylor's series and then the process of "telescoping" permits one to obtain a power series containing fewer terms which can be used to compute the function to the same accuracy over the particular range of interest.

The telescoping process has attracted so much attention in the literature of the recent past that the important concepts associated with the idea of trigonometrical interpolation (which were stressed by Lanzcos (Ref. 1)) have not received sufficient emphasis. In Eq. (8-8) we showed that the coefficients in the expansion in a series of Chebyshev polynomials could be obtained by a process of numerical integration.

The most promising approach to the determination of of the A_n in Eq. (8-34) is associated with an orthogonality property of the Fourier series which, as Lanzcos has emphasized, is peculiar to the Fourier series when one examines the Fourier series as being an example of expansions in terms of orthogonal functions. We refer the reader to the paper by Lanzcos (Ref. 1) and the discussion by Clenshaw (Ref. 11) for more details. the result which we want to employ is the following:

$$NA_r = f(x_0)+2f(x_1)T_1(x_r)+\cdots+2f(x_{N-1})T_{N-1}(x_r)+f(x_N)T_N(x_r) \quad (8\text{-}35)$$

where $r = 0,1,2,\ldots,N$, and the arguments x_j appearing in $f(x_j)$

are defined by

$$x_j = \cos(j\pi/N) \tag{8-36a}$$

and the arguments x_r of the Chebyshev polynomials $T_j(x_r)$ are defined by

$$x_r = \cos(r\pi/N) \tag{8-36b}$$

In the discussion above we have made no distinction in notations between the A_n which appear in the infinite series of Eq. (8-7) and the A_n which appear in the <u>truncated</u> series of Eq. (8-12) or Eq. (8-34). The reader will find this discussed by Clenshaw in his tables (Ref. 11) and in a recent paper (Ref. 3). Further study of the relationships are needed, but it will suffice for our purposes to point out that we are assured by certain aspects of the theory of Fourier series that as $N \to \infty$ that the A_n which are computed on the basis of Eq. (8-35) will converge to the A_n employed in Eq. (8-7) and defined by an integral in Eq. (8-8).

It is interesting to observe that the formula in Eq. (8-35) is itself a truncated Chebyshev series and hence it can be summed by use of the Clenshaw algorithm. All we need to do is to be able to compute the value of $f(x)$ at $N + 1$ points x_j defined by Eq. (8-36a). This aspect of the determination of the A_n emphasizes the <u>interpolating</u> nature of the <u>truncated</u> series of Chebyshev polynomials. An ALGOL 60 procedure for the calculation of these coefficients was given by Schwarz (Ref. 29) in his tutorial paper on this programming language. This is quite a sophisticated program in that the degree of approximation which the user feeds in as an input to the ALGOL procedure determines the degree N of the truncated Chebyshev series, and a random number generator is employed to check that the resulting series does indeed provide the desired interpolating polynomial.

In this discussion, we will be content with the much simpler program for the evaluation of Eq. (8-35) as given in Program 8-8.

Program 8-8

```
      SUBRØUTINE CLNSHW(FUNCTN,A,B,N,CØEF)
      DØUBLE PRECISIØN A,B, CØEF(N),BPA,BMA,DFLØAT,DCØS,PI,PIØN,
     A        ZERØ,ØNE,TWØ,X(100),T(100),E(100),F(100)
      DATA PI,ZERØ,ØNE,TWØ/3.141592653589793D0,0.0D0,1.0D0,2.0D0/
  199 FØRMAT(37H SUBRØUTINE CLNSHW CANNØT HANDLE A = ,D25.16,
     A        5H B = ,D25.16,5H N = ,I4)
      DFLØAT(KK) = DBLE(FLØAT(KK))
      IF(N.LT.2) GØ TØ 98
      IF(N.GT.98) GØ TØ 98
      IF(DABS(A-B).LT.(1.0D-8)) GØ TØ 98
      BPA = B + A
      BMA = B - A
      PIØN = PI/DFLØAT(N-1)
      DØ 1 I=1,N
      T(I) = DCØS(PIØN*DFLØAT(I-1))
      X(I) = (BPA + BMA*T(I))/TWØ
      CALL FUNCTN(X(I),E(I))
    1 CØNTINUE
      F(N+2) = ZERØ
      F(N+1) = ZERØ
      E(N)   = E(N)/TWØ
      DØ 2 K =1,N
      DØ 3 L =N,1,-1
      F(L) = TWØ*T(K)*F(L+1) - F(L+2) + E(L)
    3 CØNTINUE
      COEF(K) = (F(1) - F(3))/DFLØAT(N-1)
    2 CØNTINUE
      RETURN
   98 WRITE( ,199) A,B,N
      CALL EXIT
      END
```

There are a number of aspects of Program 8-8 which require discussion. In the argument list, FUNCTN refers to another SUBRØUTINE which it is assumed the reader will have provided. It must be of the form:

```
      SUBRØUTINE FUNCTN(ARG,VALUE)
      DØUBLE PRECISIØN ARG, VALUE
       .
      VALUE = ...
      RETURN
      END
```

When the statement CALL FUNCTN(ARG,VALUE) appears the function that is served is that of obtaining a value for f(x) where the relation

between the mathematical quantities x and f(x) and the FORTRAN symbols are

$$x = ARG$$
$$f(x) = VALUE$$

The A and B which appear in Program 8-8 give the range of the variable x over which the function is to be approximated, i.e,

$$A < ARG < B$$

Therefore, if $A \neq -1$, $B \neq +1$, the user will have to arrange to use Eq. (8-4) to employ the coefficients obtained from the program. The coefficients A_n are contained in the array COEF(N), i.e.,

$$A_n = \text{COEF(I), with } n = (I-1), \text{ for } n = 0,1,2,\ldots,(N-1)$$

The appearance of the statement "DØ 3 L=N,1,-1" reveals that this program must be modified if the compiler employed does not permit a DØ-loop to proceed from "high" values to "low" values.

In Program 8-9 we give a test program which was used to test the algorithm in Program 8-8.

Program 8-9

```
      DØUBLE PRECISIØN COEF(100)
      EXTERNAL FUNC
  110 FØRMAT(10X,I4,D25.16)
      CALL CLNSHW(FUNC,-1.0D0,+1.0D0,18,COEF)
      WRITE(6,110) (I,COEF(I),I=1,18)
      CALL EXIT
      END
      SUBRØUTINE FUNC(ARG,VALUE)
      DØUBLE PRECISIØN ARG, VALUE, DEXP
      VALUE = DEXP(ARG)
      RETURN
      END
```

In Table 8-1 we present the output from the use of Programs 8-8 and 8-9. In the right hand column we present, for comparison, the Chebyshev coefficients for f(x) = exp(x) on $-1 < x < +1$ which are given by Clenshaw(Ref. 11).

Table 8-1

CHEBYSHEV COEFFICIENTS FOR EXP(X)

N	From Program 8-9	From Clenshaw (Ref.11)	
1	.25321317555040014+01	.253213175550401667120	+01
2	.11303182079849631+01	.113031820798497005442	+01
3	.27149533953407577+00	.27149533953407656237	+00
4	.44336849848663651-01	.4433684984866380495	-01
5	.54742404420939781-02	.547424044209373265	-02
6	.54292631191400011-03	.54292631191394375	-03
7	.44977322954210721-04	.4497732295429515	-04
8	.31984364624658501-05	.319843646240199	-05
9	.19921248049862861-06	.19921248066728	-06
10	.11036771836317291-07	.1103677172552	-07
11	.55058954781038511-09	.55058960797	-09
12	.24979782948013741-10	.2497956617	-10
13	.10391165052597521-11	.103915223	-11
14	.40046397570596811-13	.3991263	-13
15	.13453290768987181-14	.142376	-14
16	.65307236742656231-16	.4741	-16
17	.0000000000000000	.148	-17
18	.0000000000000000	.4	-18

A study of the output from Program 8-9 with the more precise results printed alongside reveals that the initial results (N near 1) are accurate to about 15 decimal places, but that as N increases there is a growth of the error and by the time one reaches N = 14 that there is only one significant figure and that this is in the 14th decimal place. These results are quite encouraging and suggest that Eq. (8-35) can be quite useful for the determination of the coefficients in the Chebyshev series which is truncated as in Eq. (8-34).

8.7 The Future for Diffraction Theory

From the many references at the end of this section, the author has assembled a set of BLOCK DATA statements which contain (on punched cards) virtually all the Chebyshev coefficients which have been published to the present date. It was originally part

of the author's plan for this report that these would be contained in the Appendices. However, the compilation runs to many pages and the results are most useful when they can be read directly into an electronic computer. Therefore, the author plans to run further checks upon these listings (to insure their accuracy) and to then submit the data (in the form of a strip of magnetic tape) to the SHARE organization so that it will be unnecessary for programmers who have access to this tape to punch the data directly from the published tables.*

Before we discuss some possible applications of the Chebyshev series in diffraction theory, let us pause to point out a field which needs further investigation. The results in Eqs. (8-34) and (8-35) permit us to interpolate (using the values of the function at N+1 points) with efficiency and accuracy a function $f(x)$ defined on $-1 < x < +1$. In diffraction theory (and in many other fields) we meet the need to interpolate when the function is defined in the complex plane and therefore we find ourselves dealing with a function of two (or more) variables. The ideas advanced by Lanzcos could be extended to such problems by "brute force." For example, if $F(z) = F(x,y)$ is defined on $-1 < x < +1$, $-1 < y < +1$, we could write

$$F(x,y) = \tfrac{1}{2}f_0(y) + f_1(y)T_1(x) + f_2(y)T_2(x) + \cdots + \tfrac{1}{2}f_N(y)T_N(x)$$

where

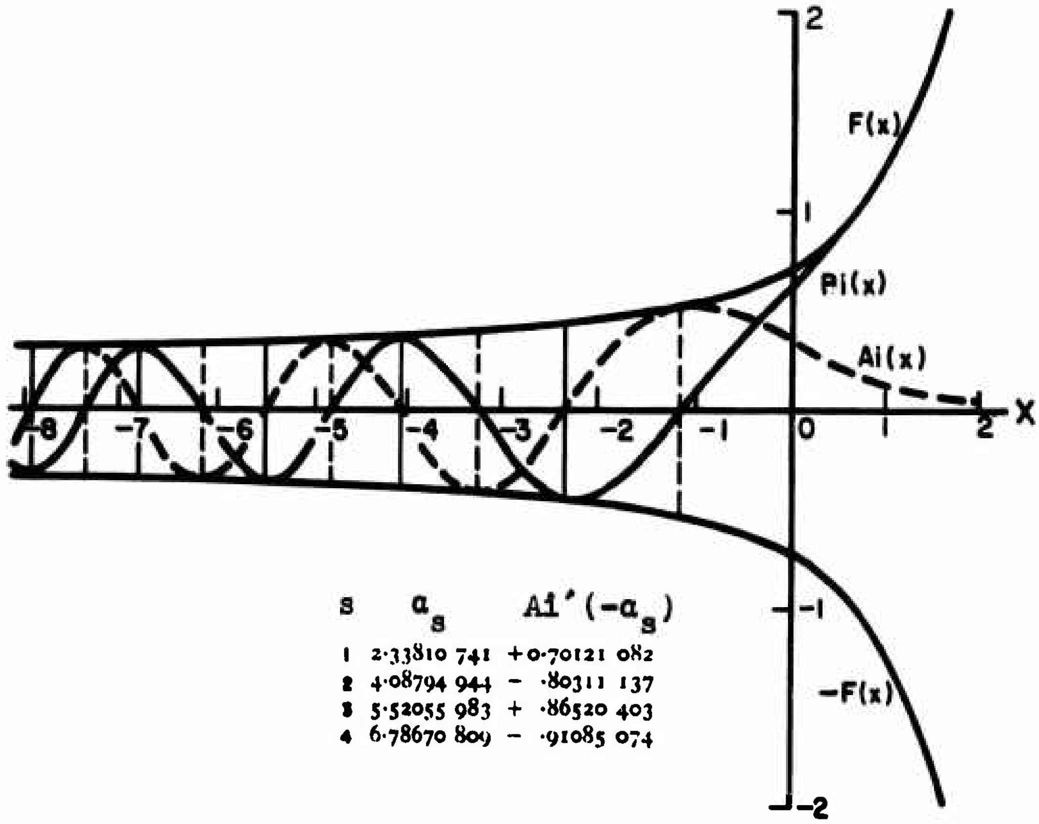
$$f_r(y) = \tfrac{1}{2}A_0^r + A_1^r T_1(y) + A_2^r T_2(y) + \cdots + A_{M-1}^r T_{M-1}(y) + \tfrac{1}{2}A_M^r T_M(y)$$

*The questions related to the status of the "copyright" on these tables has also enterred the decision to omit these DATA BLOCKS from this report. However, since keypunching and verifying is relatively expensive, it is hoped that arrangements can be made whereby this file (consisting of several thousand IBM cards) can be shared with other scientists who employ the Chebyshev series.

where the coefficients A_p^r could be determined from a result derived from Eq. (8-35). We observe that this approach would require the knowledge of the value of the function $F(x,y)$ at $(N+1)(M+1)$ points in the domain of interest, i.e., the square $-1 < x < +1$, $-1 < y < +1$. It is quite conceiveable that as we push foward in the field of the constructive theory of functions that we will find means to interpolate over the rectangle which require many fewer points than the $(N+1)(M+1)$ that the above approach requires. The interpolation problem bears some resemblance to the problem of choosing points for the evaluation of multiple integrals. This has begun to be a field which is attracting considerable interest. One author recently pointed out that the use of a Gauss 3-point formula for an n-dimensional integral would require 3^n points, whereas comparable accuracy can be obtained with about $2n^2$ points. Results such as this for the interpolation of functions of more than one variable are badly needed. However, the problem is sufficiently difficult that we may have to wait years for significant advances. In the meantime, the outstanding paper on interpolation of one-dimensional functions which was written in 1938 by Lanzcos (Ref. 1) has only now begun to be properly appreciated by people who face applied problems.

The outstanding advantages which will accrue to diffraction theorists with the further development of the Chebyshev series representations for special functions will be demonstrated by considering several types of problems. Let us start with the Airy functions. The behavior of the Airy functions Ai(x) and Bi(x) are depicted in Fig. 8-1. An essential point to be borne in mind when seeking representations in terms of truncated Chebyshev polynomials is that the coefficients in the expansion can be expected to decrease rapidly with increasing order only if the function which is to be represented is "smooth."

If we recall that we know Taylor series (See Eq. (B-5)) for these functions, we can recognize the possibility of using the terms in



s	a_s	$Ai'(-a_s)$
1	2·33810 741	+0·70121 082
2	4·08794 944	− ·80311 137
3	5·52055 983	+ ·86520 403
4	6·78670 809	− ·91085 074

Fig. 8-1 Behavior of the Airy Functions

these series in an algorithm (such as Program 8-7) to obtain efficient representations of the form

$$Ai(x) = \frac{1}{2} A_0 + A_1 T_1(x) + \cdots + A_N T_N(x) \tag{8-37a}$$

$$Bi(x) = \frac{1}{2} B_0 + B_1 T_1(x) + \cdots + B_M T_M(x) \tag{8-37b}$$

where N need not be the same as M, and where $-1 < x < +1$. It might even be worthwhile to expand the range to $-2 < x < +2$, and employ series of the form

$$g(x) = \frac{1}{2} C_0 + C_1 T_1\left(\frac{x}{2}\right) + \cdots + C_K T_K\left(\frac{x}{2}\right) \tag{8-38}$$

The use of either Eq. (8-37) or Eq. (8-38) would be merely a conventional use of the Chebyshev polynomials to "telescope" a power

series. However, if one expected to compute the Airy functions many times on an electronic computer, it would be economically sound to seek these more efficient series.

Before leaving the subject of the Airy functions near the origin, the author would like to point out that it may prove to be advantageous to consider the form of Eq. (B-5) and express $Ai(x)$ and $Bi(x)$ in the form

$$Ai(x) = \alpha f_1(\xi) - \beta x f_2(\xi) \quad , \quad Bi(x) = \sqrt{3}\,[\alpha f_1(\xi) + \beta x f_2(\xi)] \quad (8\text{-}39)$$

where

$$\xi = x^3/A^3$$

and $f_1(\xi)$ and $f_2(\xi)$ are expressed in the form of Chebyshev series

$$f_j(\xi) = \frac{1}{2} E_0 + E_1 T_1(\xi) + \cdots + E_N T_N(\xi) \quad , \quad j = 1,2$$

and the expansion would be employed for the range $-A < x < +A$. The author hopes that this alternative causes the reader to see that the employment of the Chebyshev representations is not purely "mechanical." One will often find that some preliminary analytical work will suggest an argument to be employed in the Chebyshev polynomial which is most appropriate for the problem under consideration.

Let us turn now to an aspect of the Chebyshev representation of the Airy functions which is more than just "financially economic" in the sense that we will consider representations which cannot be "telescoped" because they start from asymptotic series. Miller has shown (See Eq. (B-3)) that the Airy functions can be expressed in terms of certain auxiliary functions $F(x)$ and $\chi(x)$ when x is large and negative. We recall that the forms are

$$Ai(-\eta) = F(-\eta)\sin[\chi(-\eta)] \quad , \quad Bi(-\eta) = F(-\eta)\cos[\chi(-\eta)] \quad (8\text{-}40)$$

where asymptotic expansions for $F(-\eta)$ and $\chi(-\eta)$ are given in Eq.

(6-9). Suppose we seek a representation for $A < \eta < \infty$. Since this is a "one-sided" range, we would probably choose to use the shifted (or T-star) polynomials $T_n^*(x)$. An inspection of Eq. (6-9) suggests that we express $F(-\eta)$ and $\chi(-\eta)$ in the forms

$$F^2(-\eta) = \frac{f_1(\xi)}{\pi\sqrt{\eta}} \quad , \quad \chi(-\eta) - \frac{\pi}{4} = \frac{2}{3}\sqrt{\eta^3}\, f_2(\xi) \ , \ \xi = A^3/\eta^3 \qquad (8\text{-}41)$$

where $f_1(\xi)$ and $f_2(\xi)$ are to be expressed in the form of series of shifted Chebyshev polynomials

$$f(x) = \frac{1}{2} A_0 + A_1T_1^*(x) + A_2T_2^*(x) + \dots + A_NT_N^*(x) \qquad (8\text{-}42a)$$

where the counterpart of Eq. (8-35) is

$$NA_r = f(x_0)+2f(x_1)T_1^*(x_r)+\dots+2f(x_{N-1})T_{N-1}^*(x_r)+f(x_N)T_N^*(x_r) \qquad (8\text{-}42b)$$

and the x_r and x_j $(j = 0,1,2,\dots,N)$ are given by

$$x_r = \cos(\frac{j\pi}{2N}) \quad , \quad x_j = \cos(\frac{r\pi}{2N}) \ , \ \left.\begin{matrix} r \\ j \end{matrix}\right\} = 0,1,2,\dots,N \qquad (8\text{-}42c)$$

In arriving at the choice of this form of expansion we have been guided by the asymptotic theory which led to Eq. (6-9), but unless we take A extremely large (in which case we can evaluate the asym-totic series in Eq. (6-9) by considering only a few <u>decreasing</u> terms) we cannot convert the series of inverse powers in Eq. (6-9) into series of Chebyshev polynomials by means of Program 8-7. If, however, we use Eq. (6-9) only for those values of η for which the asymptotic series yields the desired accuracy, and then by "brute force" (such as by numerical integration from large values of η to smaller values, or vice versa) obtain the values of $F(-\eta)$ and $\chi(-\eta)$ at $(N + 1)$ points such as those required in Eq. (8-35) and Program 8-8, we can determine a set of values of A_n for use in Eq. (8-42). In this way we will obtain a polynomial which

possesses many of the features of the asymptotic series, but which will possess the advantage of being able to be used down to very small values of the variable (η in Eq. (8-41)) which were completely inconceiveable with the asymptotic series. The procedures suggested above depend upon the analytical study of the asymptotic behavior to provide the motivation for which portion of the function will be sufficiently "smooth" to possess a rapidly decreasing set of coefficients A_n in the Chebyshev series. Furthermore, we have used the asymptotic theory to provide a motivation for the selection of a "argument" to employ in the Chebyshev polynomial.*

A similar set of relations can be developed for the computation of the Airy functions for large, positive values of the argument by using the relations in Eqs. (6-52a) and (6-52b). However, this would so closely parallel the result in Eq. (8-41) that we will choose instead to consider the asymptotic expansions

$$\mathrm{Ai}(x) \sim \frac{1}{2}\pi^{-1/2}x^{-1/4}e^{-\xi}\left(1 - \frac{3\cdot 5}{1!\,216}\frac{1}{\xi} + \frac{5\cdot 7\cdot 9\cdot 11}{2!(216)^2}\frac{1}{\xi^2} - \frac{7\cdot 9\cdot 11\cdot 13\cdot 15\cdot 17}{3!\,(216)^3}\frac{1}{\xi^3} + \cdots\right)$$

$$\mathrm{Bi}(x) \sim \pi^{-1/2}x^{-1/4}e^{\xi}\left(1 + \frac{3\cdot 5}{1!\,216}\frac{1}{\xi} + \frac{5\cdot 7\cdot 9\cdot 11}{2!\,(216)^2}\frac{1}{\xi^2} + \frac{7\cdot 9\cdot 11\cdot 13\cdot 15\cdot 17}{3!\,(216)^3}\frac{1}{\xi^3} + \cdots\right)$$

where

$$\xi = \frac{2}{3}\sqrt{x^3}$$

*This field is so new that there has not been accumulated (in the literature or in the experience of the present author) enough experience to be certain that the natural argument suggested by the asymptotic theory is the "best" argument to employ. The reader who undertakes work along the lines suggested above should enter this field with an "open mind" and be prepared to "experiment."

to suggest that representations be sought for Ai(x) and Bi(x) which are of the form

$$2\sqrt{\pi}\sqrt{x}\ \mathrm{Ai}(x) = \exp(-\xi) f_1(\eta) \ , \quad \sqrt{\pi}\sqrt{x}\ \mathrm{Bi}(x) = \exp(+\xi) f_2(\eta) \qquad (8\text{-}43)$$

where

$$\eta = \sqrt{(A/x)^3} \ , \qquad A < x < \infty$$

and the functions $f_1(\eta)$ and $f_2(\eta)$ are series of the form given in Eq. (8-42)

We can look foward to finding many expansions similar to those above in the literature of the next several years. The reader's attention is directed to the papers of wimp (Ref. 67) and of Luke and Wimp (Ref. 68) as being typical of some of the work which has been done to date.

The present author has begun work on the obtaining of Chebyshev polynomial representations for the roots $t_s(q)$ of the Airy function combination

$$w_1'(t_s) - q\, w_1(t_s) = 0$$

where arg q is held fixed (for certain values of arg q which occur in practical problems) and the Chebyshev expansions are with respect to the modulus of q.

Some more interesting applications, however, arise in the case of some of the diffraction functions. Consider the case of the asymptotic expansion for the current distribution function $g(\xi)$.

$$g(\xi) \xrightarrow[\xi \to -\infty]{} 2\exp\left(-i\frac{\xi^3}{3}\right)\left\{1 + i\frac{1}{4\xi^3} - \frac{1}{\xi^6} - i\frac{469}{64}\frac{1}{\xi^9} + \frac{5005}{64}\frac{1}{\xi^{12}} + i\frac{11\,22121}{1024}\frac{1}{\xi^{15}}\right.$$

$$\left. - \frac{24\,33368}{128}\frac{1}{\xi^{18}} - i\frac{16102\,89919}{4096}\frac{1}{\xi^{21}} + \cdots\right\} \qquad (8\text{-}44)$$

We know a wide variety of representations for $g(\xi)$. If we use

one of the following representations

$$g(\xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\exp(i\xi t)}{w_1'(t)} dt = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\exp(i\xi t)}{Bi'(t) + i\,Ai'(t)} dt = 2\sqrt{\pi}\, i \sum_{s=1}^{\infty} \frac{\exp(i\xi t_s^o)}{t_s^o w_1(t_s^o)}$$

$$= \sum_{s=1}^{\infty} \frac{\exp\left(-\frac{\sqrt{3}-i}{2}\beta_s \xi\right)}{\beta_s\, Ai(-\beta_s)} = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\exp\left(\frac{\sqrt{3}-i}{2} p\,\xi\right)}{Ai'(p)} dp$$

$$= \frac{1}{\pi} \int_0^{\infty} \frac{\exp\left(-\frac{\sqrt{3}+i}{2}\xi t\right)}{Bi'(t) - i\,Ai'(t)} dt + \frac{1}{\pi} \int_0^{\infty} \frac{\exp(i\xi t)}{Bi'(t) + i\,Ai'(t)} dt = \sum_{n=0}^{\infty} g^{(n)}(0)\frac{\xi^n}{n!} \quad (8\text{-}45)$$

or make use of the fact that $g(\xi)$ satisfies the integral equation

$$g(\xi) = 2\exp\left(-i\,\frac{1}{3}\xi^3\right) - \frac{1}{2}\,\frac{\exp\left(-i\frac{\pi}{4}\right)}{2\sqrt{\pi}} \int_{-\infty}^{\xi} g(x) \exp\left[-i\,\frac{1}{12}(\xi - x)^3\right]\sqrt{\xi - x}\; dx \quad (8\text{-}46)$$

we can devise "brute force" means of evaluating $g(\xi)$ for all values of ξ. We can then use the interpolation properties of the Chebyshev polynomials to find a representation in terms of Chebyshev polynomials which will be more convenient (as well as more economical) to employ when further values of $g(\xi)$ are required for arbitrary values of ξ.

Let us start with the case in which ξ is large and negative. Although it might turn out to be convenient to find Chebyshev polynomial representations for the real and the imaginary parts of the quantity in braces in Eq. (8-44), the present author suggests that Eq. (8-44) be replaced by

$$g(\xi) = 2\exp(-i\tfrac{1}{3}\xi^3)M(\eta)\exp[-i\frac{P(\eta)}{4(-\xi)^3}] \quad (8\text{-}47)$$

where $M(\eta)$ is an amplitude function and $P(\eta)$ is a "modified" (since it has been divided by $4(-\xi)^3$) phase function. Both $M(\eta)$ and $P(\eta)$ tend to unity as $\xi \to -\infty$. In order to find a representation for these functions for ξ in the range $-\infty < \xi < -A < 0$, it is proposed that one take η to be defined by

$$\eta = (-A/\xi)^3$$

and seek representations for $M(\eta)$ and $P(\eta)$ in the form of the series contained in Eq. (8-42).

We can draw many parallels between methods to be employed for the functions of which $g(\xi)$ is an example, and the methods which we have suggested for the representation of the Airy functions. The form for $g(\xi)$ in Eq. (8-47) is the analog of Eq. (8-40). Near the origin, we can use an analog of Eq. (8-37) and seek to represent $g(\xi)$ in the form of a Chebyshev series

$$g(\xi) = \frac{1}{2} A_0 + A_1T_1(x) + A_2T_2(x) + \cdots + A_NT_N(x) , \quad x = \xi/A$$

where $-A < \xi < +A$, and the result in Eq. (8-35) or the method of telescoping a power series can be used to determine the A_n.

For large positive values of ξ we know that

$$g(\xi) \xrightarrow[\xi \to \infty]{} \frac{\exp(-\frac{\sqrt{3}-i}{2}\beta_1\xi)}{\beta_1 Ai(-\beta_1)} = G_\infty(\xi) \tag{8-48}$$

where β_1 is the root of $Ai'(-\beta_1) = 0$ which has the value $\beta_1 = 1.101879\cdots$. For smaller values of ξ, but $\xi \geq 0$, we have to replace the approximation in Eq. (8-48) by a sum over a finite number of the roots β_s. If $g(\xi)$ is to be computed very frequently, it might be advantageous to express $g(\xi)$ in the form

$$g(\xi) = G_\infty(\xi)M(\eta)\exp[iP(\eta)] \tag{8-49}$$

where $M(x)$ and $P(x)$ are expressed in the form of series of the type defined in Eq. (8-42) and the variable η is defined by

$\eta = (A/\xi)$ when the range of ξ is $0 < A < \xi < \infty$.

The present author believes that the development of methods such as those described above can transform diffraction theory from an interesting theoretical exercise into a practical tool for engineers and scientists who are concerned with the problems that arise in connection with the diffraction and scattering of waves by convex surfaces. Most of the theory leads to representations which permit the evaluation of the functions that occur in only limited regions. For example, it is characteristic of most of these problems that there is a "gap" between the largest values of a variable for which it is practical to employ a power series representation and the smallest values of the variable for which the asymptotic series can be employed. If this "gap" lies in the region for which engineering results are required one is forced to considering graphs of the functions and the use of "artistic license" in drawing the curves so as to close the "gap." However, the use of such an approach requires considerable manual manipulation of graphs and "French curves." If a "brute force" method is used to close the "gap" in order to determine the coefficients in the Chebyshev polynomial representations, one will then be able to handle the functions in a manner which will permit the use of an electronic computer to assist in the practical applications.*

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*The author has been seeking to determine "brute force" methods to close the "gap" by employing methods of numerical analysis in order to obtain relatively accurate representations for the diffraction functions. However, in a practical problem when one is faced with the need to obtain values in the "gap" to employ in either Eq. (8-35) or Eq. (8-42), it should be borne in mind that an "educated guess" can sometimes be made by drawing a curve of the function and attempting to see if a "French curve" will provide an approximate means of bridging the "gap."

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Appendix A

FORTRAN AND NUMERICAL ANALYSIS

At the present time, numerical methods are assuming a role of increasing importance in applied science; the actual, rather than the theoretical, production of a solution is of paramount interest. Diffraction theory provides many interesting challenges to the numerical analyst. Although very few of the papers which have come to the attention of the present author have actually been written by people engaged in studies in diffraction theory, the author finds that almost every new issue of journals devoted to numerical analysis contains papers which are of direct interest to those who face problems in diffraction theory.

We are now at the threshold of an era in which the electronic digital computer is becoming as common an engineering tool as the slide rule. We are reaching the point where even relatively simple problems which are traditionally performed by persons operating manual-type desk calculators can be more economically (as well as more accurately---in the sense of the elimination of the chances of human error in transcribing numbers from computation sheets to the keyboard and from the registers to the computation sheets) executed on the electronic computer. The problem of setting up a set of computation sheets for the hand-computer is hardly more difficult than the devising of the FORTRAN program for the electronic computer. Much of the inertia which causes scientists and engineers to continue the traditional hand-computations is due to the fact that although the task might require only 30 seconds (perhaps a charge of three to five dollars for computer time), there is present the delays in getting the IBM cards punched by the "keypunch pool" and the "turn around" time in getting the problem onto and off the electronic computer.

We are also at the threshold of a new era in which even the use of the electronic computer is being revolutionized through the introduction of "time-sharing" concepts and other innovations. These new advances will soon make the use of the manually-operated desk machine less attractive.

The present author is a strong advocate of the "open shop" operation of computing facilities; that is, he advocates that the programming be done by the scientist or engineer who devises the theoretical formulae (or by one of their close assistants) instead of being "farmed out" to a "closed shop" facility where specialists in programming and numerical analysis translate the formulae into output in the form of tables or graphs. The "do-it-yourself" programmer will build up a background of experience in the particular types of calculations which his field of research is centered around and he will be able to draw upon his experience with previous computations to devise methods for the computation of new problems. Furthermore, since new problems are often only relatively small perturbations of problems which have already been considered, the engineer and scientist will begin to develop a nucleus of programs which often require but minor changes in order to undertake a new problem. Moreover, since these engineers and scientists have an understanding of the importance of the particular problem in their field, they are in a position to be able to recognize when a program that has been devised will be valuable to his colleagues. The reporting of these computing programs can become a vital part of the task of preparing the results of the theory and the calculations for inclusion in technical reports.

We are now able to undertake problems which would have been far too complex to have attempted to do by hand on manually-operated desk machines. However, we are also able, in a matter of seconds, to produce thousands of meaningless results because of the use of

an algorithm which is not appropriate for the problem under consideration for the range of values of the parameters which are of interest. A method of calculation (an algorithm) which may have worked well for a hand computation for a small value of a parameter may lead to mere computer noise when the electronic computer is used to evaluate the same physical phenomena for a large value of the parameter. While doing a problem by hand, the human operator would generally recognize when the results were fading away into the roundoff noise (such as when finding the difference between two numbers which are identical to all of the significant digits). However, when using the electronic computers which are available today, most of the intermediate results are "out-of-sight" and the user must be alert to the possibility that the results which he obtains may be meaningless in spite of the fact that a "correct" program has been written and has been used for a value of the parameter for which results are known from hand computations or from the results published by a previous author.

The experience of the present author has offered a good example of where "mathematically correct" procedures can lead to erroneous results when used with the finite word size of the modern electronic computer. In 1955 the author supervised the preparation of a set of data for the scattering properties of perfectly conducting spheres with diameters up to about seven wavelengths. Curves based upon these data were published in an appendix of a monograph on diffraction theory, namely

> R.W.P. King and T.T. Wu, The Scattering and Diffraction of Waves, Cambridge, Mass., Harvard University Press, 1959

and have been reproduced in the Encyclopedia of Physics and in J. van Bladel's Electromagnetic Fields (New York, McGraw-Hill, 1964). On three occasions the author has received long distance telephone calls from individuals who have programmed the exact

solution for the sphere and have had difficulty in obtaining results which correspond to those presented in the curves which were based upon the author's data. On each occasion, the author was able to resolve the problem on the telephone by inquiring as to how the spherical Bessel functions had been obtained. In each case, the investigators had made use of the property that the first two $j_n(x)$ functions are

$$j_0(x) = \frac{\sin(x)}{x} \quad , \quad j_1(x) = \frac{\sin(x)}{x^2} - \frac{\cos(x)}{x}$$

and the other values obtained from the recursion formula

$$j_{n+1}(x) = - j_{n-1}(x) + \frac{2n+1}{x} j_n(x)$$

When x is larger than unity, and n is allowed to range through values that exceed the magnitude of x, the calculation (using the finite word size of the computer) deteriorates rapidly and eventually one even gets results which show the calculated values of $j_n(x)$ increasing with increasing values of n instead of possessing the correct property of decreasing with increasing values of n. The author was able to help his telephone callers by referring them to literature in which it is shown that the $j_n(x)$ functions should be calculated by "backward recurrence" by means of

$$j_{n-1}(x) = - j_{n+1}(x) + \frac{2n+1}{x} j_n(x)$$

There is, by now, a rather extensive literature upon this subject. For example:

I. A. Stegun and M. Abramowitz, "Generation of Bessel Functions on High Speed Computers," Math. Tables, Vol. 11, Oct., 1957, pp. 255-257

J. B. Randels and R. F. Reeves, "Note on Empirical Bounds for Generating Bessel Functions," Comm. Assoc. Comput. Mach., Vol, 1, May, 1958, pp. 3-5

M. Goldstein and R. M. Thaler, "Recurrence Techniques for the Calculation of Bessel Functions," Math. Tables, Vol. 13, April, 1959, pp. 102-108

F. J. Corbató and J. L. Uretsky, "Generation of Bessel Functions in Digital Computers," J. Assoc. Comput. Mach., Vol. 6, July, 1959, pp. 366-375

W. Gautschi, "Recursive Computation of the Repeated Integrals of the Error Function," Math. Comp., Vol. 15, July, 1961, pp. 227-232

F. W. J. Olver, "Error Analysis of Miller's Recurrence Algorithm," Math. Comp., Vol. 18, Jan., 1964, pp. 65-74

The present author considers this problem of the computation of the Bessel functions to be a good example of why the individual research worker in diffraction theory cannot afford to "divorce himself" from the field of numerical analysis. The textbooks from which we learn our basic applied mathematics present us with devices such as the recurrence formulae for the Bessel functions, but we often fail to realize that these exact mathematical properties may have to be experimented with before they can be used on a calculating device which works with only a finite number of digits.

The author highly recommends that the reader acquaint himself with the paper

I. A. Stegun and M. Abramowitz, "Pitfalls in Computation," J. Soc. Indust. Appl. Math., Vol. 4, Dec., 1956, pp. 207-219

Among the related papers on this subject of difficulties associated with translating exact mathematical formulae into numerically meaningful results, the author recommends the following:

A. S. Householder, "Generation of Errors in Digital Computation," Bull. Amer. Math. Soc., Vol. 60, June, 1954, pp. 234-247

G. E. Forsythe, "Singularity and Near Singularity in Numerical Analysis," Amer. Math. Monthly, Vol. 65, April, 1958, pp. 229-240

Mathematics Research Center, The University of Wisconsin, Some Considerations in Practical Computation, by A. S. Chai, and H. J. Wertz, MRC Technical Summary Report No. 469, Contract No. DA-11-022-ORD-2059, Madison, Wisc., May, 1964

To date, the author knows of no book which has been written expressly for the engineer and scientist who is interested in the diffraction and the propagation of radio waves. However, such books are beginning to appear in other fields. For example:

R. S. Ledley, Use of Computers in Biology and Medicine, New York, McGraw-Hill, 1965

L. Lapidus, Digital Computation for Chemical Engineers, New York, McGraw-Hill, 1962

An examination of these books will reveal, however, that they are primarily concerned with "classical" numerical analysis and that they are primarily a collection of methods which are not very different from those to be found in books which are concerned with numerical analysis and which refrain from the use of "digital" or "computer" in their titles. However, the author does recommend the book by Ledley for its introductory material which will

provide the reader with some photographs and descriptions of some of the digital equipment which is currently available.

A different type of book which is beginning to appear is to be found in the efforts of certain authors to make available to other scientists the actual computer programs which have been developed in specific fields. Examples of books of this type are:

M. A. Melkanoff, D. S. Saxon, J. S. Nodvik, and D. G. Cantor, A FORTRAN Program for Elastic Scattering Analyses with the Nuclear Optical Model, Berkeley, University of California Press, 1961

P. A. D. de Maine and R. D. Seawright, Digital Computer Programs for Physical Chemistry, New York, Macmillan, 1963

F. Herman and S. Skillman, Atomic Structure Calculations, Englewood Cliffs, N.J., Prentice-Hall, 1963

The book by de Maine and Seawright is unique in that it presents the programs in both FORTRAN II and ALGOL 60. The present author has had no experience with the use of ALGOL 60 and, therefore, is in no position to pass judgement upon the relative merits of the use of these languages. However, since ALGOL 60 has been adopted as a "publication" language by several journals, the scientist who wishes to adapt for his useage the published algorithms of other scientists will often find the need to be able to read an ALGOL 60 procedure. The reader who wishes to acquaint himself with this programming language will probably find that one of the best sources for "self-study" is the following manual:

D. D. McCracken, A Guide to ALGOL Programming, New York, John Wiley, 1962

Other books which deal with ALGOL are:

E. W. Dijkstra, _A Primer of ALGOL 60 Programming_, New York, 1962

R. Wooldridge and J. F. Ractliffe, _An Introduction to ALGOL Programming_, Princeton, N.J., Van Nostrand, 1963

C. M. Reeves and M. Wells, _A Course on Programming in ALGOL 60_, Chicago, Educational Methods Inc., 1964

The reader who is not familiar with FORTRAN and who wishes to set out to learn this programming language will find himself confronted with the making of a choice between a number of texts if he wishes to begin a "self-study" program to acquaint himself with the features of the language.* The suggestion as to what text a particular person would find most helpful as a starting point is dependent upon the background, interests, and temperament of the individual. In the opinion of the present author, none of the texts presently available provide the mature scientists and engineers (assumed to have an education in electromagnetic theory and advanced calculus which includes some graduate courses) with an adequate "marriage" between the teaching of programming concepts and the use of these in numerical methods. One of the best efforts in this direction is the text

W. Prager, _Introduction to Basic FORTRAN Programming and Numerical Methods_, New York, Blaisdell, 1965

This text was written for a sophomore course for students of applied mathematics. It provides a readable account of the elements of FORTRAN (essentially FORTRAN II) and has the advantage of introducing at an early stage some flow charts and programs

*In the San Francisco area one of the local TV stations has been conducting a course in FORTRAN during the early morning hours. A least one course (of twenty lessons) has been taped and is available for classroom use in certain universities.

which will serve to answer the demands of the reader who starts his study and very soon becomes impatient and starts saying to himself: "Okay, so FORTRAN provides a set of precise rules for the writing of mathematical expressions. Now show me how you employ these rules in a typical scientific problem." However, after giving a few programs, the emphasis of the text shifts to the presentation (in a rather lucid manner) of a "minimum" knowledge of classical numerical methods. The present author is of the opinion that this text will be useful to many persons (engaged in electromagnetic studies) who work with "closed shop" programmers or with younger assistants who have been "college trained" in computer programming and numerical methods. The study of this text will help to open up the lines of communication between the scientist-engineer and his programming assistant.

For the person who wishes to persue his study of FORTRAN to the point of undertaking to do his own programming, the present author highly recommends either of the following texts:

J. T. Golden, FORTRAN IV Programming and Computing, Englewood Cliffs, N.J., Prentice-Hall, 1965

D. D. McCracken, A Guide to FORTRAN IV Programming, New York, John Wiley, 1965

Either of these texts will, if diligently studied (and if help is sought from colleagues who have had more experience with programming when obscure points arise), should provide the necessary background to undertake to write FORTRAN IV programs.

The reader who wishes to "shop around" for other texts will find himself faced with a "flood" of books (some of which are rather expensive) from which to choose. A partial list includes the following:

J. N. Haag, Comprehensive FORTRAN Programming, New York, Hayden, 1965

L. D. Harris, Numerical Methods Using FORTRAN, Columbus, Ohio, Charles E. Merrill, 1964

J. M. McCormick and M. G. Salvadori, Numerical Methods in FORTRAN, Englewood Cliffs, N.J., Prentice-Hall, 1964

D. D. McCracken, A Guide to FORTRAN Programming, New York, John Wuley, 1961

D. D. McCracken and W. S. Dorn, Numerical Methods and FORTRAN Programming, New York, John Wiley, 1964

E. I. Organick, A FORTRAN Primer, Reading, Mass., Addison-Wesley, 1963

R. H. Pennington, Introductory Computer Methods and Numerical Analysis, New York, Macmillan, 1965

S. V. Pollack, A Guide to FORTRAN IV, New York, Columbia University Press, 1965

The present author has also derived some benefits from sections in the book

B. W. Arden, An Introduction to Digital Computing, Reading, Mass., Addison-Wesley, 1963.

However, this book contains programs which are written in the MAD (Michigan Algorithm Decoder) language which resembles FORTRAN (FORmula TRANslation) in many respects, and yet has some of the features of ALGOL 60, while at the same time having features not found in either of the two more common languages.

Several of the titles above reveal that many writers are trying

teach numerical methods and computer programming with the same text. Although this is very desirable in that it affords the scientist-engineer the opportunity to see the programming rules applied to the type of problems which he is likely to encounter in his work, the result is generally that the book is an inferior text on numerical analysis.

The present author feels that the two books which he personally feels to be the most valuable books on numerical analysis are

C. Lanczos, Applied Analysis, Englewood Cliffs, N.J., Prentice-Hall, 1956

A. Ralston, A First Course in Numerical Analysis, New York, McGraw-Hill, 1965

In many respects, the book by Lanczos is "unorthodox" since its emphasis is somewhat different from a host of other books. However, Lanczoz has presented many "tricks of the trade" which he has found useful during his long and highly successful career as a physicist and a consultant in applied mathematics. The book by Ralston, although it is newly off the press, promises to become a classic. Ralston acknowledges his endebtedness to his teachers who wrote the excellent treatises:

F. B. Hildebrand, Introduction to Numerical Analysis, New York, McGraw-Hill, 1956

Z. Kopal, Numerical Analysis, 2nd ed., London, Chapman and Hall, 1961

and to his former colleague who wrote the very useful book:

R. W. Hamming, Numerical Methods for Scientists and Engineers, New York, McGraw-Hill, 1962

Two other books which the author has found quite useful are:

Staff of National Physical Laboratory, _Modern Computing Methods_, London, Her Majesty's Stationery Office, 1961

G. N. Lance, _Numerical Methods for High Speed Computers_, New York, Gordon and Breach, 1960

Three books have appeared which have been the result of the joint efforts of a number of authors and which contain much valuable reference material. These are:

L. Fox, ed., _Numerical Solution of Ordinary and Partial Differential Equations_, Reading, Mass., Addison-Wesley, 1962

A. Ralston and H. S. Wilf, eds., _Mathematical Methods for Digital Computers_, New York, John Wiley, 1960

J. Todd, ed., _Survey of Numerical Analysis_, New York, McGraw-Hill, 1962

The collection of papers edited by Ralston and Wilf is especially recommended to the reader who wishes to study some examples of the preparation of flow charts which can often be of great help in the explanation of the methods which underlie a computer program.

There are many other textbooks available which provide the reader with a collection of techniques for use in numerical analysis. Much of the material which is included is classical and can be found in some of the older texts, such as

E.T. Whittaker and G. Robinson, _The Calculus of Observations; a Treatise on Numerical Mathematics_, London, Blackie, 1924

Among the newer texts, the author has consulted the following:

P. J. Davis, Interpolation and Approximation, New York, Blaisdell, 1963

C. E. Fröberg, Introduction to Numerical Analysis, Reading, Mass., Addison-Wesley, 1965

D. R. Hartree, Numerical Analysis, Oxford, Oxford University Press, 1958

P. Henrici, Elements of Numerical Analysis, New York, John Wiley, 1964

J. G. Herriot, Methods of Mathematical Analysis and Computation, New York, John Wiley, 1963

A. S. Householder, Principles of Numerical Analysis, New York, McGraw-Hill, 1953

C. Lanczos, Linear Differential Operators, New York, Van Nostra nd, 1961

B. Noble, Numerical Methods: 1 and 2, New York, Interscience, 1964

R. G. Stanton, Numerical Methods for Science and Engineering, Englewood Cliffs, N.J., Prentice-Hall, 1961

E. L. Stiefel, An Introduction to Numerical Mathematics, New York, Academic Press, 1963

Since two of the most important problems which face the numerical analyst are those of evaluating integrals and of solving differential equations, the author wishes to direct attention to the two books:

V. I. Krylov (trans. by A. H. Stroud), Approximate Calculation of Integrals, New York, Macmillan, 1962

P. Henrici, Discrete Variable Methods in Ordinary Differential Equations, New York, John Wiley, 1962

This survey of some of the recent books which have appeared on FORTRAN programming and numerical methods serves to indicate that the scientist-engineer (whose formal education has included no courses in programming and numerical methods) who wishes to acquire skills which will permit him to make use of the electronic digital computer will find a wealth of source material available to him. The problem which the individual will have to resolve is that of choosing where he is going to start. The author feels that most of the above books are excellent and the choice of which books are to be singled out for use by any given individual will depend largely upon the background of that person. However, with apologies to all the authors whose works are slighted, the present author recommends to most of his colleagues in the field of electromagnetic theory that they consider as a basic library the three books:

W. Prager, Introduction to Basic FORTRAN Programming and Numerical Methods, New York, Blaisdell, 1965

J. T. Golden, FORTRAN IV Programming and Computing, Englewood Cliffs, N.J., Prentice-Hall, 1965

A. Ralston, A First Course in Numerical Analysis, New York, McGraw-Hill, 1965

For the use of any given computer, the potential user will have to consider the manuals on FORTRAN which have been prepared by the manufacturer. Although these manuals are generally written in such terse language that they do not constitute a good text

for the learning of FORTRAN, they constitute a vital reference once one has begun to acquire a familiarity with the language. A great deal of care has been taken in the preparation of most of these manuals.

The present author feels that one of the most difficult concepts to master in the FORTRAN language is that of the FORMAT statements for the input and output of the data. When one has mastered these concepts it is possible to devise a display of the output data which is essentially the format which one would like to have reproduced as a table in a technical report. One of the best ways to master FORMAT is to actually write programs and gradually attempt to be more sophisticated in the layout of the output data. In connection with the INPUT/OUTPUT statements, the author would like to direct attention to one manufacturer's manual in which he believes that an exceptionally good job has been done in illustrating the concepts associated with this important aspect of computer programming. The manual is:

> Honeywell Electronic Data Processing, AUTOMATH 1800 Language Manual, Wellesley Hills, Mass., 1964

The AUTOMATH 1800 language is a "dialect" of FORTRAN IV. However, most of the features of the FORMAT specifications are identical with those of the features of most of the FORTRAN IV compilers for the larger electronic computers. Through the use of skillfully-executed drawings and illustrations this manual succeeds (in the opinion of the present author) in clearing up a lot of points which one might be forced to learn to master from experience if one tries to follow some of the more terse descriptions given in some of the other manuals and texts. It is extremely important to master FORMAT because in many instances one can have a program which is working "perfectly" from a mathematical point of view but one will be forced to re-run the job because of a mistake in format specification.

Appendix B

TABLES OF THE AIRY FUNCTION ON THE REAL AXIS

In this appendix we will present some tables of the Airy functions which we have constructed for real values of the argument. The definitions which we use in this section are those employed by Miller[#].

$$Ai(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt \qquad \text{(B-1)}$$

$$Bi(x) = \frac{1}{\pi}\int_0^\infty \left[\sin\left(\frac{t^3}{3} + xt\right) + \exp\left(-\frac{t^3}{3} + xt\right)\right] dt \qquad \text{(B-2)}$$

We recall that Miller also defined the auxiliary functions defined by

$$Ai(x) = F(x)\sin[\chi(x)] \quad , \quad Bi(x) = F(x)\cos[\chi(x)] \qquad \text{(B-3)}$$

$$Ai'(x) = G(x)\sin[\psi(x)] \quad , \quad Bi'(x) = G(x)\cos[\psi(x)] \qquad \text{(B-4)}$$

For small values of x, one can evaluate these functions by means of the Taylor series

$$Ai(x) = \alpha y_1 - \beta y_2 \qquad Bi(x) = 3^{1/2}(\alpha y_1 + \beta y_2) \qquad \text{(B-5)}$$

$$y_1 = 1 + \frac{1}{3!}x^3 + \frac{1\cdot 4}{6!}x^6 + \frac{1\cdot 4\cdot 7}{9!}x^9 + \ldots \qquad y_2 = x + \frac{2}{4!}x^4 + \frac{2\cdot 5}{7!}x^7 + \frac{2\cdot 5\cdot 8}{10!}x^{10} + \ldots$$

[#] J.C.P. Miller, The Airy Integral, British Association Mathematical Tables, Part-Volume B, Cambridge, Cambridge University Press, 1946

where α and β are the constants

$$\alpha = 3^{-\frac{2}{3}}/(-\tfrac{1}{3})! = 0.355028053887817239256\cdots$$

$$\beta = 3^{-\frac{1}{3}}/(-\tfrac{2}{3})! = 0.258819403792806798405\cdots$$

The tables given by Miller include the following

$Ai(x)$ and $Ai'(x)$. $x = -20.00(0.01) + 2.00$. 8D

Log_{10} $Ai(x)$ and $Ai'(x)/Ai(x)$. $x = 0.0(0.1)25.0(1)75$. 7-8D

Zeros and Turning-Values of $Ai(x)$ and $Ai'(x)$. The first 50 of each. 8D

$Bi(x)$ and Reduced Derivatives. $x = -10.0(0.1) + 2.5$. 7-8D

Zeros and Turning-Values of $Bi(x)$ and $Bi'(x)$. The first 20 of each. 8D

Log_{10} $Bi(x)$ and $Bi'(x)/Bi(x)$. $x = 0.0(0.1)10.0$. 7-8D

Auxiliary Functions. $F(x)$, $\chi(x)$, $G(x)$, $-\psi(x)$. $x = -80(1) - 30.0(0.1) + 2.5$. 8D

Values of $Ai(z)$, $Bi(z)$, $Ai'(z)$, and $Bi'(z)$ have been given by Woodward and Woodward* for complex values of $z = x + iy$ for which the magnitudes of x and y are less than 2.4. If one makes use of the relationships between these Airy functions and the modified Hankel functions which were computed by the staff of the Harvard Computation Laboratory**, one can obtain values of these functions for complex values of z for which the modulus of z is less than 6.

We have programmed (using data from the Harvard tables**) Eq. (B-3) and Eq. (B-4) so as to be able to evaluate the Airy functions in

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*P. M. Woodward and A. M. Woodward, "Four-Figure Tables of the Airy Function in the Complex Plane," Phil. Mag., Vol. 37, pp. 236-261

**Harvard University Computation Laboratory, Tables of the Modified Hankel Functions of Order One-Third and of Their Derivatives, Annals of the Computation Laboratory of Harvard University, Vol. II, Cambridge, Mass., Harvard University Press, 1945

the complex plane under the condition that the modulus of z be less than 6.1. The program is known as

SUBRØUTINE AIBIZ(Z,AI,BI,AIP,BIP)

and is listed in Program B-1. The relation of the FORTRAN variables to the mathematical definitions are as follows:

$$z = Z$$
$$Ai(z) = AI$$
$$Bi(z) = BI$$
$$Ai'(z) = AIP$$
$$Bi'(z) = BIP$$

The presence of the type declaration

CØMPLEX Z,AI,BI,AIP,BIP

reveals that the complex arithmetic features of FORTRAN IV are utilized. However, the program actually employs double precision arithmetic. For values of z which are somewhat smaller than the limit of 6.1 which has been placed on the modulus of z, the program can be easily modified* so as to obtain more than the eight

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*For example, let z = ZR + iZI and add "DØUBLE PRECISION ZR,ZI" while retaining "CØMPLEX Z." Omit the statements defining AI, BI, AIP, and BIP. Then rename the subprogram as follows: "SUBRØUTINE AIBIZ(ZR,ZI,DAIR,DAII,DAIPR,DAIPI,DBIR,DBII,DBIPR,DBIPI)." Let the first executable statement be "Z = -CMPLX(ZR,ZI)." Replace the statements defining DRZ and DIZ with "DRZ = -ZR" and "DIZ = -ZI." The "CALL EXIT" which will occur when CABS(Z) is greater than 6.1 will have to be modified according to what accuracy the user wishes to achieve. Since the greatest loss of accuracy occurs when computing Ai(x) and Ai'(x) for positive values of x, a comparison of the output of these quantities with the tabulated values given in Block B-2 will provide the basis for a criterion for the number of significant figures available.

decimal digits which are available when employing the "CØMPLEX" features of FORTRAN IV. We have employed this feature in the argument list of the subprogram because we had in mind the needs of a user who is employing complex arithmetic in the calling program.

If the modulus of z exceeds 6.1, the program writes a message

AIBIZ CANNOT HANDLE Z = ...

(where the dots indicate that the value of z is printed) and then executes a CALL EXIT.

The NPREC subroutines (described in the Preface) have been employed to compute values of the Airy functions on the real axis which are accurate to from 15 to 21 significant figures. In Block B-2 we present the values of the Airy functions and their derivatives for the interval x = 0.1(0.1)10.0. These data have been arranged in the form of a BLOCK DATA subprogram with the data stored in a labeled CØMMØN block referred to by the FORTRAN name EQUADP for its relation to the EQUal-interval Airy function in Double Precision. If we let the mathematical variable x be related to the FORTRAN index I by means of the relationship

x = 0.1*FLØAT(I)

then the functions that are tabulated are

Ai(x) = EAIRYF(I)
Bi(x) = EBIRYF(I)
Ai'(x) = EAIRYD(I)
Bi'(x) = EBIRYD(I)

The values of Bi(x) and Bi'(x) are accurate to 20 figures throughout most of the range, but the values of Ai(x) and Ai'(x) deteriorate to about 15 or 16 figures in the middle of the range. The author has been unable to devise an algorithm which will achieve more accuracy for Ai(x) and Ai'(x) for x in the vicinity of 5 when doing the calculations with a "word length" of 21 decimal

digits. Since the need for higher accuracy has not arisen, the author has not undertaken the computation of the auxiliary functions $r(x)$, $\mu(x)$, $p(x)$, and $\lambda(x)$ in the representations

$$2\sqrt{\pi}\ \mathrm{Ai}(x) = \sqrt{r(x)}\ \exp[-\mu(x)] \qquad \text{(B-6a)}$$

$$\sqrt{\pi}\ \mathrm{Bi}(x) = \sqrt{r(x)}\ \exp[+\mu(x)] \qquad \text{(B-6b)}$$

$$-2\sqrt{\pi}\mathrm{Ai}'(x) = \sqrt{p(x)}\ \exp[-\lambda(x)] \qquad \text{(B-7a)}$$

$$\sqrt{\pi}\mathrm{Bi}'(x) = \sqrt{p(x)}\ \exp[+\lambda(x)] \qquad \text{(B-7b)}$$

where the differential equations for the functions have been given in Eqs. (6-38), (6-50), (6-53) and (6-54). Accurate values for these auxiliary functions can be obtained for small values of x from the Taylor series in Eq. (B-5) and for large values of x from the asymptotic expansions of Eqs. (6-52) and (6-55). If step-by-step integration is employed from the small and large values of x, it is quite possible that one can close the "gap" that occurs around $x = 5$ with very precise values (whose accuracy will be apparent from the degree of overlap between the values obtained by starting at the two edges of the "gap").

In Table B-1 we give the values of Ai(x) and Bi(x) for $x = 10.0(0.1)25.0$. In Table B-2 we give the values of Ai′(x) and Bi′(x) over the same range. The reader should observe that for x greater than 25.0 one will soon reach the point where the values of the Airy functions will lie outside the range 10^{-38} to 10^{+38} which is employed by most FORTRAN compilers. The manner in which the Airy functions continue to grow with increasing x is displayed in Tables B-3 and B-4 where we present the values of these functions for $(1/x)$ in the range $0.001(0.001)0.100$. Since problems occur in which we require values (for large values of x) for products such as $\mathrm{Ai}(x)\mathrm{Bi}'(x)$ or ratios such as $\mathrm{Bi}'(x)/[\mathrm{Bi}'(x) - q\mathrm{Bi}(x)]$, it is not practical to use the Airy functions from Tables B-3 and B-4. These products and ratios are well within the range of the FORTRAN numbers and therefore it is convenient to employ

tables of the auxiliary functions $r(x)$, $\mu(x)$, $p(x)$ and $\lambda(x)$ which are defined in Eq. (B-7). In order to tabulate a function which will permit accurate and easy interpolation, it is convenient to consider the asymptotic expansions in Eqs. (6-52) and (6-55) and define auxiliary functions which have the property of approaching unity for x tending to infinity. Let us first introduce the quantity $\xi = \xi(x)$ which is defined by

$$\xi = \frac{2}{3}(\sqrt{x^3}) \tag{B-8}$$

We then define the auxiliary functions

$$Q(x) = \sqrt{x}\, r(x) = 2\pi\sqrt{x}\, Ai(x)Bi(x) \tag{B-9a}$$

$$M(x) = \xi^{-1}\mu(x) \tag{B-9b}$$

$$P(x) = \frac{-1}{\sqrt{x}}\, p(x) = -\frac{2\pi}{\sqrt{x}} Ai'(x)Bi'(x) \tag{B-10a}$$

$$\Lambda(x) = \xi^{-1}\lambda(x) \tag{B-10b}$$

Tables of $Q(x)$ and $M(x)$ are given in Table B-5 for $(1/x) = 0.001\,(0.001)\,0.100$. Tables over the same range are given for $P(x)$ and $\Lambda(x)$ in Table B-6.

Fox* has also considered the case of Airy functions for large, positive values of the argument. The auxiliary functions which he employs are suggested by the asymptotic expansions which are of the form described by Eq. (6-48). Let us supplement Eq. (6-48) by giving the explicit form for the asymptotic expansions for $Ai'(x)$ and $Bi'(x)$. Let ξ be as defined in Eq. (B-8) and let us express the asymptotic expansions in the form

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*L. Fox, Tables of Weber Parabolic Cylinder Functions and Other Functions for Large Arguments, National Physical Laboratory Mathematical Tables, Vol. 4, London, Her Majesty's Stationery Office, 1960

$$2\sqrt{\pi}\ \mathrm{Ai}(x) = x^{-1/4}\exp(-\xi)L(-\xi)\ , \quad \sqrt{\pi}\ \mathrm{Bi}(x) = x^{-1/4}\exp(\xi)L(\xi) \quad \text{(B-11)}$$

$$2\sqrt{\pi}\ \mathrm{Ai}'(x) = -x^{1/4}\exp(-\xi)K(-\xi)\ , \quad \sqrt{\pi}\ \mathrm{Bi}'(x) = x^{1/4}\exp(\xi)K(\xi) \quad \text{(B-12)}$$

where $L(\xi)$ and $K(\xi)$ are the asymptotic series

$$L(\xi) = \sum_{s=0}^{\infty} \frac{u_s}{\xi^s} = 1 + \frac{3\cdot 5}{1!216}\frac{1}{\xi} + \frac{5\cdot 7\cdot 9\cdot 11}{2!(216)^2}\frac{1}{\xi^2} + \frac{7\cdot 9\cdot 11\cdot 13\cdot 15\cdot 17}{3!(216)^3}\frac{1}{\xi^3} + \cdots \quad \text{(B-13)}$$

$$M(\xi) = \sum_{s=0}^{\infty} \frac{v_s}{\xi^s} = 1 - \frac{3\cdot 7}{1!216}\frac{1}{\xi} - \frac{5\cdot 7\cdot 9\cdot 13}{2!(216)^2}\frac{1}{\xi^2} - \frac{7\cdot 9\cdot 11\cdot 13\cdot 15\cdot 19}{3!(216)^3}\frac{1}{\xi^3} - \cdots \quad \text{(B-14)}$$

in which $u_o = 1$ and

$$u_s = \frac{(2s+1)(2s+3)(2s+5)\ldots(6s-1)}{s!(216)^s}\ , \quad v_s = -\frac{6s+1}{6s-1}u_s \quad \text{(B-15)}$$

The functions tabulated by Fox can be defined as follows:

$$R(z) = L(-\tfrac{1}{z}) \quad \text{(B-16)}$$

$$S(z) = L(\tfrac{1}{z}) \quad \text{(B-17)}$$

$$W(z) = -K(-\tfrac{1}{z}) \quad \text{(B-18)}$$

$$X(z) = K(\tfrac{1}{z}) \quad \text{(B-19)}$$

Fox showed that $R(z)$ and $S(z)$ satisfy the differential equation

$$y''(z) + 2\left(\frac{1}{z} \pm \frac{1}{z^2}\right)y'(z) + \frac{5}{36z^2}y(z) = 0 \quad \text{(B-20)}$$

where the + sign is used for $R(z)$ and the - sign for $S(z)$. He also showed that $W(z)$ and $X(z)$ satisfy the differential equation

$$y''(z) + 2\left(\frac{1}{z} \pm \frac{1}{z^2}\right)y'(z) - \frac{7}{36z^2}y(z) = 0 \quad \text{(B-21)}$$

where the + sign is used for $W(z)$ and the - sign is used for $X(z)$. These differential equations are very promising for the

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purpose of developing a method of numerical integration which can be used to obtain the values of the Airy functions in the "gap" between the region where the Taylor series of Eq. (B-5) become impractical* and the smallest values of x for which the asymptotic series of Eq. (B-13) can be employed to compute the functions to the desired accuracy. A very promising approach would be that of employing the asymptotic series of Eqs. (B-13) and (B-14) for large values of ξ (i.e., for large values of x or small values of $z = \xi^{-1}$) and developing a program for the step-by-step integration of Eqs. (B-20) and (B-21) which would carry the computation to smaller values of x and ξ (or to larger values of z).

In Tables B-7 and B-8 we present our results for the functions R(z), S(z), W(z), and X(z). The tables are given as a function of (1/x), where $x = (\frac{2}{3}z)^{-\frac{2}{3}}$, for (1/x) = 0.001(0.001)0.100. The tables are accurate to about 20 decimals.

Since Fox has tabulated the functions as a function of $z = \xi^{-1}$, there is only one point in common in the two sets of calculations, namely x = 25 where (1/x) = 0.04 and z = 0.012. The values given by Fox are

---

*The Taylor series for the Airy function Ai(x) presents the same numerical difficulties for large positive values of x as does the Taylor series for the more familiar function exp(-x). The terms in the series so completely cancel each other that when one employs a computer word length of 21 decimal digits the output has become "computer noise" by the time one reaches $x \approx 10$. The use of step-by-step integration starting from x = 0 and going towards x = 10 was found to yield essentially the same accuracy as that obtained with the Taylor series. However, step-by-step integration starting from large values of x and working towards smaller values is a more stable numerical process and one can preserve a relatively high degree of accuracy.

$$R(0.012) = 0.9991719494\ , \quad S(0.012) = 1.0008387474$$
$$-W(0.012) = 1.0011604193\ , \quad X(0.012) = 0.9988269392$$

The reader can readily see for himself that these agree with our results in Tables B-7 and B-8 for $(1/x) = 0.040$.

Fox computed these auxiliary functions to 10 decimals over the range $z = 0.000(0.001)0.050$. In terms of the argument x of the Airy functions $Ai(x)$ and $Bi(x)$, this range is $x_0 < x < \infty$, where $x_0 \approx 9.65$. This is essentially the same range as that of Tables B-7 and B-8, namely $10 < x < \infty$.

The tables which have been given as Tables B-5 through B-8 provide excellent examples of slowly varying functions which can be represented very efficiently by Chebyshev polynomial series of the form given in Eq. (8-42). A representation in terms of a series of the form $\Sigma a_n T_n^*(\eta)$ would require only a small number of coefficients a_n in order to compute these functions to high accuracy over the range employed in Tables B-5 through B-8. If x lies in the range $A < x < \infty$, then we would take η to be in the form $\eta = (A/x)^\nu$, where appropriate choices of ν might be $\nu = 3$ for Tables B-5 and B-6 and $\nu = 1.5$ for Tables B-7 and B-8.

Let us now direct our attention to negative values of x. In Tables B-9 and B-10 we present values for the Airy functions and their derivatives for the range $-10 < x < 0$ at increments of 0.1. However, we have followed the convention employed by Miller in <u>The Airy Integral</u> (see footnote on p. B-1) and we give tables of $Ai(-x)$, $Bi(-x)$, $Ai'(-x)$ and $Bi'(-x)$. Therefore, the arguments in the tables are positive and $x = 0(0.1)10.0$. When these calculations were compared with the results obtained from the asymptotic series the agreement was found to exist to 19 decimals. Therefore, we assume that throughout most of these tables the results are good to approximately 20 decimals.

Since the facilities to work with 21 decimal digits are not generally available (and since the NPREC subroutines are not especially suited for writing output on magnetic tape to be later reread into the computer), we formed the auxiliary functions $F(-x)$, $\chi(-x)$, $G(-x)$, and $\psi(-x)$ at the time of computing the Airy functions in the range $0 < x < 10.0$. The auxiliary functions are defined by means of the relations

$$\mathrm{Ai}(-x) = F(-x)\sin[\chi(-x)] \quad , \quad \mathrm{Ai}'(-x) = G(-x)\sin[\psi(-x)] \qquad \text{(B-22)}$$

$$\mathrm{Bi}(-x) = F(-x)\cos[\chi(-x)] \quad , \quad \mathrm{Bi}'(-x) = G(-x)\cos[\psi(-x)] \qquad \text{(B-23)}$$

or

$$F^2(-x) = \mathrm{Ai}^2(-x) + \mathrm{Bi}^2(-x) \quad , \quad \chi(-x) = \tan^{-1}[\,\mathrm{Ai}(-x)/\mathrm{Bi}(-x)\,] \qquad \text{(B-24)}$$

$$G^2(-x) = \mathrm{Ai}'^2(-x) + \mathrm{Bi}'^2(-x) \quad , \quad \psi(-x) = \tan^{-1}[\mathrm{Ai}'(-x)/\mathrm{Bi}'(-x)] \qquad \text{(B-25)}$$

In Tables B-11 and B-12 we present our output for these auxiliary functions for $x = 0.0(0.1)10.0$. Unfortunately, the computer program was written without adequate forethought and the angles defined as $\chi(-x)$ and $\psi(-x)$ which appear in these tables require a correction for those cases in which the angle lies in either the second or the third quadrants since the tabular values are in the range from $-\pi/2$ to $+\pi/2$. The reader who wishes to use these values can readily determine if a quadrant correction is required by consulting the corresponding values of $\mathrm{Ai}(-x)$, $\mathrm{Bi}(-x)$, $\mathrm{Ai}'(-x)$ and $\mathrm{Bi}'(-x)$ which are given in Tables B-9 and B-10.

For x between 10 and 25, we computed the auxiliary functions by means of the asymptotic series given in Eq. (6-9). We then used Eqs. (B-22) and (B-23) to compute the Airy functions. The Airy functions are given in Tables B-13 and B-14 and the auxiliary functions are given in Tables B-15 and B-16. In Tables B-15 and B-16 the angles $\chi(-x)$ and $\psi(-x)$ are given in radians, but the reader will observe that they are not confined to the range from

0 to 2π radians. The functions which we have tabulated (in radians) will have the behavior depicted in Fig. B-1 if we extend the tables back towards the origin. Since the quantities tabulated in the tables are $\chi(-x)$ and $\psi(-x)$, where $10 < x < 25$, the reader will observe that the tables represent the behavior of the functions plotted in Fig. B-1 in the region far to the left of the

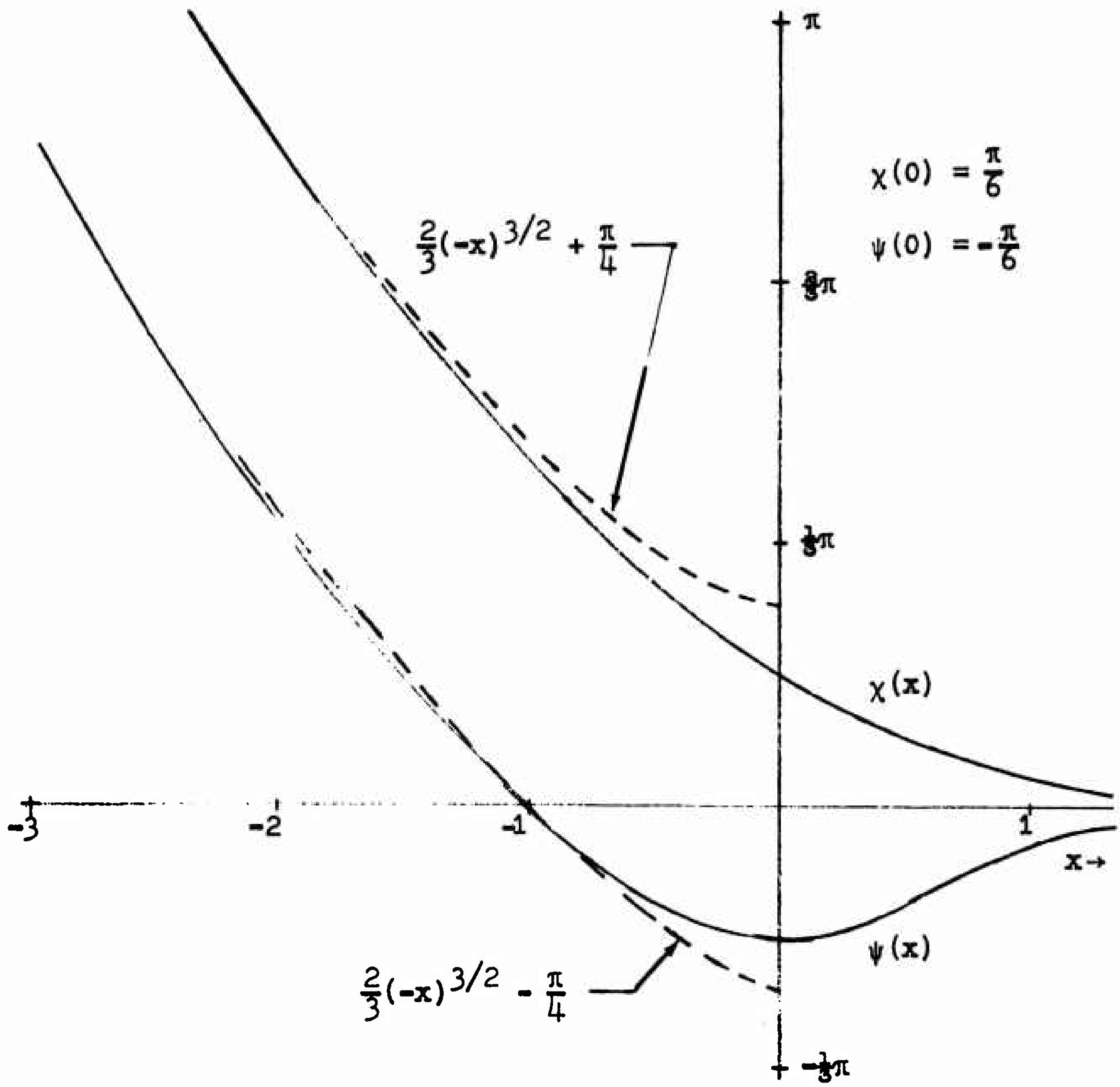

Fig. B-1 The Auxiliary Functions $\chi(x)$ and $\psi(x)$

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portions of the curves which are depicted in the figure.

For large values of x, the auxiliary functions possess the asymptotic expansions

$$|F(-x)|^2 \sim \frac{1}{\pi x^{1/2}}\left(1 - \frac{1\cdot3\cdot5}{1!\,96}\frac{1}{x^3} + \frac{1\cdot3\cdot5\cdot7\cdot9\cdot11}{2!\,96^2}\frac{1}{x^6} - \frac{1\cdot3\cdot5\cdot7\cdot9\cdot11\cdot13\cdot15\cdot17}{3!\,96^3}\frac{1}{x^9} + \ldots\right) \quad \text{(B-26)}$$

$$\chi(-x) - \frac{1}{4}\pi \sim \frac{2}{3}x^{3/2}\left(1 - \frac{5}{32}\frac{1}{x^3} + \frac{1105}{6144}\frac{1}{x^6} - \frac{82825}{65536}\frac{1}{x^9} + \frac{1282031525}{58720256}\frac{1}{x^{12}} - \ldots\right) \quad \text{(B-27)}$$

$$|G(-x)|^2 \sim \frac{1}{\pi}x^{1/2}\left(1 + \frac{1\cdot3}{1!\,96}\frac{7}{x^3} - \frac{1\cdot3\cdot5\cdot7\cdot9}{2!\,96^2}\frac{13}{x^6} + \frac{1\cdot3\cdot5\cdot7\cdot9\cdot11\cdot13\cdot15}{3!\,96^3}\frac{19}{x^9} - \ldots\right) \quad \text{(B-28)}$$

$$\psi(-x) + \frac{1}{4}\pi \sim \frac{2}{3}x^{3/2}\left(1 + \frac{7}{32}\frac{1}{x^3} - \frac{1463}{6144}\frac{1}{x^6} + \frac{4\,95271}{3\,27680}\frac{1}{x^9} - \frac{206530429}{8388608}\frac{1}{x^{12}} + \ldots\right) \quad \text{(B-29)}$$

It would be easier to interpolate in the tables if the quantities which were tabulated were the slowly varying functions $f(x)$, $\theta(x)$, $g(x)$ and $\varphi(x)$ which are defined as follows:

$$f^2(x) = \pi\sqrt{\pi}\ F^2(-x) \quad \text{(B-30)}$$

$$\theta(x) = \frac{3}{2}x^{\frac{2}{3}}\ [\chi(-x) - \pi/4] \quad \text{(B-31)}$$

$$g^2(x) = \frac{\pi}{\sqrt{x}}\ G^2(-x) \quad \text{(B-32)}$$

$$\varphi(x) = \frac{3}{2}x^{\frac{2}{3}}\ [\psi(-x) + \pi/4] \quad \text{(B-33)}$$

There was not adequate time under the present contract period to

prepare tables of these function. The author plans to return to this problem and to compute the functions for $(1/x) = 0.000(0.001)(0.100)$. The functions $f(x)$, $\theta(x)$, $g(x)$ and $\varphi(x)$ should also be developed into series of the Chenbyshev polynomials $T_n^*(\eta)$ where $\eta = (A/x)^3$ and $A < x < \infty$.

The asymptotic expansions for $f(x)$ and $g(x)$, namely

$$f^2(x) = 1 - \frac{1\times3\times5}{96\,x^3} + \frac{1\times3\times5\times7\times9\times11}{2!(96)^2x^6} - \frac{1\times3\times5\times7\times9\times11\times13\times15\times17}{3!(96)^3\,x^9} + \ldots \qquad \text{(B-34)}$$

$$g^2(x) = 1 + \frac{1\times3\times7}{96\,x^3} - \frac{1\times3\times5\times7\times9\times13}{2!(96)^2x^6} + \frac{1\times3\times5\times7\times9\times11\times13\times15\times19}{3!(96)^3\,x^9} + \ldots \qquad \text{(B-35)}$$

can be readily extended to higher order terms. However, it is not so obvious as to how to develop the further terms in the expansions for $\theta(x)$ and $\varphi(x)$, namely

$$\theta(x) = 1 - \frac{5}{32\,x^3} + \frac{1105}{6144\,x^6} - \frac{82825}{65536\,x^9} + \frac{1282031525}{58720256\,x^{12}} + \ldots \qquad \text{(B-37)}$$

$$\varphi(x) = 1 + \frac{7}{32\,x^3} - \frac{1463}{6144\,x^6} + \frac{495271}{327680x^9} - \frac{206530429}{8388608\,x^{12}} + \ldots \qquad \text{(B-38)}$$

However, the properties

$$\pi F^2(-x)\,\frac{d\chi(-x)}{dx} = 1 \quad , \quad \pi G^2(-x)\,\frac{d\psi(-x)}{dx} = x \qquad \text{(B-39)}$$

permit us to assume a series representation for $\theta(x)$ and $\varphi(x)$

$$\theta(x) = 1 + \sum_{n=1}^{\infty} A_n x^{-3n} \quad , \quad \varphi(x) = 1 + \sum_{n=1}^{\infty} B_n x^{-3n} \qquad \text{(B-40)}$$

and to determine the relationship between the known coefficients in the expansions of $f^2(x)$ and $g^2(x)$ and the unknown coefficients in the expansions of $\theta(x)$ and $\varphi(x)$.

Block B-1

THE COEFFICIENTS FOR ASYMPTOTIC EXPANSION OF $\theta(x)$ AND $\varphi(x)$

```
 DOUBLE PRECISION CHI(18), PSI(18)
 DATA (CHI(I),PSI(I),I=1,18)/
A -0.15625000000000000000D+00 ,+0.21875000000000000000D+00 ,
B +0.17985026041666666667D+00 ,-0.23811848958333333333D+00 ,
C -0.12638092041015625000D+01 ,+0.15114471435546875000D+01 ,
D +0.21832866753850664411 3D+02 ,-0.24620345711708068847 7D+ 2 ,
E -0.69682822536884082688 2D+03 ,+0.76086294204038050439 6D+ 3 ,
F +0.35553981581391285131 3D+05 ,-0.38092128868188640229 7D+ 5 ,
G -0.26532584171176254655 6D+07 ,+0.28082562244080195275 9D+ 7 ,
H +0.27258321302415599342 3D+09 ,-0.28607946453262385636 6D+ 9 ,
I -0.36894027011098001901 6D+11 ,+0.38480790905747179385 8D+11 ,
J +0.63630204766133750347 9D+13 ,-0.66050619870875374085 1D+13 ,
K -0.13622439180500760352 1D+16 ,+0.14087107203786266895 5D+16 ,
L +0.35446828581078074793 6D+18 ,-0.36542984737250173263 8D+18 ,
M -0.11017990287911060070 0D+21 ,+0.11329639450011994200 7D+21 ,
N +0.40320823577747335533 8D+23 ,-0.41371744404875883016 5D+23 ,
O -0.17159646692716793386 0D+26 ,+0.17574349642737663059 0D+26 ,
P +0.84031035110193005527 5D+28 ,-0.85924160506996514557 4D+28 ,
Q -0.46916652616593043608 9D+31 ,+0.47906551025465751515 7D+31 ,
R +0.29625489083114947131 5D+34 ,-0.30213269421511401499 7D+34 /
```

In Block B-1 we present numerical values for the coefficients. The relationship between A_n and B_n and the CHI(I) and PSI(I) is

$$A_I = \text{CHI(I)} \quad , \quad B_I = \text{PSI(I)}$$

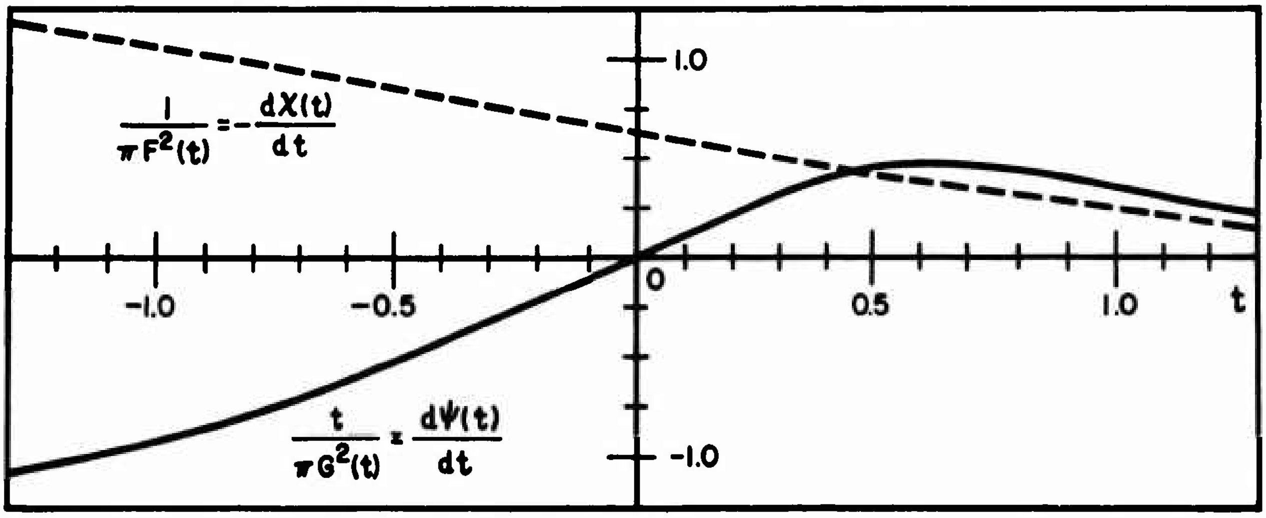

Fig. B-2 The Behavior of $\chi'(t)$ and $\psi'(t)$

The behavior of the derivatives of $\chi(t)$ and $\psi(t)$ is depicted in Fig. B -2.

In Program B-2 we present a FORTRAN subroutine which evaluates the Airy functions and their derivatives by using the classical expansions

$$\sqrt{\pi}Ai(x) = \frac{1}{2} x^{-1/4} e^{-\xi} \left(1 - \frac{3\cdot 5}{1!\ 216} \frac{1}{\xi} + \frac{5\cdot 7\cdot 9\cdot 11}{2!\ (216)^2} \frac{1}{\xi^2} - \frac{7\cdot 9\cdot 11\cdot 13\cdot 15\cdot 17}{3!\ (216)^3} \frac{1}{\xi^3} + \cdots\right) \quad \text{(B-41)}$$

$$\sqrt{\pi}Bi(x) = x^{-1/4} e^{\xi} \left(1 + \frac{3\cdot 5}{1!\ 216} \frac{1}{\xi} + \frac{5\cdot 7\cdot 9\cdot 11}{2!\ (216)^2} \frac{1}{\xi^2} + \frac{7\cdot 9\cdot 11\cdot 13\cdot 15\cdot 17}{3!\ (216)^3} \frac{1}{\xi^3} + \cdots\right) \quad \text{(B-42)}$$

where

$$\xi = \frac{2}{3} x^{3/2}$$

The reader will note that Program B-2 will set the Airy functions equal to the following

$$Ai(x) \approx 0 \quad , \quad Ai'(x) \approx 0$$

$$Bi(x) \approx 10^{+38} \qquad Bi'(x) \approx 10^{+38}$$

when ξ becomes large enough that the FORTRAN subroutine cannot compute $\exp(\pm\xi)$. In Table B-17 we present a table of values of $\exp(\pm\xi)$. The reader will observe that the overflow and underflow occur for x between 25.5 and 26.0.

In Table B-18 we give tables of values of the sine and cosine of $(\xi + \pi/4)$ which will be helpful in considerations associated with the employment of the asymptotic expansions

$$\sqrt{\pi}\ Ai(-x) = x^{-1/4}[P(x)\sin(\xi + \pi/4) - Q(x)\cos(\xi + \pi/4)] \quad \text{(B-43)}$$

$$\sqrt{\pi}\ Bi(-x) = x^{-1/4}[P(x)\cos(\xi + \pi/4) + Q(x)\sin(\xi + \pi/4)] \quad \text{(B-44)}$$

where, if we let

$$\xi = \frac{2}{3} x^{3/2}$$

the functions $P(\xi)$ and $Q(\xi)$ are the asymptotic expansions

$$P(x) = 1 - \frac{5\times7\times9\times11}{2!(216)^2\xi^2} + \frac{9\times11\times13\times15\times17\times19\times21\times23}{4!(216)^4\xi^4} + \cdots \qquad \text{(B-43)}$$

$$Q(x) = \frac{3\times5}{1!(216)^1\xi^1} - \frac{7\times9\times11\times13\times15\times17}{3!(216)^3\xi^3} + \cdots \qquad \text{(B-44)}$$

The reader will observe that Eqs. (B-41) and (B-42) and Eqs. (B-43) and (B-44) involve the expressions

$$A_1(x) = \frac{3\times5}{1!(216)^1\xi^1}$$

$$A_2(x) = \frac{5\times7\times9\times11}{2!(216)^2\xi^2}$$

$$A_3(x) = \frac{7\times9\times11\times13\times15\times17}{3!(216)^3\xi^3}$$

In Table B-19 we present the values of $A_n(x)$ for n = 1(1)30 for x = 4.0(1.0)8.0. These tables have been retyped from computer output so as to present the numbers in "fixed point" format, i.e., as decimal digits which are not multiplied at powers of 10. This display of the behavior of the $A_n(x)$ provides a graphic picture of the asymptotic nature of series in the classical asymptotic expansions for the Airy functions. In Table B-20 we present the values of $A_n(x)$ for n = 1(1)14 for x = 1.0(1.0)20.0. These results have been reproduced from computer output and appear as "floating point" numbers. It is hoped that the availability of these tables which present the values of the successive terms in the series will attract the interest of people who have been interested in summation techniques for asymptotic series.

AIRY FUNCTIONS ON POSITIVE AXIS

```
      BLOCK DATA
C     DOUBLE PRECISION EAIRYF(100),EBIRYF(100),EAIRYD(100),EBIRYD(100)
      COMMON/EQUADP/   EAIRYF(100),EBIRYF(100),EAIRYD(1C0),EBIRYD(100)
      DOUBLE PRECISION EAIRYF,EBIRYF,EAIRYD,EBIRYD
      DATA (EAIRYF(I),I= 1,20 )/
     A +0.32920312994353810016 7D+00, +0.303703154286381994888D+00,       BIRYF 51
     B +0.278806481955004921937D+00, +0.254742354295676346079D+00,       BIRYF 52
     C +0.231693606480833489766D+00, +0.209800061666379468346D+00,       BIRYF 53
     D +0.189162400398150073303D+00, +0.169846317444364859218D+00,       BIRYF 54
     E +0.151886803640544361094D+00, +0.135292416312881415525D+00,       BIRYF 55
     F +0.120049427355397667971D+00, +0.106125762263312542736D+00,       BIRYF 56
     G +0.934746657715027098618D-01, +0.820380498076106776300D-01,       BIRYF 57
     H +0.717494970081054096755D-01, +0.625369079689319603243D-01,       BIRYF 58
     I +0.543247927329194677511D-01, +0.470362168668458052271D-01,       BIRYF 59
     J +0.405944200315294966642D-01, +0.349241304232743791285D-01/       BIRYF 60
      DATA (EAIRYF(I),I=21,40 )/
     A +0.299526021158665266071D-01, +0.256104044217732195414D-01,       BIRYF 61
     B +0.218319931806226323690D-01, +0.185560936229754673564D-01,       BIRYF 62
     C +0.157259233804704900029D-01, +0.132892825296714841680D-01,       BIRYF 63
     D +0.111985354510658809471D-01, +0.941050691492396296066D-02,       BIRYF 64
     E +0.788631230412123083992D-02, +0.659113935746071911486D-02,       BIRYF 65
     F +0.549399634918683731736D-02, +0.456743927403982090691D-02,       BIRYF 66
     G +0.378728842682675326918D-02, +0.313234277990420779553D-02,       BIRYF 67
     H +0.258409878698963494754D-02, +0.212647868263817083143D-02,       BIRYF 68
     I +0.174557200060997913522D-02, +0.142939289210779079508D-02,       BIRYF 69
     J +0.116765487299144948805D-02, +0.951563851204801873486D-03/       BIRYF 70
      DATA (EAIRYF(I),I=41,60 )/
     A +0.773629663781597186934D-03, +0.627495868309163374767D-03,       BIRYF 71
     B +0.507787168156149292581D-03, +0.409973586386962155966D-03,       BIRYF 72
     C +0.330250323514308983621D-03, +0.265432123924450242886D-03,       BIRYF 73
     D +0.212860921358597521838D-03, +0.170325523286434948469D-03,       BIRYF 74
     E +0.135992117015067536940D-03, +0.108344428136074417337D-03,       BIRYF 75
     F +0.861324270647884410113D-04, +0.683285559252481008964D-04,       BIRYF 76
     G +0.540905310134005671123D-04, +0.427298616941166203245D-04,       BIRYF 77
     H +0.336853119085998144214D-04, +0.265006132968499709854D-04,       BIRYF 78
     I +0.208058177132606861127D-04, +0.163017505858772529274D-04,       BIRYF 79
     J +0.127470945091844875592D-04, +0.994769436025288956907D-05/       BIRYF 80
```

Block B-2 (Cont'd)

AIRY FUNCTIONS ON POSITIVE AXIS

```
      DATA (EAIRYF(I),I=61,80 )/
     A +0.77477310324484275354 2D-05, +0.60224607196881982164 6D-05,      BIRYF 81
     B +0.46722608205742871624 4D-05, +0.36177623188518029924 2D-05,      BIRYF 82
     C +0.27958823432049135851 4D-05, +0.21565999525969199854 6D-05,      BIRYF 83
     D +0.16603434781875353773 5D-05, +0.12758794168766681483 8D-05,      BIRYF 84
     E +0.97861133392660376300 5D-06, +0.74921288639971670798 8D-06,      BIRYF 85
     F +0.57253228858776572578 4D-06, +0.43671663591422622860 5D-06,      BIRYF 86
     G +0.33251378244377575998 2D-06, +0.25271719392667524446 4D-06,      BIRYF 87
     H +0.19172560675134307514 2D-06, +0.14519461748012537057 8D-06,      BIRYF 88
     I +0.10976168502027784231 7D-06, +0.82829600662268901746 0D-07,      BIRYF 89
     J +0.62396400972839404779 8D-07, +0.46922076160992316251 2D-07/      BIRYF 90
      DATA (EAIRYF(I),I=81,100)/
     A +0.35224356235735678845 7D-07, +0.26397418340282783290 1D-07,      BIRYF 91
     B +0.19748617496676931362 7D-07, +0.14749354994719218683 2D-07,      BIRYF 92
     C +0.10997009755195506508 5D-07, +0.81855061513456239669 2D-08,      BIRYF 93
     D +0.60826082187745444968 8D-08, +0.45124405191537035544 4D-08,      BIRYF 94
     E +0.33420610425187034858 1D-08, +0.24711684308724898430 8D-08,      BIRYF 95
     F +0.18242282535640260751 9D-08, +0.13444621833707132745 2D-08,      BIRYF 96
     G +0.98926866131642674939 5D-09, +0.72674117707792016823 7D-09,      BIRYF 97
     H +0.53302637046174916382 2D-09, +0.39032335304151354798 3D-09,      BIRYF 98
     I +0.28537159314931001137 0D-09, +0.20831028224308731191 2D-09,      BIRYF 99
     J +0.15181958141049115394 3D-09, +0.11047532552898705397 7D-09/      BIRYF100
      DATA (EBIRYF(I),I= 1,20 )/
     A +0.65986169019418923365 9D+00, +0.70546420291866123540 0D+00,      BIRYF  1
     B +0.75248558508731563800 4D+00, +0.80177300001359723983 5D+00,      BIRYF  2
     C +0.85427704310315549329 4D+00, +0.91106334169494048181 3D+00,      BIRYF  3
     D +0.97332865587816593668 2D+00, +0.10424221712315607004 1D+01,      BIRYF  4
     E +0.11198728131344472440 1D+01, +0.12074235949528712594 5D+01,      BIRYF  5
     F +0.13070742475163069549 6D+01, +0.14211336756103480695 0D+01,      BIRYF  6
     G +0.15522841623445437464 5D+01, +0.17036597115386808825 1D+01,      BIRYF  7
     H +0.18789415037478950009 4D+01, +0.20824741712541118898 0D+01,      BIRYF  8
     I +0.23194075069389249473 0D+01, +0.25958693567439062901 2D+01,      BIRYF  9
     J +0.29191768630421089531 6D+01, +0.32980949999782147103 5D+01/      BIRYF 10
      DATA (EBIRYF(I),I=21,40 )/
     A +0.37431535649561751381 6D+01, +0.42670365817666447300 2D+01,      BIRYF 11
     B +0.48850615818356446867 2D+01, +0.56157706541205973649 5D+01,      BIRYF 12
     C +0.64816607384605786082 1D+01, +0.75100876980822723653 2D+01,      BIRYF 13
     D +0.87343876499889138895 8D+01, +0.10195270377637055196 9D+02,      BIRYF 14
```

AIRY FUNCTIONS ON POSITIVE AXIS

```
E +0.119425540677505369932D+02, +0.140373289637302320321D+02,      BIRYF 15
F +0.165546609224694953683D+02, +0.195869757334411851242D+02,      BIRYF 16
G +0.232483032629415794797D+02, +0.276796094236590113663D+02,      BIRYF 17
H +0.330555067546114794157D+02, +0.395927150740250136500D+02,      BIRYF 18
I +0.475607474995894428570D+02, +0.572954309710521601291D+02,      BIRYF 19
J +0.692160432014470796382D+02, +0.838470714084681399249D+02/      BIRYF 20
 DATA (EBIRYF(I),I=41,60 )/
A +0.101845887523051137965D+03, +0.124038009868642106852D+03,      BIRYF 21
B +0.151462108837072244104D+03, +0.185427548398557588035D+03,      BIRYF 22
C +0.227588081835599718470D+03, +0.280036398801291454043D+03,      BIRYF 23
D +0.345425630757233572149D+03, +0.427125767580848003197D+03,      BIRYF 24
E +0.529425358022231954379D+03, +0.657792044171171182495D+03,      BIRYF 25
F +0.819209658679961277475D+03, +0.102261511691363748730D+04,      BIRYF 26
G +0.127946559546844104016D+04, +0.160447607824134502484D+04,      BIRYF 27
H +0.201658003865953139453D+04, +0.254018283758149890535D+04,      BIRYF 28
I +0.320679972423471169607D+04, +0.405719997396124701563D+04,      BIRYF 29
J +0.514421815428079881453D+04, +0.653644610480986345432D+04/      BIRYF 30
 DATA (EBIRYF(I),I=61,80 )/
A +0.832308942401549333788D+04, +0.106203661136178511168D+05,      BIRYF 31
B +0.135799506914193075287D+05, +0.174001355680848214708D+05,      BIRYF 32
C +0.223406077183969981600D+05, +0.287420429010405230442D+05,      BIRYF 33
D +0.370521293733575018648D+05, +0.478601855742919822443D+05,      BIRYF 34
E +0.619432824867711327537D+05, +0.803277907094302470138D+05,      BIRYF 35
F +0.104371639380453998523D+06, +0.135874428498139178489D+06,      BIRYF 36
G +0.177225055164428126350D+06, +0.231599940898915527818D+06,      BIRYF 37
H +0.303229615112533402322D+06, +0.397757777803424096130D+06,      BIRYF 38
I +0.522725663987833761361D+06, +0.688226433707225850508D+06,      BIRYF 39
J +0.907790616061993809150D+06, +0.119958600412445993103D+07/      BIRYF 40
 DATA (EBIRYF(I),I=81,100)/
A +0.158804612792942995453D+07, +0.210608370993170481278D+07,      BIRYF 41
B +0.279810375106714662483D+07, +0.372411124729598102323D+07,      BIRYF 42
C +0.496531954147130198151D+07, +0.663181887920608279615D+07,      BIRYF 43
D +0.887307746960718106309D+07, +0.118923424547171342873D+08,      BIRYF 44
E +0.159664181202323064819D+08, +0.214728688914353490950D+08,      BIRYF 45
F +0.289274889032649731325D+08, +0.390359877364338344898D+08,      BIRYF 46
G +0.527653891539910475720D+08, +0.714428035887130369638D+08,      BIRYF 47
H +0.968922655804510928330D+08, +0.131624577044260557824D+09,      BIRYF 48
I +0.179101064677735531068D+09, +0.244100550504570744478D+09,      BIRYF 49
J +0.333230648256769068418D+09, +0.455641153548225141085D+09/      BIRYF 50
```


+45

Block B-2 (Cont'd)

AIRY FUNCTIONS ON POSITIVE AXIS

```
      DATA (EAIRYD(I),I= 1,20 )/                                          AIRYF 51
     A -0.257130421907586188694D+00, -0.252405470285619536113D+00,        AIRYF 51
     B -0.245146364219054803435D+00, -0.235832034419208217275D+00,        AIRYF 52
     C -0.224910532664683893136D+00, -0.212793259389158520513D+00,        AIRYF 53
     D -0.199851191582280475174D+00, -0.186412863807271709023D+00,        AIRYF 54
     E -0.172763843461634673866D+00, -0.159147441296793212789D+00,        AIRYF 55
     F -0.145766407345016252835D+00, -0.132785378557226174207D+00,        AIRYF 56
     G -0.120333865590183582473D+00, -0.108509590480139163658D+00,        AIRYF 57
     H -0.973820128423013192198D-01, -0.869959084481041318130D-01,        AIRYF 58
     I -0.773748895253250280858D-01, -0.685247801186109345660D-01,        AIRYF 59
     J -0.604367817857565401535D-01, -0.530903844336536317152D-01/        AIRYF 60
      DATA (EAIRYD(I),I=21,40 )/
     A -0.464559940326745994697D-01, -0.404972632444531352690D-01,        AIRYF 61
     B -0.351731227208180631303D-01, -0.304395201289725926882D-01,        AIRYF 62
     C -0.262508810359032303567D-01, -0.225613108861087476304D-01,        AIRYF 63
     D -0.193255606923776375447D-01, -0.164997809949151397315D-01,        AIRYF 64
     E -0.140420893877864247334D-01, -0.119129767059513185141D-01,        AIRYF 65
     F -0.100755760323168795191D-01, -0.849581721856859588562D-02,        AIRYF 66
     G -0.714248778588473799250D-02, -0.598721902511379379381D-02,        AIRYF 67
     H -0.500441396795258264592D-02, -0.417113174441938102365D-02,        AIRYF 68
     I -0.346694074902762820044D-02, -0.287375106090479106000D-02,        AIRYF 69
     J -0.237563473756225221852D-02, -0.195864095020417889956D-02/        AIRYF 70
      DATA (EAIRYD(I),I=41,60 )/
     A -0.161061146122698677855D-02, -0.132100066388768655519D-02,        AIRYF 71
     B -0.108070330522464033083D-02, -0.881892086491768072369D-03,        AIRYF 72
     C -0.717866567557508888609D-03, -0.582914177810333171072D-03,        AIRYF 73
     D -0.472183639986264239997D-03, -0.381570728688738440491D-03,        AIRYF 74
     E -0.307615996337649743297D-03, -0.247413890868462475971D-03,        AIRYF 75
     F -0.198532547881805241308D-03, -0.158943452645947516444D-03,        AIRYF 76
     G -0.126960123336596717283D-03, -0.101184956556993612269D-03,        AIRYF 77
     H -0.804633913055651433702D-04, -0.638445812461772346787D-04,        AIRYF 78
     I -0.505478116845372109444D-04, -0.399339524370989583921D-04,        AIRYF 79
     J -0.314812971171127642363D-04, -0.247652003970349547513D-04/        AIRYF 80
      DATA (EAIRYD(I),I=61,80 )/
     A -0.194409853751029541248D-04, -0.152296516969415666723D-04,        AIRYF 81
     B -0.119059704599572708114D-04, -0.928860344486298290344D-05,        AIRYF 82
     C -0.723193146660179255897D-05, -0.561931944434578581449D-05,        AIRYF 83
     D -0.435758416329777174177D-05, -0.337246477537639185199D-05,        AIRYF 84
```

AIRY FUNCTIONS ON POSITIVE AXIS

```
E -0.26049260870862621552 7D-05, -0.200815089473879199093D-05,       AIRYF 85
F -0.154510036678976894710D-05, -0.118654107171763965162D-05,        AIRYF 86
G -0.909454038883345944514D-06, -0.695755540208059307280D-06,        AIRYF 87
H -0.531271395972054468417D-06, -0.404916820450777976994D-06,        AIRYF 88
I -0.308042390010359286234D-06, -0.233913869900323839021D-06,        AIRYF 89
J -0.177299583294303527417D-06, -0.134143929790678657413D-06/        AIRYF 90
 DATA (EAIRYD(I),I=81,100)/
A -0.101309720326608340505D-06, -0.763753298418617912636D-07,        AIRYF 91
B -0.574753973633800902711D-07, -0.431760275048587314789D-07,        AIRYF 92
C -0.323772544044760225520D-07, -0.242369921224153161011D-07,        AIRYF 93
D -0.181118760461761228033D-07, -0.135113493599557286259D-07,        AIRYF 94
E -0.100621099218369226948D-07, -0.748064138965894639571D-08,        AIRYF 95
F -0.555203734438591352518D-08, -0.411371244280792002093D-08,        AIRYF 96
G -0.304289987561865077457D-08, -0.224707555705066904925D-08,        AIRYF 97
H -0.165663945937406655257D-08, -0.121933377816811225959D-08,        AIRYF 98
I -0.895994584899315423995D-09, -0.657325041947166966809D-09,        AIRYF 99
J -0.481449519646824493519D-09, -0.352063367673892395487D-09/        AIRYF100
 DATA (EBIRYD(I),I= 1,20 )/
A +0.451512631149646501273D+00, +0.461789284362150930392D+00,        AIRYF  1
B +0.480049028752448022161D+00, +0.507281676050622443423D+00,        AIRYF  2
C +0.544572564140592301826D+00, +0.593144478634285707492D+00,        AIRYF  3
D +0.654405919172140030291D+00, +0.730006901615251767032D+00,        AIRYF  4
E +0.821903890307209026080D+00, +0.932435933392775632969D+00,        AIRYF  5
F +0.106441464824199110938D+01, +0.122123139870489501378D+01,        AIRYF  6
G +0.140698585385631747316D+01, +0.162664116078045225903D+01,        AIRYF  7
H +0.188621225484816548873D+01, +0.219299543725376537142D+01,        AIRYF  8
I +0.255584935690043801617D+01, +0.298554005084659907290D+01,        AIRYF  9
J +0.349516586628916143290 3D+01, +0.410068204993288988948D+01/      AIRYF 10
 DATA (EBIRYD(I),I=21,40 )/
A +0.482154992571749535555D+01, +0.568154176958525216724D+01,        AIRYF 11
B +0.670974081272382604949D+01, +0.794178587973531898047D+01,        AIRYF 12
C +0.942142331733430175577D+01, +0.112024454678439209301D+02,        AIRYF 13
D +0.133511161523309309182D+02, +0.159492107200737569778D+02,        AIRYF 14
E +0.190978328828365926439D+02, +0.229222149663821701858D+02,        AIRYF 15
F +0.275777652077707314471D+02, +0.332576989805450989635D+02,        AIRYF 16
G +0.402026851208845424802D+02, +0.487130600819514070777D+02,        AIRYF 17
H +0.591643195813609870371D+02, +0.720268003348870573039D+02,        AIRYF 18
I +0.878907272628334108906D+02, +0.107498142389073103603D+03,        AIRYF 19
J +0.131783674369121777630D+03, +0.161926683504613401847D+03/        AIRYF 20
```

Block B-2 (Cont'd)

AIRY FUNCTIONS ON POSITIVE AXIS

```
 DATA (EBIRYD(I),I=41,60 )/                                           
A +0.199418067433209753432D+03, +0.246145991711785592620D+03,        AIRYF 21
B +0.304506088857535057244D+03, +0.377543343700777899685D+03,        AIRYF 22
C +0.469135077327966397969D+03, +0.584227322325565711454D+03,        AIRYF 23
D +0.729140668553345279570D+03, +0.911966643689746054289D+03,        AIRYF 24
E +0.114308226536499485663D+04, +0.143581908021798251879D+04,        AIRYF 25
F +0.180733448135136778445D+04, +0.227974829358333524240D+04,        AIRYF 26
G +0.288162777214314625196D+04, +0.364993093183071595662D+04,        AIRYF 27
H +0.463255373313904242067D+04, +0.589167408620813233525D+04,        AIRYF 28
I +0.750814891132999784872D+04, +0.958730503037030788426D+04,        AIRYF 29
J +0.122665777617769432806D+05, +0.157256026219304768407D+05/        AIRYF 30
 DATA (EBIRYD(I),I=61,80 )/
A +0.201995688493040284085D+05, +0.259969166536126015233D+05,        AIRYF 31
B +0.335228276885502190973D+05, +0.433104535885386788018D+05,        AIRYF 32
C +0.560624958425228607535D+05, +0.727066535687063982786D+05,        AIRYF 33
D +0.944696781543122509382D+05, +0.122976430308445474993D+06,        AIRYF 34
E +0.160382583230962694397D+06, +0.209552670873971319527D+06,        AIRYF 35
F +0.274299338264356465031D+06, +0.359705317489760324411D+06,        AIRYF 36
G +0.472557386398703350731D+06, +0.621932135876414137470D+06,        AIRYF 37
H +0.819987835358799621008D+06, +0.108303650793106090343D+07,        AIRYF 38
I +0.143299752823736503372D+07, +0.189937143786448514206D+07,        AIRYF 39
J +0.252192411395678168341D+07, +0.335434231274453887692D+07/        AIRYF 40
 DATA (EBIRYD(I),I=81,100)/
A +0.446921942430834953142D+07, +0.596486543242387316053D+07,        AIRYF 41
B +0.797462208449489026205D+07, +0.106796233866350354521D+08,        AIRYF 42
C +0.143263010306620583354D+08, +0.192504337957505324078D+08,        AIRYF 43
D +0.259102347607551755717D+08, +0.349319380279469601489D+08,        AIRYF 44
E +0.471726967264458809626D+08, +0.638074897809082138594D+08,        AIRYF 45
F +0.864493723334492879784D+08, +0.117316098337319166702D+09,        AIRYF 46
G +0.159461323560196940922D+09, +0.217095870563978843178D+09,        AIRYF 47
H +0.296034763868005038668D+09, +0.404320141561829412565D+09,        AIRYF 48
I +0.553090447234023111777D+09, +0.757794958003258271823D+09,        AIRYF 49
J +0.103989320206398250522D+10, +0.142923613448286577638D+10/        AIRYF 50
 END
```

```
      SUBROUTINE AIBIZ(Z,AI,BI,AIP,BIP)
      COMPLEX          Z,AI,BI,AIP,BIP
      DOUBLE PRECISION AZERO,A(22),BZERO,B(22),CZERO,C(22),DZERO,D(22)
      DOUBLE PRECISION DRZ,DIZ,D2RZ,D2IZ,D3RZ,D3IZ,FR,FI,FPR,FPI,GR,GI,
     A       GPR,GPI,AREAL(23),AIMAG(23),ZERO,TWO,CA,CB,
     B       DAIR,DAII,DAIPR,DAIPI,DBIR,DBII,DBIPR,DBIPI
      DATA AZERO,BZERO,(A(M),B(M),M=1,11) /
     A          0.93043671693    D  0 ,        0.67829872514    D  0 ,
     B         31.01455723097    D  0 ,       11.30497875240    D  0 ,
     C        206.76371487316    D  0 ,       53.83323215431    D  0 ,
     D        574.34365242545    D  0 ,      119.62940478735    D  0 ,
     E        870.21765519008    D  0 ,      153.37103177865    D  0 ,
     F        828.77871922864    D  0 ,      127.80919314888    D  0 ,
     G        541.68543740434    D  0 ,       74.74221821572    D  0 ,
     H        257.94544638302    D  0 ,       32.35593862152    D  0 ,
     I         93.45849506631    D  0 ,       10.78531287384    D  0 ,
     J         26.62635187074    D  0 ,        2.85325737403    D  0 ,
     K          6.121000430056   D  0 ,        0.613603736351   D  0 ,
     L          1.159280384480   D  0 ,        0.109376780098   D  0 /
      DATA (A(M),B(M),M=12,22)                          /
     A          0.184012759441   D  0 ,        0.16422939955    D- 1 ,
     B          0.24833030964    D- 1 ,        0.2105505122     D- 2 ,
     C          0.2884208010     D- 2 ,        0.233167788      D- 3 ,
     D          0.291334142      D- 3 ,        0.22528289       D- 4 ,
     E          0.25827495       D- 4 ,        0.1915671        D- 5 ,
     F          0.2025686        D- 5 ,        0.144470         D- 6 ,
     G          0.141557         D- 6 ,        0.9729           D- 8 ,
     H          0.8870           D- 8 ,        0.589            D- 9 ,
     I          0.501            D- 9 ,        0.32             D-10 ,
     J          0.26             D-10 ,        0.2              D-11 ,
     K          0.1              D-11 ,        0.0              D  0 /
      DATA CZERO,DZERO,(C(M),D(M),M=1,11) /
     A          0.46521835846    D  0 ,        0.67829872514    D  0 ,
     B          6.20291144619    D  0 ,       45.21991500962    D  0 ,
     C         25.84546435915    D  0 ,      376.83262508015    D  0 ,
     D         52.21305931140    D  0 ,     1196.29404787350    D  0 ,
     E         62.15840394215    D  0 ,     1993.82341312250    D  0 ,
     F         48.75168936639    D  0 ,     2044.94709038206    D  0 ,
     G         27.08427187022    D  0 ,     1420.10214609865    D  0 ,
```

Program B-1 (Cont'd)

SUBROUTINE AIBIZ(Z,AI,BI,AIP,BIP)

```
     H           11.21501940796   D  0 ,      711.83064967351   D  0 ,
     I            3.59455750255   D  0 ,      269.63282184603   D  0 ,
     J            0.91815006451   D  0 ,       79.89120647290   D  0 ,
     K            0.191281263439  D  0 ,       19.021715826880  D  0 ,
     L            0.33122296699   D- 1 ,        3.718810523339  D  0 /
      DATA (C(M),D(M),M=12,22)
     A            0.4842441038    D- 2 ,        0.607648778323  D  0 ,
     B            0.605683682     D- 3 ,        0.84220204896   D- 1 ,
     C            0.65550182      D- 4 ,        0.10026214869   D- 1 ,
     D            0.6198599       D- 5 ,        0.1036301278    D- 2 ,
     E            0.516550        D- 6 ,        0.93867869      D- 4 ,
     F            0.38220         D- 7 ,        0.7512435       D- 5 ,
     G            0.2528          D- 8 ,        0.535074        D- 6 ,
     H            0.150           D- 9 ,        0.34135         D- 7 ,
     I            0.8             D-11 ,        0.1962          D- 8 ,
     J            0.0             D  0 ,        0.102           D- 9 ,
     K            0.0             D  0 ,        0.5             D-11 /
      DATA ZERO,TWO,CA,CB/0.0D0,2.0D0,0.66090107608300,0.381571414184D0/
  110 FORMAT(25H AIBIZ CANNOT HANDLE Z = ,(E15.8,E15.8))
      IF(CABS(Z).GT.(6.1)) GO TO 99
      M=3+INT(20.*CABS(Z)/6.1)
      DRZ=DBLE(REAL(-Z))
      DIZ=DBLE(REAL((0.,1.)*Z))
      D2RZ=DRZ*DRZ-DIZ*DIZ
      D2IZ=TWO*DRZ*DIZ
      D3RZ=DRZ*D2RZ-DIZ*D2IZ
      D3IZ=DRZ*D2IZ+DIZ*D2RZ
      D3RZ = D3RZ/(-200.0D0)
      D3IZ = D3IZ/(-200.0D0)
      DO 39  I=1,23
      AREAL(I)= ZERO
      AIMAG(I)= ZERO
   39 CONTINUE
      IF(M.GT.22)  M=22
      DO 1 N=M,1,-1
      AREAL(N)=D3RZ*AREAL(N+1) - D3IZ*AIMAG(N+1) + A(N)
      AIMAG(N)=D3RZ*AIMAG(N+1) + D3IZ*AREAL(N+1)
    1 CONTINUE
```

SUBROUTINE AIBIZ(Z,AI,BI,AIP,BIP)

```
      FR = D3RZ*AREAL(1)-D3IZ*AIMAG(1)+AZERO
      FI = D3RZ*AIMAG(1)+D3IZ*AREAL(1)
      DO 2 N=M,1,-1
      AREAL(N)=D3RZ*AREAL(N+1) - D3IZ*AIMAG(N+1) + B(N)
      AIMAG(N)=D3RZ*AIMAG(N+1) + D3IZ*AREAL(N+1)
    2 CONTINUE
      GPR= D3RZ*AREAL(1)-D3IZ*AIMAG(1)+BZERO
      GPI= D3RZ*AIMAG(1)+D3IZ*AREAL(1)
      GR = DRZ*GPR-DIZ*GPI
      GI = DRZ*GPI+DIZ*GPR
      DO 3 N=M,1,-1
      AREAL(N)=D3RZ*AREAL(N+1)-D3IZ*AIMAG(N+1)+C(N)
      AIMAG(N)=D3RZ*AIMAG(N+1)+D3IZ*AREAL(N+1)
    3 CONTINUE
      GPR = D3RZ*AREAL(1)-D3IZ*AIMAG(1)+CZERO
      GPI = D3RZ*AIMAG(1)+D3IZ*AREAL(1)
      FPR =-D2RZ*GPR + D2IZ*GPI
      FPI =-D2RZ*GPI - D2IZ*GPR
      DO 4 N=M,1,-1
      AREAL(N)=D3RZ*AREAL(N+1)-D3IZ*AIMAG(N+1)+D(N)
      AIMAG(N)=D3RZ*AIMAG(N+1)+D3IZ*AREAL(N+1)
    4 CONTINUE
      GPR = D3RZ*AREAL(1)-D3IZ*AIMAG(1)+DZERO
      GPI = D3RZ*AIMAG(1)+D3IZ*AREAL(1)
      WRITE(6,1000) FR,FI,GR,GI,FPR,FPI,GPR,GPI
C-----NOW CONSTRUCT MILLER AIRY FUNCTIONS
      DAIR   = CB*(FR + GR)
      DAII   = CB*(FI + GI)
      DAIPR  =-CB*(FPR + GPR)
      DAIPI  =-CB*(FPI + GPI)
      DBIR   = CA*(FR - GR)
      DBII   = CA*(FI - GI)
      DBIPR  =-CA*(FPR - GPR)
      DBIPI  =-CA*(FPI - GPI)
      AI     = CMPLX(SNGL(DAIR),SNGL(DAII))
      AIP    = CMPLX(SNGL(DAIPR),SNGL(DAIPI))
      BI     = CMPLX(SNGL(DBIR),SNGL(DBII))
      BIP    = CMPLX(SNGL(DBIPR),SNGL(DBIPI))
      RETURN
   99 WRITE(6,110)  Z
      CALL EXIT
      END
```

```
      SUBROUTINE ASYPOS(X,AL,AM,AI,AIP,BL,BM,BI,BIP,ERROR,FLOVER)
C-----THIS SUBROUTINE EVALUATES THE CLASSICAL ASYMPTOTIC EXPANSION OF
C-----THE AIRY FUNCTION. THE LOGICAL VARIABLE AUNDER IS TRUE WHEN THE
C-----ARGUMENT X IS SO LARGE THAT AI AND ADP ARE LESS THAN 1.0E-38.
C-----IF THE TERMS IN THE ASYMPTOTIC EXPANSIONS DO NOT DECEASE TO
C-----1.0D-17, THE VALUE OF THE LAST TERM EMPLOYED WILL BE FOUND STORED
C-----IN ERROR.
      LOGICAL FLOVER
      DOUBLE PRECISION  X,AL,AM,AI,AIP,BL,BM,BI,BIP,ERROR
      DOUBLE PRECISION DFLOAT,DSQRT,DEXP,Z,XI,D72XI,RL,RM,SL,SM,TL,TM,W
      DOUBLE PRECISION ZERO,ONE,D1P5,D5,D7,D203,D10M17,D88,PI,D72
      DATA             ZERO,ONE,D1P5,D5,D7,D203,D10M17,D88,PI,D72/0.0D0,
     1 1.0D0,1.5D0,5.0D0,7.0D0,0.66666666666666667D0,1.0D-18,88.0D0,
     2 3.14159265358979320D0,72.0D0/
  110 FORMAT(6H      I =,I2,8H     TL =  ,D22.16,8H     TM =  ,D22.16)
  111 FORMAT(6H      I =,I2,8H     SL =  ,D22.16,8H     SM =  ,D22.16)
  120 FORMAT(11H0   FOR X = ,D22.16,59H SUMMING 40 TERMS DID NOT LEAD TO
     ATERMS AS SMALL AS 1.0D-17//8H     TL = ,D22.16,8H     TM = ,D22.16/
     B                           8H     AL = ,D22.16,8H     AM = ,D22.16)
  121 FORMAT(11H0   FOR X = ,D22.16,59H SUMMING 40 TERMS DID NOT LEAD TO
     ATERMS AS SMALL AS 1.0D-17//8H     SL = ,D22.16,8H     SM = ,D22.16/
     B                           8H     BL = ,D22.16,8H     BM = ,D22.16)
  199 FORMAT(11H0   FOR X = ,D22.16,33H TERMS STARTED TO GROW AT INDEX =,
     A  I3,41H AND PROGRAM LEFT THE DO-LOOP WITH VALUES//
     B                           8H     RL = ,D22.16,8H     RM = ,D22.16/
     C                           8H     TL = ,D22.16,8H     TM = ,D22.16)
  996 FORMAT(69H1   PROGRAM EXIT BECAUSE SUBROUTINE ASYPOS(X,...) WILL NO
     AT ACCEPT X = ,D22.16)
      DFLOAT(N) = DBLE(FLOAT(N))
      IF(X.LT.D10M17) GO TO 999
C-----X MUST BE POSITIVE
      FLOVER = .FALSE.
      Z      = DSQRT(X)
      XI     = D203*(Z**3)
      D72XI  =-D72*XI
      AL     = ONE
      AM     = ONE
```

```
      BL       = ONE
      BM       = ONE
      TL       = D5/D72XI
      TM       =-D7/D72XI
      SL       =-TL
      SM       =-TM
      AL       = AL + TL
      AM       = AM + TM
      BL       = BL + SL
      BM       = BM + SM
      ERROR    = DABS(TL)
      DO 10 K=1,40
      INDEX    = K
      RL       = DFLOAT((6*K+1)*(6*K+5))/(D72XI*DFLOAT(K+1))
      RM       = DFLOAT((6*K-1)*(6*K+7))/(D72XI*DFLOAT(K+1))
      IF(DABS(RL).GT.ONE) GO TO 99
C-----TRANSFER OUT WHEN TERMS START TO INCREASE
      TL       = RL*TL
      TM       = RM*TM
      SL       = -RL*SL
      SM       = -RM*SM
      AL       = AL + TL
      AM       = AM + TM
      BL       = BL + SL
      BM       = BM + SM
      IF(DABS(TL).LT.D10M17) GO TO 89
C-----TRANSFER OUT WHEN TERMS ARE SMALLER THAN 1.0D-17
      ERROR    = DABS(TL)
      WRITE(6,110) INDEX,TL,TM
      WRITE(6,111) INDEX,SL,SM
   10 CONTINUE
      WRITE(6,120) X,TL,TM,AL,AM
      WRITE(6,121) X,SL,SM,BL,BM
C-----OUTPUT A MESSAGE THAT COMPLETION OF SUM OF 40 TERMS DID NOT
C-----INCLUDE SUFFICIENT TERMS TO HAVE DABS(TL) GO BELOW 1.0E-17.
   99 WRITE(6,199) X,INDEX,RL,RM,TL,TM
```

```
   89 IF(XI.GT.86.0D0) GO TO 20
      XI      = DEXP(XI)
      W       = DSQRT(Z*PI)
      Z       = 2.0D0*W*XI
      CALL OVERFL(JOVER)
      IF(JOVER.EQ.1) GO TO 20
      W       = XI/W
      AI      = AL/Z
      AIP     =-(AM/Z)*DSQRT(X)
      BI      = BL*W
      BIP     = BM*W*DSQRT(X)
      RETURN
   20 AI      = ZERO
      AIP     = ZERO
      BI      = 1.0D+38
      BIP     = 1.0D+38
C-----WHEN AIRY FUNCTIONS UNDERFLOW THEY ARE SET EQUAL TO ZERO
      FLOVER = .TRUE.
      RETURN
  999 WRITE(6,998) X
      CALL EXIT
      END
```

Table B-1

THE AIRY FUNCTIONS (10 < x < 25)

x	Ai(x)	Bi(x)
10.0	.110475325528986859354E -9	.455641153548225140835E 9
10.1	.802650470290959510384E-10	.624020807875100558670E 9
10.2	.582256739314597136640E-10	.855991952142939724341E 9
10.3	.421728192262601761880E-10	.117606462781201857215E 10
10.4	.304988818659400130198E-10	.161837835592598853249E 10
10.5	.220227451928340164378E-10	.223055444113669522882E 10
10.6	.158780907846761020495E-10	.307911609741163370741E 10
10.7	.114305570400661900122E-10	.425712499253557560813E 10
10.8	.821640341535987879103E-11	.589495535241436546451E 10
10.9	.589717854235583209735E-11	.817551556012362078094E 10
11.0	.422627586496035959278E-11	.113557825304304762810E 11
11.1	.302430294339025847710E-11	.157973084083029614418E 11
11.2	.216097346420640172470E-11	.220094965796322737891E 11
11.3	.154181773655640718300E-11	.307110749536208389145E 11
11.4	.109844631776013069817E-11	.429175361113039160819E 11
11.5	.781429018396285434761E-12	.600656801588960365482E 11
11.6	.555095615024136498269E-12	.841912414627147490078E 11
11.7	.393746092799132774141E-12	.118182563620701733575E 12
11.8	.278892938693867579416E-12	.166143364026631481787E 12
11.9	.197257784302520279984E-12	.233912189241969675323E 12
12.0	.139318468887536083916E-12	.329807225829074176155E 12
12.1	.982570348972166959969E-13	.465695641579935721054E 12
12.2	.691992701366569063961E-13	.658530881386804951696E 12
12.3	.486658957563709191193E-13	.932565597282615388988E 12
12.4	.341771385120404093962E-13	.132254131472876654684E 13
12.5	.239682782607804993681E-13	.187829193562205186697E 13
12.6	.167853540284906328650E-13	.267139640130579937797E 13
12.7	.117386739477356562230E-13	.380480511320195002716E 13
12.8	.819794032816253390974E-14	.542678737845923755319E 13
12.9	.571728616652078710946E-14	.775116548633197763185E 13
13.0	.398177607883333536368E-14	.110867067190594047452E 14
13.1	.276928527454303941924E-14	.158798728260683566178E 14
13.2	.192337838326519232428E-14	.227770767368572048498E 14
13.3	.133404390443613224813E-14	.327154633403950921574E 14
13.4	.924029923125637737763E-15	.470554511226251063246E 14
13.5	.639167387674186665182E-15	.677748902657079072076E 14
13.6	.441527701799410115984E-15	.977506683278796963400E 14
13.7	.304591778119552643572E-15	.141117834890440166596E 15
13.8	.209844377837890372860E-15	.204178075433459808551E 15
13.9	.144376826840785879547E-15	.295692656996284091124E 15
14.0	.992020549119237726828E-16	.428805361786534149450E 15

Table B-1 (Cont'd)

THE AIRY FUNCTIONS (10 < x < 25)

x	Ai(x)	Bi(x)
14.0	.99202054911923772682 8E-16	.62268146675227724165 3E 15
14.1	.68072056878493414214 6E-16	.62268146675227724165 3E 15
14.2	.46649188699217451491 4E-16	.90543148677546649543 3E 15
14.3	.31926278665956251526 7E-16	.13183388389184787072 3E 16
14.4	.21821441453045464354 7E-16	.192[illegible]0984419442095902E 16
14.5	.14895374549659271958 4E-16	.28061248320050404417 2E 16
14.6	.10154398124274345279 9E-16	.41021481999426487387 0E 16
14.7	.69134304839317009478 9E-17	.60046714595756731728 0E 16
14.8	.47007974549452896948 0E-17	.88011351540339[illegible]612038E 16
14.9	.31921982487469206899 2E-17	.12916881583852275329 3E 17
15.0	.21649625207379922994 7E-17	.18982099567493589680 3E 17
15.1	.14664083918048589676 5E-17	.27931640928367285103 9E 17
15.2	.99198548744269999211 1E-18	.41154029216842117679 8E 17
15.3	.67019827808940721329 3E-18	.60714171609358602084 5E 17
15.4	.45222094093655530275 3E-18	.89686628805320948188 0E 17
15.5	.30475381524560126842 6E-18	.13265492278009282768 6E 18
15.6	.20511644705673016694 5E-18	.19646049695182558249 3E 18
15.7	.13788162287347052483 5E-18	.29132749197516991105 7E 18
15.8	.92569635645463635769 0E-19	.43255392125947808577 6E 18
15.9	.62070990076353263458 6E-19	.64305747837896110895 0E 18
16.0	.41568888289170243945 9E-19	.95721239060491865261 5E 18
16.1	.27804136961807302362 8E-19	.14266388566183744841 7E 19
16.2	.18574337755095416832 0E-19	.21289490114021627019 2E 19
16.3	.12393152430345889190 9E-19	.31809747362901470662 8E 19
16.4	.82587893048584169927 5E-20	.47587974192997146263 1E 19
16.5	.54969111729670607642 7E-20	.71281127151950813126 8E 19
16.6	.36541836402301733573 4E-20	.10690316541409244048 5E 20
16.7	.24262352206306747175 2E-20	.16052533099410898636 4E 20
16.8	.16089697306148330499 0E-20	.24134146175720067495 6E 20
16.9	.10657045527532456820 9E-20	.36329048823462927495 6E 20
17.0	.70501972983886145431 6E-21	.54753038113305869818 1E 20
17.1	.46584639164238295888 5E-21	.82621470691784049311 9E 20
17.2	.30744165996521946592 0E-21	.12482642624231324592 2E 21
17.3	.20265750869292588665 0E-21	.18881978353923866405 9E 21
17.4	.13342713476133049273 6E-21	.28596589248012361239 7E 21
17.5	.87742208232947097371 2E-22	.43361591835430069985 5E 21
17.6	.57631158548261311866 3E-22	.65829249317587619928 4E 21
17.7	.37808719414485980172 5E-22	.10005845105134786129 6E 22
17.8	.24775010074757542129 8E-22	.15226789026733373132 3E 22
17.9	.16215277295333902869 1E-22	.23199630888176504589 7E 22
18.0	.10600466825247955656 9E-22	.35389182503565686697 9E 22

Table B-1 (Cont'd)

THE AIRY FUNCTIONS (10 < x < 25)

x	Ai(x)	Bi(x)
18.0	.106004668252479556569E-22	.353891825035656866979E 22
18.1	.692176842161265116230E-23	.540474526324040891267E 22
18.2	.451442021548144010205E-23	.826406488524846671057E 22
18.3	.294090577881388059609E-23	.126509912416052574242E 23
18.4	.191362047645285272726E-23	.193894866042865782519E 23
18.5	.124373376697194045779E-23	.297520959111072231734E 23
18.6	.807415896027368370647E-24	.457063894124628186543E 23
18.7	.523560345022757982345E-24	.702979994427667652976E 23
18.8	.339107262365315512574E-24	.108246617442008372776E 24
18.9	.219386367536543900949E-24	.166874373649704369106E 24
19.0	.141770437779335271884E-24	.257553555223445854590E 24
19.1	.915095679970875393665E-25	.397966656714802327415E 24
19.2	.590001868284209467362E-25	.615638430900988868981E 24
19.3	.379968497345214649698E-25	.953461869446631873350E 24
19.4	.244427673798422745413E-25	.147835314814324184241E 25
19.5	.157059056151781837797E-25	.229482196854864866637E 25
19.6	.100806063153014384290E-25	.356627195427772272298E 25
19.7	.646282826485243200913E-26	.554847051820414897862E 25
19.8	.413877756396473509949E-26	.864219768505250258610E 25
19.9	.264750186892370283040E-26	.134761529914774766915E 26
20.0	.169167286867054031329E-26	.210376504965110381481E 26
20.1	.107972611148515300070E-26	.328788681557529046869E 26
20.2	.688381387028602595842E-27	.514426774414172293774E 26
20.3	.438393496762402792880E-27	.805779334100291031133E 26
20.4	.278881510801988761086E-27	.126355258865505297776E 27
20.5	.177213635431494210350E-27	.198359929862883354653E 27
20.6	.112485745214810135618E-27	.311743062321866211659E 27
20.7	.713217397495334314911E-28	.490479300961905262920E 27
20.8	.451722354958226535600E-28	.772546135309276695631E 27
20.9	.285790363311651692469E-28	.121816703817634209032E 28
21.0	.180613844247038337893E-28	.192294462486380835187E 28
21.1	.114020531328895726437E-28	.303880666246597862256E 28
21.2	.719026264127932400624E-29	.480744773505085303756E 28
21.3	.452936387392152928570E-29	.761377682662431374138E 28
21.4	.285010932756958239155E-29	.120714281033582144510E 29
21.5	.179150801726944136460E-29	.191597119702290587854E 29
21.6	.112488983696001255802E-29	.304431788340486276995E 29
21.7	.705563806817001060242E-30	.484239946720569689753E 29
21.8	.442077811985405880216E-30	.771080497483682132865E 29
21.9	.276693089647270999206E-30	.122915378863818866915E 30
22.0	.172996024035369828514E-30	.196145814091862286920E 30

Table B-1 (Cont'd)

THE AIRY FUNCTIONS (10 < x < 25)

x	Ai(x)	Bi(x)
22.0	.172996024035369828514E-30	.196145814091862286920E 30
22.1	.108047116424744690369E-30	.313340911729386914638E 30
22.2	.674109610528497203946E-31	.501094146811929577183E 30
22.3	.420135285336424635698E-31	.802203705318530443379E 30
22.4	.261571313128137732769E-31	.128561831061175369958E 31
22.5	.162680085685162156032E-31	.206253078441225230845E 31
22.6	.101070170695543960172E-31	.331244589548109735947E 31
22.7	.627273482057724934700E-32	.532544713624646993627E 31
22.8	.388899306256067356723E-32	.857079814105482206269E 31
22.9	.240860011402533986301E-32	.138083993977994646176E 32
23.0	.149018609576720609125E-32	.222700547962286089419E 32
23.1	.921012556949555995971E-33	.359545698270528538025E 32
23.2	.568644443471534853285E-33	.581086485732068488485E 32
23.3	.350725392461375990915E-33	.940113582617548287527E 32
23.4	.216095505004479745698E-33	.152255071398365133662E 33
23.5	.133007862893943276068E-33	.246839191489190158363E 33
23.6	.817829734113617523939E-34	.400595805695179662706E 33
23.7	.502346533234157006111E-34	.650800155505538849720E 33
23.8	.308247656155050173329E-34	.105836849856814775385E 34
23.9	.188952629102646620004E-34	.172295214057834582311E 34
24.0	.115708108539854246546E-34	.280773183681073145343E 34
24.1	.707836998698931221413E-35	.458018689682726109010E 34
24.2	.432575820285657310276E-35	.747919743604813718455E 34
24.3	.264089804803899827980E-35	.122255973895435077904E 35
24.4	.161065431548029546006E-35	.200044930628918696415E 35
24.5	.981330379746299477985E-36	.327662289156333824001E 35
24.6	.597298231555962029287E-36	.537236991628094207048E 35
24.7	.363187735241054462147E-36	.881749018891367053629E 35
24.8	.220615464672314862753E-36	.144864772539046811883E 36
24.9	.133877141633755278742E-36	.238242053223789420319E 36
25.0	.811602682469138668487E-37	.392203077804138177325E 36

Table B-2

DERIVATIVE OF THE AIRY FUNCTION (10 < x < 25)

x	Ai′(x)	Bi′(x)
10.0	-.352063367673892363533E -9	.142923613448286577692E 10
10.1	-.257036072748846387539E -9	.196740716238251062303E 10
10.2	-.187358789021275579840E -9	.271241636128147866228E 10
10.3	-.136352894995670240492E -9	.374530497077283683286E 10
10.4	-.990759951950108036761E-10	.517945020616542516815E 10
10.5	-.718769678145156708944E-10	.717369224528329918220E 10
10.6	-.520632962821416485765E-10	.995087851932571811777E 10
10.7	-.376528208686528534456E-10	.138240963122361104302E 11
10.8	-.271888353493916459796E-10	.192338311212363513003E 11
10.9	-.196026217007170914481E-10	.268006719340776907645E 11
11.0	-.141114412466285173343E-10	.374001681969269770188E 11
11.1	-.101429758155983816715E-10	.522692904797488189087E 11
11.2	-.727946252450190530376E-11	.731580388504505598723E 11
11.3	-.521646387179624041028E-11	.102545631351554279899E 12
11.4	-.373249361707886810024E-11	.143949188987245501545E 12
11.5	-.266667996750453140542E-11	.202365072766383857488E 12
11.6	-.190236835053486654447E-11	.284900706747623729896E 12
11.7	-.135510550276357766030E-11	.401680287838070182323E 12
11.8	-.963849540422801512153E-12	.567144804250655898710E 12
11.9	-.684551044188868002757E-12	.801920458292784187969E 12
12.0	-.485473655498530846105E-12	.113550750244337074287E 13
12.1	-.343788607225013804111E-12	.161015473690776814904E 13
12.2	-.243100574560611553690E-12	.228644970156900944678E 13
12.3	-.171653181219169350850E-12	.325139468318448890123E 13
12.4	-.121029638216071935143E-12	.463008889057847137266E 13
12.5	-.852134656467385644268E-13	.660264868136429539267E 13
12.6	-.599106334933158660424E-13	.942874575068032770823E 13
12.7	-.420611679840498836960E-13	.134832383858282565653E 14
12.8	-.294878419392747857881E-13	.193079763200973492481E 14
12.9	-.206438824906816728690E-13	.276872159461025515857E 14
13.0	-.144320805739726260420E-13	.397575449699083454108E 14
13.1	-.100753029085771972421E-13	.571683093709411447142E 14
13.2	-.702394775707173076768E-14	.823160385407441846648E 14
13.3	-.488990917590951962661E-14	.118687429457867220864E 15
13.4	-.339953102870762817933E-14	.171361788249212270537E 15
13.5	-.236014254392431128473E-14	.247747978649419038682E 15
13.6	-.163629178056036442168E-14	.358666671217253971620E 15
13.7	-.113289279093531600068E-14	.519941652567138096400E 15
13.8	-.783293745803420104969E-15	.754742530147430011883E 15
13.9	-.540842103467590409731E-15	.109703787692095207595E 16
14.0	-.372931011001790067898E-15	.159669141158800278892E 16

Table B-2 (Cont'd)

DERIVATIVE OF THE AIRY FUNCTION (10 < x < 25)

x	Ai'(x)	Bi'(x)
14.0	-.372931011001790067898E-15	.159699141158800278892E 16
14.1	-.256803437804027701846E-15	.232699219222501791059E 16
14.2	-.176599566070206014615E-15	.339579493084730082477E 16
14.3	-.121282183267192788724E-15	.496202126064718553459E 16
14.4	-.831811737648166827696E-16	.726013144363931786151E 16
14.5	-.569738820618578060986E-16	.106364603606365231391E 17
14.6	-.389718797482950699416E-16	.156032349615269043694E 17
14.7	-.266227976563968532810E-16	.229189739147063522656E 17
14.8	-.181628846924093636467E-16	.337082983764942639641E 17
14.9	-.123750435850749231677E-16	.496406868409789010027E 17
15.0	-.842056795401777276489E-17	.731974920340701049732E 17
15.1	-.572230113023400326210E-17	.108071275789039096117E 18
15.2	-.388361908657038719602E-17	.159763741339151116339E 18
15.3	-.263233428425207203717E-17	.236482091789810624187E 18
15.4	-.178191026281872938783E-17	.350484348534871958621E 18
15.5	-.120468320445344374176E-17	.520100884035879501899E 18
15.6	-.813399615049179806911E-18	.772775639717304038684E 18
15.7	-.548505485507717093205E-18	.114964674372645543364E 19
15.8	-.369407493269601595505E-18	.171245382270399544710E 19
15.9	-.248473414594476579026E-18	.255396600820210487493E 19
16.0	-.166918867683818095517E-18	.381374350712186265760E 19
16.1	-.111991216141294907995E-18	.570198837153071058912E 19
16.2	-.750447218812812018426E-19	.853567789541781502297E 19
16.3	-.502235082061330036532E-19	.127933692660791080863E 20
16.4	-.335702848776859470705E-19	.191984194401749204689E 20
16.5	-.224110854252529725688E-19	.288454985378880417268E 20
16.6	-.149428113011277509772E-19	.433931402928068885809E 20
16.7	-.995095925827409089460E-20	.653571351908070188981E 20
16.8	-.661854371221130383194E-20	.985580907742082647083E 20
16.9	-.439669581352420798102E-20	.148804946833208216459E 21
17.0	-.291714821929331379220E-20	.224940029106572692805E 21
17.1	-.193312647218127341685E-20	.340438687224393856879E 21
17.2	-.127947946419387368802E-20	.515860593635818687887E 21
17.3	-.845822474639819997197E-21	.782610378626513480762E 21
17.4	-.558469585938764676797E-21	.118871347197492022127E 22
17.5	-.368294962879009668983E-21	.180769701178078440958E 22
17.6	-.242588107629833848014E-21	.275226040900256853687E 22
17.7	-.159596081558193161737E-21	.419534227803661370943E 22
17.8	-.104871005874897652191E-21	.640262173687551042641E 22
17.9	-.688288674550328064774E-22	.978271609876441700670E 22
18.0	-.451200186068194188678E-22	.149647965032878506923E 23

Table B-2 (Cont'd)

DERIVATIVE OF THE AIRY FUNCTION (10 < x < 25)

x	Ai′(x)	Bi′(x)
18.0	-.451200186068194188678E-22	.149647965032878506923E 23
18.1	-.295428804264887189014E-22	.229187302251998860769E 23
18.2	-.193206963192365262157E-22	.351412563742740517818E 23
18.3	-.126206150973546449297E-22	.539448071632784365475E 23
18.4	-.823431555209787357858E-23	.829060605502091608608E 23
18.5	-.536617882341472770778E-23	.127563328556779909413E 24
18.6	-.349296979861062651655E-23	.196502012337324119170E 24
18.7	-.227100434869718153847E-23	.303045914865188251441E 24
18.8	-.147480950125854257976E-23	.467895453475392041898E 24
18.9	-.956643405946357676276E-24	.723247386754521478153E 24
19.0	-.619814582713001505565E-24	.111923500633958878160E 25
19.1	-.401118226383766064136E-24	.173400672908348013245E 25
19.2	-.259288489287757465750E-24	.268951567813594147069E 25
19.3	-.167415489980768475800E-24	.417628306665201991831E 25
19.4	-.107971942225061616387E-24	.649227736852334287326E 25
19.5	-.695553223646362425475E-25	.101040212694630985947E 26
19.6	-.447564152553792789525E-25	.157427373632910751995E 26
19.7	-.287664969090279585933E-25	.245557857440084573018E 26
19.8	-.184682902838145125674E-25	.383454457731479848594E 26
19.9	-.118433810008075039405E-25	.599458205274357870353E 26
20.0	-.758639162574835495532E-26	.938183933613396435559E 26
20.1	-.485407641738001915529E-26	.146994081138312857531E 27
20.2	-.310235140347328980716E-26	.230564955148026726872E 27
20.3	-.198056738529886949184E-26	.362048845316527828166E 27
20.4	-.126300109920378222084E-26	.569141550079938628266E 27
20.5	-.804515679375548958203E-27	.895677198104916149584E 27
20.6	-.511897694621916819229E-27	.141110618917740115788E 28
20.7	-.325350067626899870719E-27	.222558245219023286826E 28
20.8	-.206556504873923717234E-27	.351400489433666270624E 28
20.9	-.130993054953951399461E-27	.555436728409775256354E 28
21.0	-.829813025830044574252E-28	.878899605284316392040E 28
21.1	-.525092678368820007868E-28	.139224375987128229705E 29
21.2	-.331907114796451232764E-28	.220780913543930495785E 29
21.3	-.209567138861677843860E-28	.350491035946830083655E 29
21.4	-.132177325238434689872E-28	.557006627069181907760E 29
21.5	-.832758375165568946491E-29	.886157240513378081889E 29
21.6	-.524095521039701057495E-29	.141132530703163474088E 30
21.7	-.329482522733166544808E-29	.225013365759704309281E 30
21.8	-.206912110558766860222E-29	.359131331033594800644E 30
21.9	-.129799232105214802276E-29	.573800626830740172540E 30
22.0	-.813377403844754027080E-30	.917762786908182597479E 30

Table B-2 (Cont'd)

DERIVATIVE OF THE AIRY FUNCTION (10 < x < 25)

x	Ai'(x)	Bi'(x)
22.0	-.813377403844754027080E-30	.917762786908182597479E 30
22.1	-.509151381457080439650E-30	.146946937006262074943E 31
22.2	-.318374049162866549573E-30	.235532191162482658238E 31
22.3	-.198868211274338516367E-30	.377918912719522546737E 31
22.4	-.124088375664058961103E-30	.607022216607614340186E 31
22.5	-.773456518055552540449E-31	.976038939404641387162E 31
22.6	-.481593724269869776536E-31	.157103297210591046103E 32
22.7	-.299548312633107599125E-31	.253138416269339055819E 32
22.8	-.186120883232010398308E-31	.408304750427087470643E 32
22.9	-.115522484120786590826E-31	.659270192850050294597E 32
23.0	-.716278857286630380774E-32	.106559974370519480668E 33
23.1	-.443652515828674492955E-32	.172415165670092483886E 33
23.2	-.274504815190346398542E-32	.279258594067754200415E 33
23.3	-.169669655805052771703E-32	.452779130387328190077E 33
23.4	-.104762672646452756687E-32	.734876017637694172617E 33
23.5	-.646186852331631329218E-33	.119395682742405410672E 34
23.6	-.398161987187901188789E-33	.194182061992621446461E 34
23.7	-.245082650887335127394E-33	.316136468290461383036E 34
23.8	-.150701438880702904404E-33	.515209856141382226506E 34
23.9	-.925710603529363355180E-34	.840498384951380869874E 34
24.0	-.568050616012267829181E-34	.137256159691979614415E 35
24.1	-.348220015788500586208E-34	.224371728410589565028E 35
24.2	-.213243725472911048247E-34	.367151112602355714287E 35
24.3	-.130453446109963475861E-34	.601396768344890356898E 35
24.4	-.797246349442357737978E-35	.986088659676636779442E 35
24.5	-.486730015619838165884E-35	.161848464434329873300E 36
24.6	-.296854227395966080957E-35	.265912078278692804920E 36
24.7	-.180866765756120044706E-35	.437324219725806996113E 36
24.8	-.110086897987992595411E-35	.719952937361442789172E 36
24.9	-.669383062186468254957E-36	.118642128982540942490E 37
25.0	-.406608933724328100332E-36	.195707350832333089799E 37

Table B-3

THE AIRY FUNCTIONS $(0.001 < \frac{1}{x} < 0.100)$

(1/x)	Ai(x)	Bi(x)
0.001	.930693306317955485086E-9157	.540771183919494721773E 9155
0.002	.544129859120752869406E-3238	.130807477732797324889E 3237
0.003	.628495943845674274504E-1763	.138730581429762156905E 1762
0.004	.742354028122558385645E-1145	.415336299525649304717E 1144
0.005	.915362430845268422205E-820	.122945336104471014137E 819
0.006	.843886040856128434502E-624	.146087133812480004021E 623
0.007	.353675793005622612602E-495	.376499010337000788020E 494
0.008	.197761599674377005772E-405	.719818813693086571432E 404
0.009	.688251558785490068674E-340	.219378585612474715465E 339
0.010	.263448215208818446772E-290	.604122399667020144904E 289
0.011	.100223403470256699800E-251	.166551066397672661217E 251
0.012	.521747391010095647771E-221	.334157017394593106380E 220
0.013	.441298907252905824563E-196	.411205679886961034969E 195
0.014	.159633504358041380102E-175	.117966932906271127016E 175
0.015	.247943568628726911840E-158	.736163981505047776209E 157
0.016	.877056551990095995018E-144	.229537054145294573065E 143
0.017	.242606426528231114448E-131	.855347567794050344634E 130
0.018	.132964520680748812308E-120	.160590927951391626029E 120
0.019	.294422824659537106049E-111	.745119488764483099884E 110
0.020	.458494172407482346967E-103	.490909969944421933808E 102
0.021	.777323070927649153251E-96	.296707722326800278356E 95
0.022	.203325983063809095209E-89	.116101896494173887393E 89
0.023	.108701251814561191839E-83	.222049771104671550471E 83
0.024	.149348765049003460452E-78	.165091629745794345171E 78
0.025	.636574265855291489975E-74	.395313930243859353963E 73
0.026	.983835924349333975296E-70	.260846684987749558534E 69
0.027	.628037236809010073494E-66	.416406936452967796262E 65
0.028	.184753909602422154057E-62	.144147461850162556849E 62
0.029	.274783174471007016260E-59	.980349092330160281365E 58
0.030	.223580135030987306677E-56	.123296110696707439005E 56
0.031	.106475496052052563139E-53	.263180270091308497214E 53
0.032	.314539756224823825237E-51	.905152590712662808589E 50
0.033	.606083635040168055854E-49	.477031417498577739606E 48
0.034	.795749241546771408435E-47	.368795849088851102785E 46
0.035	.739508174745689560384E-45	.402637373610171014764E 44
0.036	.502940972912168266167E-43	.600423291522738395540E 42
0.037	.257770557835916177955E-41	.118765723883184810174E 41
0.038	.102171324895982292746E-39	.303659329763126651487E 39
0.039	.320444892242718533067E-38	.980851154939333821367E 37
0.040	.811602682469138668487E-37	.392203077804138177325E 36

Table B-3 (Cont'd)

THE AIRY FUNCTIONS ($0.001 < \frac{1}{x} < 0.100$)

(1/x)	Ai(x)	Bi(x)
0.040	.811602682469138668487E-37	.392203077804138177325E 36
0.041	.169033497730313117822E-35	.193653138875607804086E 35
0.042	.294222788718245739519E-34	.110859648726721681373E 34
0.043	.434267308488206756539E-33	.759980447626766135501E 32
0.044	.550653380567940534247E-32	.606280977710626440167E 31
0.045	.606930146970480553907E-31	.556280535636208725464E 30
0.046	.587681466299607758940E-30	.580849554019758834664E 29
0.047	.504723259749611857292E-29	.683632638849326447096E 28
0.048	.387837223664883116157E-28	.899080893714537202545E 27
0.049	.268758568323820156783E-27	.131088272264116393018E 27
0.050	.169167286867054031329E-26	.210376504965110381481E 26
0.051	.973558351512335486084E-26	.369191963451837984260E 25
0.052	.515352063867567102862E-25	.704250769221475439925E 24
0.053	.252305843477184230322E-24	.145224826170895419895E 24
0.054	.114820364188157618548E-23	.322113393168156120520E 23
0.055	.487964063862984578483E-23	.764935705773403810749E 22
0.056	.194483331106111069868E-22	.193661675586031172714E 22
0.057	.729802982911303486560E-22	.520672332238329842059E 21
0.058	.258779065683937461418E-21	.148121546259055823347E 21
0.059	.869971719908677752125E-21	.444380454729049109594E 20
0.060	.278148946542974734485E-20	.140162877885854483746E 20
0.061	.848185996614385401642E-20	.463456914602543340970E 19
0.062	.247343953643841145151E-19	.160225215428276470237E 19
0.063	.691482180053791226938E-19	.577731964518804108917E 18
0.064	.185749660301352992256E-18	.216770358400911382943E 18
0.065	.480475735262335138371E-18	.844547350296630152667E 17
0.066	.119916034756960821130E-17	.340984288885812278960E 17
0.067	.289303356285707724085E-17	.142404755352202877036E 17
0.068	.675857108174873905539E-17	.614102779301469485853E 16
0.069	.153139511916235170473E-16	.273010742754822367001E 16
0.070	.337063058274944800157E-16	.124934187995900039288E 16
0.071	.721681285135679272488E-16	.587662774161686865432E 15
0.072	.150511765311079361152E-15	.283753542363377502893E 15
0.073	.326147607861233071412E-15	.140467911690914787537E 15
0.074	.608048752257388506856E-15	.712073997836399275432E 14
0.075	.118052218205220256179E-14	.369237058137592350778E 14
0.076	.224280155334573760389E-14	.195643721022054371310E 14
0.077	.417363875295484712244E-14	.105823389519819948278E 14
0.078	.761461644902346229964E-14	.583783286300729838475E 13
0.079	.136323091301867811296E-13	.328169161236655114695E 13
0.080	.239682782607804993681E-13	.187829193562205186697E 13

Table B-3 (Cont'd)

THE AIRY FUNCTIONS $(0.001 < \frac{1}{x} < 0.100)$

(1/x)	Ai (x)	Bi (x)
0.080	.239682782607804993681E-13	.187829193562205186697E 13
0.081	.414179180248762741008E-13	.109373096504218355772E 13
0.082	.703952272592320827752E-13	.647472505108678947116E 12
0.083	.117761665554870568585E-12	.389398359784379339714E 12
0.084	.194024489084984085448E-12	.237762594367360676178F 12
0.085	.315044613678942864574E-12	.147298831024328822390E 12
0.086	.504437487453983406699E-12	.925348367031405238548E 11
0.087	.796904123840931864865E-12	.589139949888266753300E 11
0.088	.124279484569302524896E-11	.379934192624903466277E 11
0.089	.191428692225799409340E-11	.248059612412914477204E 11
0.090	.291365059323629366300E-11	.163890399534411438215E 11
0.091	.438418730979491590476E-11	.103522412252059187517E 11
0.092	.652455055188680163687E-11	.739973951822263685519E 10
0.093	.960730423873868311470E-11	.505259864170708644191E 10
0.094	.140027371677518759019E-10	.348519955909064472880E 10
0.095	.202091343658702181221E-10	.242768637538500222579E 10
0.096	.288909075667412502784E-10	.170708359875517152212E 10
0.097	.409262066194499579656E-10	.121134162404667382622E 10
0.098	.574658703684580241021E-10	.867136092690895012927E 9
0.099	.800058277690785450579E-10	.626011306649738338512E 9
0.100	.110475325528986859354E -9	.455641153548225140835E 9

+55

Table B-4

DERIVATIVE OF THE AIRY FUNCTIONS $(0.001 < \frac{1}{x} < 0.100)$

(1/x)	Ai′(x)	Bi′(x)
0.001	-.294313391791033556947E-9155	.171005511462461529197E 9157
0.002	-.121673855855365817945E-3236	.292487871436271181403E 3238
0.003	-.114751848483867295165E-1761	.253221053681685407282E 1763
0.004	-.383219596064204652518E-1144	.656662810523992184636E 1145
0.005	-.129463235922188230280E-818	.173855590184728749853E 820
0.006	-.108957874027362158564E-622	.188575759343950813137E 624
0.007	-.422785296857667304437E-494	.449936477075528954271E 495
0.008	-.221143724756057205190E-404	.804637871235991751959E 405
0.009	-.725635617044024895672E-339	.231195946911369024353E 340
0.010	-.263514036160440990479E-289	.603971274531060297541E 290
0.011	-.955868073726703874321E-251	.158754376383094731660E 252
0.012	-.476444421498503581523E-220	.304941897329927950690E 221
0.013	-.387188035091644490679E-195	.360517473202482920175E 196
0.014	-.134970749430353308475E-174	.996589239833852660034E 175
0.015	-.202537948421418344109E-157	.641605052015505629262E 158
0.016	-.693724465467949295872E-143	.181373043312667457539E 144
0.017	-.186173600582014430403E-130	.655657609795401226977E 131
0.018	-.991656463286670803678E-120	.119625035100523868843E 121
0.019	-.213736389600757719256E-110	.540211979210612599290E 111
0.020	-.324433181982879928376E-102	.346879877954597673761E 103
0.021	-.536811201667025962536E-95	.2045917111175979143285E 96
0.022	-.137193952144782077531E-88	.782118957312420935700E 89
0.023	-.717378505454632665636E-83	.146287324752663923608E 84
0.024	-.964936154793420411866E-78	.106466902311600864725E 79
0.025	-.403001797760067803252E-73	.249770796817069688103E 74
0.026	-.610786966945587812018E-69	.161600249276788420967E 70
0.027	-.382634048014761128079E-65	.253135332030817950335E 66
0.028	-.110540530879865367826E-61	.860433843845897573295E 62
0.029	-.161556843436399897723E-58	.578486803664697738897E 59
0.030	-.129751195868075733481E-55	.710922679367021066501E 56
0.031	-.605562573286067749903E-53	.149271682670983592763E 54
0.032	-.176083808305908618789E-50	.505268945396998064963E 51
0.033	-.334136222594521250151E-48	.262202113857394864991E 49
0.034	-.432229379328568511469E-46	.199692914471756014262E 47
0.035	-.395928193559257603049E-44	.214864965144589735634E 45
0.036	-.265523901156624403731E-42	.315908148332784099759E 43
0.037	-.134245920177997981804E-40	.616330075220920694193E 41
0.038	-.525094020680003857333E-39	.155484122343726435955E 40
0.039	-.162574525566226067321E-37	.495712301440351924833E 38
0.040	-.406608933724328100332E-36	.195707350832333089799E 37

Table B-4 (Cont'd)

DERIVATIVE OF THE AIRY FUNCTIONS $(0.001 < \frac{1}{x} < 0.100)$

(1/x)	Ai'(x)	Bi'(x)
0.040	-.406608933724328100332E-36	.195707350832333089799E 37
0.041	-.836520656030999501490E-35	.939604276725060286415E 35
0.042	-.143873313502102127459E-33	.539769325536328610677E 34
0.043	-.209886658895773666511E-32	.365673521738057225231E 33
0.044	-.263115971040898316521E-31	.288362304878726660861E 32
0.045	-.286788394877879066203E-30	.261603555301064914460E 31
0.046	-.274679623199388457299E-29	.270150342752465406174E 30
0.047	-.233400808761588565766E-28	.314527758160566859537E 29
0.048	-.177485056205858865301E-27	.409286302457679941042E 28
0.049	-.121739778019382776126E-26	.590579550672295932092E 27
0.050	-.758639162574835495532E-26	.938183933613396435559E 26
0.051	-.432331386065965277338E-25	.163006780901957671120E 26
0.052	-.226661908710396795843E-24	.307911957178197226310E 25
0.053	-.109926456337716099708E-23	.628877155062266841845E 24
0.054	-.495646316356907151508E-23	.138177204481589972839E 24
0.055	-.208734219222657938347E-22	.325109246012410064063E 23
0.056	-.824542724209578535902E-22	.815635765383640605797E 22
0.057	-.306712130747167603616E-21	.217337239578195057447E 22
0.058	-.107824178779839298745E-20	.612874325497683676482E 21
0.059	-.359433643015074662421E-20	.182287075345943374214E 21
0.060	-.113967304649805750679E-19	.570090356654233081980E 20
0.061	-.344702041910845990034E-19	.186934636996463089285E 20
0.062	-.997154889691565566807E-19	.640971921213356781391E 19
0.063	-.276571662210435567480E-18	.229254781999433071539E 19
0.064	-.737182625985756728025E-18	.853355768644593278377E 18
0.065	-.189230968475134740029E-17	.329871753206254053315E 18
0.066	-.468730692292082990727E-17	.132159210163026922405E 18
0.067	-.112247099541537374705E-16	.547745732113890684967E 17
0.068	-.260315918887213916595E-16	.234441799874861948475E 17
0.069	-.585604671579815136725E-16	.103456983669735143542E 17
0.070	-.127981075834179618974E-15	.469994454784537024208E 16
0.071	-.272108410014686783922E-15	.219490107814080063522E 16
0.072	-.563601637930054770183E-15	.105231590774304032721E 16
0.073	-.113862340257536008467E-14	.517298945198389594740E 15
0.074	-.224634313021885829049E-14	.260429171626737963428E 15
0.075	-.433251653084219100980E-14	.134124815927197217245E 15
0.076	-.817756826071794115813E-14	.705906849421171640421E 14
0.077	-.151200524709278800052E-13	.379295802380124172224E 14
0.078	-.274112268252120952532E-13	.207873536990191964063E 14
0.079	-.487671994395552091856E-13	.116099903041317726155E 14
0.080	-.852134656467385644268E-13	.660264868136429539267E 13

Table B-4 (Cont'd)

DERIVATIVE OF THE AIRY FUNCTIONS $(0.001 < \frac{1}{x} < 0.100)$

(1/x)	Ai′(x)	Bi′(x)
0.080	-.85213465646738564268E-13	.660264868136429539267E 13
0.081	-.146354755583667727203E-12	.382050005569360244412E 13
0.082	-.247253428802173484712E-12	.224759682439674223376E 13
0.083	-.411164986569351596034E-12	.134341607744334080985E 13
0.084	-.673462750454590197180E-12	.815286957702759259337E 12
0.085	-.100718876731495257755E-11	.502050328074181791250E 12
0.086	-.173079812775502557116E-11	.313519054390867627011E 12
0.087	-.271882280270428414382E-11	.198434376832673242852E 12
0.088	-.421637245777368617479E-11	.127225728507594103672E 12
0.089	-.645861697402928647589E-11	.825882901959071354759E 11
0.090	-.977666173457181014304E-11	.542548879304808233160E 11
0.091	-.146315347787940395985E-10	.360526995364261165995E 11
0.092	-.216583566590634897483E-10	.242229234423285103867E 11
0.093	-.317231603948524590943E-10	.164484734880062291083E 11
0.094	-.459952704428099147495E-10	.112840216897124317157E 11
0.095	-.660386350814962365078E-10	.781769217261657434957E 10
0.096	-.939260337825592019538E-10	.546782042248172793551E 10
0.097	-.132380646401629340827E -9	.385942680805962674046E 10
0.098	-.184950069381856421738E -9	.274829224136809255301E 10
0.099	-.256218489240527352727E -9	.197378390237140312213E 10
0.100	-.352063367673892363533E -9	.142923613448286577692E 10

Table B-5

THE AUXILIARY FUNCTIONS Q(x) AND M(x)

1/x	Q(x)		M(x)	
0.001	.100000000015625000018E	1	.100000000015625000056E	1
0.002	.100000000125000001151E	1	.100000000125000003609E	1
0.003	.100000000421875013111E	1	.100000000421875041113E	1
0.004	.100000001000000073666E	1	.100000001000000231000E	1
0.005	.100000001953125281016E	1	.100000001953125881196E	1
0.006	.100000003375000839110E	1	.100000003375002631240E	1
0.007	.100000005359377115925E	1	.100000005359381635015E	1
0.008	.100000008000004714683E	1	.100000008000014784087E	1
0.009	.100000011390634558029E	1	.100000011390654971655E	1
0.010	.100000015625017985152E	1	.100000015625056397133E	1
0.011	.100000020796906861868E	1	.10000002079697491134[illegible]	1
0.012	.100000027000053703652E	1	.100000027000168402349E	1
0.013	.100000034328211811625E	1	.100000034328397221942E	1
0.014	.100000042875135421512E	1	.100000042875424652772E	1
0.015	.100000052734579865545E	1	.100000052735017416161E	1
0.016	.100000064000301747351E	1	.100000064000946220613E	1
0.017	.100000076766059129795E	1	.100000076766986351022E	1
0.018	.100000091125611735805E	1	.100000091126918298641E	1
0.019	.100000107172721162181E	1	.100000107174528431798E	1
0.020	.100000125001151106382E	1	.100000125003609707426E	1
0.021	.100000144704667606317E	1	.100000144707962423421E	1
0.022	.100000166377039293129E	1	.100000166381395011878E	1
0.023	.100000190112037656997E	1	.100000190117724873253E	1
0.024	.100000216003447325953E	1	.100000216010779251488E	1
0.025	.100000244145016357733E	1	.100000244154396150167E	1
0.026	.100000274630556544669E	1	.100000274642425289761E	1
0.027	.100000307553843731641E	1	.100000307568729106018E	1
0.028	.100000343008668147088E	1	.100000343027183789578E	1
0.029	.100000381088824747117E	1	.100000381111680366893E	1
0.030	.100000421888113572700E	1	.100000421916125822523E	1
0.031	.100000465503401199999E	1	.100000465534444262914E	1
0.032	.100000512019315723819E	1	.100000512060578121740E	1
0.033	.100000561538857954225E	1	.100000561588489406924E	1
0.034	.100000614152791026337E	1	.100000614212160989446E	1
0.035	.100000669954946223317E	1	.100000670025597934050E	1
0.036	.100000729039162332601E	1	.100000729122828871994E	1
0.037	.100000791499285095361E	1	.100000791597907415950E	1
0.038	.100000857429172669266E	1	.100000857544913617234E	1
0.039	.100000926922686104541E	1	.100000927057955465481E	1
0.040	.100001000073697833370E	1	.100001000231170430956E	1

Table B-5 (Cont'd)

THE AUXILIARY FUNCTIONS Q(x) AND M(x)

1/x	Q(x)		M(x)	
0.041	.100001076976097172676E	1	.100001077158727049547E	1
0.042	.100001157723771840300E	1	.100001157934826551341E	1
0.043	.100001242410628484637E	1	.100001242653704530862E	1
0.044	.100001331130583227746E	1	.100001331409632662665E	1
0.045	.100001423977564221988E	1	.100001424296920459017E	1
0.046	.100001521045512220230E	1	.100001521409917071965E	1
0.047	.100001622428381159664E	1	.100001622843013139332E	1
0.048	.100001728220138759274E	1	.100001728690642674998E	1
0.049	.100001838514767131025E	1	.100001839047285003697E	1
0.050	.100001953406263404805E	1	.100001954007466740617E	1
0.051	.100002072988640367171E	1	.100002073665763816092E	1
0.052	.100002197355927113984E	1	.100002198116803545646E	1
0.053	.100002326602169716955E	1	.100002327455266745739E	1
0.054	.100002460821431904192E	1	.100002461775889895509E	1
0.055	.100002600107795754790E	1	.100002601173467344850E	1
0.056	.100002744555362407559E	1	.100002745742853569192E	1
0.057	.100002894258252783924E	1	.100002895578965471325E	1
0.058	.100003049310608325100E	1	.100003050776784730674E	1
0.059	.100003209806591743601E	1	.100003211431360200396E	1
0.060	.100003375840387789166E	1	.100003377637810352735E	1
0.061	.100003547506204029187E	1	.100003549491325773059E	1
0.062	.100003724898271643716E	1	.100003727087171703020E	1
0.063	.100003908110846235145E	1	.100003910520690633315E	1
0.064	.100004097238208652650E	1	.100004099887304946525E	1
0.065	.100004292374665831492E	1	.100004295282519610546E	1
0.066	.100004493614551647276E	1	.100004496801924923122E	1
0.067	.100004701052227785261E	1	.100004704541199308041E	1
0.068	.100004914782084624848E	1	.100004918596112163548E	1
0.069	.100005134898542139331E	1	.100005139062526763575E	1
0.070	.100005361496050811034E	1	.100005366036403212376E	1
0.071	.100005594669092561968E	1	.100005599613801453231E	1
0.072	.100005834512181700096E	1	.100005839890884331853E	1
0.073	.100006081119865881372E	1	.100006086963920715183E	1
0.074	.100006334586727087645E	1	.100006340929288666294E	1
0.075	.100006595007382620598E	1	.100006601883478676125E	1
0.076	.100006862476486111837E	1	.100006869923096952806E	1
0.077	.100007137088728549292E	1	.100007145144868769375E	1
0.078	.100007418938839320064E	1	.100007427645641870690E	1
0.079	.100007708121587269889E	1	.100007717522389940402E	1
0.080	.100008004731781779364E	1	.100008014872216128844E	1

Table B-5 (Cont'd)

THE AUXILIARY FUNCTIONS Q(x) AND M(x)

1/x	Q(x)		M(x)	
0.081	.100008308864273857112E	1	.100008319792356642781E	1
0.082	.100008620613957250052E	1	.100008632380184397938E	1
0.083	.100008940075769570951E	1	.100008952733212735298E	1
0.084	.100009267344693443436E	1	.100009280949099202189E	1
0.085	.100009602515757664670E	1	.100009617125649399206E	1
0.086	.100009945684039385871E	1	.100009961360820894065E	1
0.087	.100010296944660310877E	1	.100010313752727203523E	1
0.088	.100010656392797912979E	1	.100010674399641844529E	1
0.089	.100011024123676670224E	1	.100011043400002455837E	1
0.090	.100011400232574319411E	1	.100011420852414991324E	1
0.091	.100011784814872129071E	1	.100011806855657986350E	1
0.092	.100012177965806191299E	1	.100012201508686898484E	1
0.093	.100012579780968733760E	1	.100012604910638524034E	1
0.094	.100012990355809450339E	1	.100013017160835491826E	1
0.095	.100013409785886852483E	1	.100013438358790835758E	1
0.096	.100013838166819640421E	1	.100013868604212647687E	1
0.097	.100014275594288094921E	1	.100014307997008812310E	1
0.098	.100014722164035489803E	1	.100014756637291825713E	1
0.099	.100015177971869525513E	1	.100015214625383699355E	1
0.100	.100015643113663784071E	1	.100015682061820951333E	1

Table B-6

THE AUXILIARY FUNCTIONS P(x) AND Λ(x)

1/x	P(x)	Λ(x)
0.001	.999999999781249999763E +0	.999999999781249999335E +0
0.002	.999999998249999984762E +0	.999999998249999957345E +0
0.003	.999999994093749826412E +0	.999999994093749514119E +0
0.004	.999999985999999024668E +0	.999999985999997269999E +0
0.005	.999999972656246279398E +0	.999999972656239585865E +0
0.006	.999999952749988890330E +0	.999999952749968903522E +0
0.007	.999999924968721985538E +0	.999999924968671586191E +0
0.008	.9999998879999375[illegible]E +0	.999999887999825279027E +0
0.009	.999999840531123[illegible]88E +0	.999999840530895789688E +0
0.010	.9999997812497618[illegible]501E +0	.999999781249333488841E +0
0.011	.999999708843328155009E +0	.999999708842569230569E +0
0.012	.999999621999288974203E +0	.999999621998009792566E +0
0.013	.999999519405100631504E +0	.999999519403032836006E +0
0.014	.999999399748207047032E +0	.999999399744981384935E +0
0.015	.999999261716037623473E +0	.999999261711157825009E +0
0.016	.999999103996004930795E +0	.999999103988817421381E +0
0.017	.999998925275502219275E +0	.999998925265161355502E +0
0.018	.999998724241900760758E +0	.999998724227329280520E +0
0.019	.999998499582547018097E +0	.999998499562391395015E +0
0.020	.999998249984759642707E +0	.999998249957340034663E +0
0.021	.999997974135826300118E +0	.999997974090080781451E +0
0.022	.999997670723000323472E +0	.999997670674423089979E +0
0.023	.999997338433497194823E +0	.999997338370070430347E +0
0.024	.999996975954490854151E +0	.999996975872609947070E +0
0.025	.999996581973109835949E +0	.999996581868501633419E +0
0.026	.999996155176433233254E +0	.999996155044067020506E +0
0.027	.999995694251486488943E +0	.999995694085477380366E +0
0.028	.999995197885237014171E +0	.999995197678741442285E +0
0.029	.999994664764589633735E +0	.999994664509692621457E +0
0.030	.999994093576381858186E +0	.999994093263975759069E +0
0.031	.999993483007378982481E +0	.999993482627033372800E +0
0.032	.999992831744269010935E +0	.999992831284091416637E +0
0.033	.999992138473657408264E +0	.999992137920144548874E +0
0.034	.999991401882061676411E +0	.999991401219940907028E +0
0.035	.999990620655905756943E +0	.999990619867966388334E +0
0.036	.999989793481514258668E +0	.999989792548428434460E +0
0.037	.999988919045106510179E +0	.999988917945239318845E +0
0.038	.999987996032790437029E +0	.999987994741998935149E +0
0.039	.999987023130556263119E +0	.999987021621977085050E +0
0.040	.999985999024270035987E +0	.999985997268095263635E +0

Table B-6 (Cont'd)

THE AUXILIARY FUNCTIONS P(x) AND Λ(x)

1/x	P(x)	Λ(x)
0.041	.999984922399666975574E +0	.999984920362907940443E +0
0.042	.999983791942344646042E +0	.999983789588583334184E +0
0.043	.999982606337755950253E +0	.999982603626883678980E +0
0.044	.999981364271201946390E +0	.999981361159144979942E +0
0.045	.999980064427824486290E +0	.999980060866256255665E +0
0.046	.999978705492598674916E +0	.999978701428638265216E +0
0.047	.999977286150325150509E +0	.999977281526221717001E +0
0.048	.999975805085622184793E +0	.999975799838424956775E +0
0.049	.999974260982917602681E +0	.999974255044131131941E +0
0.050	.999972652526440520869E +0	.999972645821664829140E +0
0.051	.999970978400212904628E +0	.999970970848768181979E +0
0.052	.999969237288040942221E +0	.999969228802576445638E +0
0.053	.999967427873506236100E +0	.999967418359593034887E +0
0.054	.999965548839956810302E +0	.999965538195664021959E +0
0.055	.999963598870497933150E +0	.999963586985952090483E +0
0.056	.999961576647982754565E +0	.999961563404909941652E +0
0.057	.999959480855002757079E +0	.999959466126253148428E +0
0.058	.999957310173878019724E +0	.999957293822932453628E +0
0.059	.999955063286647293918E +0	.999955045167105507427E +0
0.060	.999952738875057890358E +0	.999952718830108039642E +0
0.061	.999950335620555375988E +0	.999950313482424462024E +0
0.062	.999947852204273080026E +0	.999947827793657895542E +0
0.063	.999945287307021408003E +0	.999945260432499617457E +0
0.064	.999942639609276962694E +0	.999942610066697922784E +0
0.065	.999939907791171470880E +0	.999939875363026394520E +0
0.066	.999937090532480514683E +0	.999937054987251576797E +0
0.067	.999934186512612066333E +0	.999934147604100044881E +0
0.068	.999931194410594825075E +0	.999931151877224865735E +0
0.069	.999928112905066354908E +0	.999928066469171442600E +0
0.070	.999924940674261021808E +0	.999924890041342736763E +0
0.071	.999921676395997729058E +0	.999921621253963859538E +0
0.072	.999918318747667449212E +0	.999918258766046027092E +0
0.073	.999914866406220551184E +0	.999914801235349870529E +0
0.074	.999911318048153920960E +0	.999911247318348093406E +0
0.075	.999907672349497874259E +0	.999907595670187468481E +0
0.076	.999903927985802859556E +0	.999903844944650165258E +0
0.077	.999900083632125949694E +0	.999899993794114399543E +0
0.078	.999896137963017120339E +0	.999896040869514395929E +0
0.079	.999892089652505313428E +0	.999891984820299653770E +0
0.080	.999887937374084283752E +0	.999887824294393506908E +0

Table B-6 (Cont'd)

THE AUXILIARY FUNCTIONS P(x) AND Λ(x)

1/x	P(x)	Λ(x)
0.081	.999883679800698226639E +0	.999883557938150966977E +0
0.082	.999879315604727184817E +0	.999879184396315839838E +0
0.083	.999874843457972232268E +0	.999874702311977104244E +0
0.084	.999870262031640432961E +0	.999870110326524541476E +0
0.085	.999865569996329572263E +0	.999865407079603604250E +0
0.086	.999860766022012658626E +0	.999860591209069512810E +0
0.087	.999855848778022193292E +0	.999855661350940565672E +0
0.088	.999850316[illegible]3034205460E +0	.999850616139350651963E +0
0.089	.999845669155052050419E +0	.999845454206500951917E +0
0.090	.999840404111389968045E +0	.999840174182610811572E +0
0.091	.999835020468656398894E +0	.999834774695867777108E +0
0.092	.999829516892737055174E +0	.999829254372376773879E +0
0.093	.999823892048777743640E +0	.999823611836108414528E +0
0.094	.999818144601166937464E +0	.999817845708846420022E +0
0.095	.999812273213518094028E +0	.999811954610134136862E +0
0.096	.999806276548651715396E +0	.999805937157220133072E +0
0.097	.999800153268577148242E +0	.999799791965002854935E +0
0.098	.999793902034474119815E +0	.999793517645974325714E +0
0.099	.999787521506674006439E +0	.999787112810162866937E +0
0.100	.999781010344640830953E +0	.999780576065074822044E +0

Table B-7

THE AUXILIARY FUNCTIONS R(z) AND S(z) FOR $0.001 < \frac{1}{x} < 0.100$

$$z = \frac{3}{2} x^{-3/2}$$

1/x	R(z)		S(z)	
0.001	.999996706044316974154E	+0	.100000329412278372087E	1
0.002	.999990683718404807046E	+0	.100000931761840078586E	1
0.003	.999982885925368675993E	+0	.100001711858635049952E	1
0.004	.999973653031312582062E	+0	.100002635766313425321E	1
0.005	.999963181959944295300E	+0	.100003683892765163054E	1
0.006	.999951605742188634138E	+0	.100004843035158859894E	1
0.007	.999939022171776845389E	+0	.100006103514383002080E	1
0.008	.999925507131627377127E	+0	.100007457842408119226E	1
0.009	.999911121769353332558E	+0	.100008900004736307109E	1
0.010	.999895916755744953029E	+0	.100010425034553319610E	1
0.011	.999879934995274825762E	+0	.100012028741678355216E	1
0.012	.999863213445223136268E	+0	.100013707530651981104E	1
0.013	.999845784388469026646E	+0	.100015458273457413725E	1
0.014	.999827676354485754845E	+0	.100017278217421491346E	1
0.015	.999808914804533461256E	+0	.100019164916695922976E	1
0.016	.999789522653448489470E	+0	.100021116180078923981E	1
0.017	.999769520674973525404E	+0	.100023130030482812003E	1
0.018	.999748927822066802836E	+0	.100025204672903738393E	1
0.019	.999727761483836496226E	+0	.100027338468728943126E	1
0.020	.999706037694367636820E	+0	.100029529914854799589E	1
0.021	.999683771304434716976E	+0	.100031777626515832827E	1
0.022	.999660976124207262133E	+0	.100034080323017483321E	1
0.023	.999637665042867553095E	+0	.100036436815770198081E	1
0.024	.999613850129878920813E	+0	.100038845998168506899E	1
0.025	.999589542721233299219E	+0	.100041306836964715242E	1
0.026	.999564753493506198034E	+0	.100043818364864897675E	1
0.027	.999539492527824228122E	+0	.100046379674133178612E	1
0.028	.999513769365449378883E	+0	.100048989911034380515E	1
0.029	.999487593056341771314E	+0	.100051648270978867275E	1
0.030	.999460972201801646059E	+0	.100054353994259506906E	1
0.031	.999433914992087499753E	+0	.100057106362291062329E	1
0.032	.999406429239746581074E	+0	.100059904694278389205E	1
0.033	.999378522409266259964E	+0	.100062748344252589558E	1
0.034	.999350201643552427022E	+0	.100065636698424505633E	1
0.035	.999321473787659500336E	+0	.100068569172813196345E	1
0.036	.999292345410121478887E	+0	.100071545211113752580E	1
0.037	.999262822822206591750E	+0	.100074564282774296520E	1
0.038	.999232912095316909374E	+0	.100077625881256528545E	1
0.039	.999202619076786880249E	+0	.100080729522457925512E	1
0.040	.999171949404247669120E	+0	.100083874743276802975E	1

Table B-7 (Cont'd)

THE AUXILIARY FUNCTIONS R(z) AND S(z) FOR $0.001 < \frac{1}{x} < 0.100$

$$z = \frac{3}{2} x^{-3/2}$$

1/x	R(z)		S(z)	
0.041	.999140908518726127682E	+0	.100087061100304058426E	1
0.042	.999109501676617385035E	+0	.100090288168627597000E	1
0.043	.999077733960652577013E	+0	.100093555540737287946E	1
0.044	.999045610289967587331E	+0	.100096862825519864757E	1
0.045	.999013135429365362601E	+0	.100100209647334512980E	1
0.046	.998980313997852957987E	+0	.100103595645161030261E	1
0.047	.998947150476524730200E	+0	.100107020471813417167E	1
0.048	.998913649215854668103E	+0	.100110483793212600048E	1
0.049	.998879814442453594916E	+0	.100113985287712712758E	1
0.050	.998845650265340694771E	+0	.100117524645475997281E	1
0.051	.998811160681773325043E	+0	.100121101567891892367E	1
0.052	.998776349582674329120E	+0	.100124715767036494032E	1
0.053	.998741220757691878780E	+0	.100128366965168710324E	1
0.054	.998705777899924227707E	+0	.100132054894260147523E	1
0.055	.998670024610330545890E	+0	.100135779295555806138E	1
0.056	.998633964401874164496E	+0	.100139539919163039123E	1
0.057	.998597600703380072953E	+0	.100143336523666833483E	1
0.058	.998560936863204280178E	+0	.100147168875768304493E	1
0.059	.998523976152586703788E	+0	.100151036749946286535E	1
0.060	.998486721768921488180E	+0	.100154939928138580901E	1
0.061	.998449176838749102530E	+0	.100158878199442375882E	1
0.062	.998411344420610184098E	+0	.100162851359832093084E	1
0.063	.998373227507731833488E	+0	.100166859211893439759E	1
0.064	.998334829030563969005E	+0	.100170901564572506961E	1
0.065	.998296151859176336202E	+0	.100174978232938854403E	1
0.066	.998257198805525870971E	+0	.100179089037961612746E	1
0.067	.998217972625603301237E	+0	.100183233806297715348E	1
0.068	.998178476021467143016E	+0	.100187412370091444487E	1
0.069	.998138711643172580217E	+0	.100191624566784543723E	1
0.070	.998098682090602120028E	+0	.100195870238936207828E	1
0.071	.998058389915204372183E	+0	.100200149234052316110E	1
0.072	.998017837621646803113E	+0	.100204461404423324714E	1
0.073	.997977027669387874810E	+0	.100208806606970277574E	1
0.074	.997935962474173559859E	+0	.100213184703098437651E	1
0.075	.997894644409462855858E	+0	.100217595558558076838E	1
0.076	.997853075807786585474E	+0	.100222039043311996706E	1
0.077	.997811258962043448125E	+0	.100226515031409384299E	1
0.078	.997769196126737014349E	+0	.100231023400865634701E	1
0.079	.997726889519157085846E	+0	.100235564033547798950E	1
0.080	.997684341320508607669E	+0	.100240136815065339549E	1

Table B-7 (Cont'd)

THE AUXILIARY FUNCTIONS R(z) AND S(z) FOR $0.001 < \frac{1}{x} < 0.100$

$$z = \frac{3}{2} x^{-3/2}$$

1/x	R(z)		S(z)	
0.081	.997641553676991098051E	+0	.100244741634665897950E	1
0.082	.997598528700831362186E	+0	.100249378385135798362E	1
0.083	.997555268471272064334E	+0	.100254046962705031426E	1
0.084	.997511775035518566080E	+0	.100258747266956478032E	1
0.085	.997468050409646279754E	+0	.100263479200739149446E	1
0.086	.997424096579470635959E	+0	.100268242670085234955E	1
0.087	.997379915501381635946E	+0	.100273037584130761143E	1
0.088	.997335509103144827961E	+0	.100277863855039680045E	1
0.089	.997290879284670435291E	+0	.100282721397931214695E	1
0.090	.997246027918752255024E	+0	.100287610130810301404E	1
0.091	.997200956851777848249E	+0	.100292529974500978088E	1
0.092	.997155667904411448657E	+0	.100297480852582577289E	1
0.093	.997110162872250931308E	+0	.100302462691328591172E	1
0.094	.997064443526460105390E	+0	.100307475419648083615E	1
0.095	.997018511614377516583E	+0	.100312518969029532363E	1
0.096	.996972368860102832165E	+0	.100317593273486990545E	1
0.097	.996926016965062207643E	+0	.100322698269508464349E	1
0.098	.996879457608552585971E	+0	.100327833896006408477E	1
0.099	.996832692448267613211E	+0	.100333000094270247786E	1
0.100	.996785723120804309165E	+0	.100338196807920838153E	1

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Table B-8

THE AUXILIARY FUNCTIONS W(z) AND X(z) FOR $0.001 < \frac{1}{x} < 0.100$

$$z = \frac{3}{2} x^{-3/2}$$

1/x	-W(z)		X(z)	
0.001	.100000461155618426153E	1	.999995388246333098952E	+0
0.002	.100001304294004069911E	1	.999986955480098149171E	+0
0.003	.100002396019651080134E	1	.999976034471457486143E	+0
0.004	.100003688692224322523E	1	.999963100438865287156E	+0
0.005	.100005154753312518936E	1	.999948427781535036163E	+0
0.006	.100006775589480561676E	1	.999932201448914782072E	+0
0.007	.100008537520485434655E	1	.999914557058525871345E	+0
0.008	.100010429933581487882E	1	.999895599552907690710E	+0
0.009	.100012444278948097354E	1	.999875413245338098656E	+0
0.010	.100014573473500965321E	1	.999854067781716996471E	+0
0.011	.100016811521420211575E	1	.999831621935281531894E	+0
0.012	.100019153259471873185E	1	.999808126153386761215E	+0
0.013	.100021594178816099418E	1	.999783624339418902219E	+0
0.014	.100024130296066914323E	1	.999758155142192453271E	+0
0.015	.100026758057364351516E	1	.999731752915223132639E	+0
0.016	.100029474265323113972E	1	.999704448447235110171E	+0
0.017	.100032276022285559193E	1	.999676269529624527106E	+0
0.018	.100035160685477649348E	1	.999647241404892908021E	+0
0.019	.100038125831037406666E	1	.999617387126355061586E	+0
0.020	.100041169224778448295E	1	.999586727850495727849E	+0
0.021	.100044288798148850243E	1	.999555283077372475300E	+0
0.022	.100047482628255224192E	1	.999523070850366009566E	+0
0.023	.100050748921108616210E	1	.999490107923711802827E	+0
0.024	.100054085997453350402E	1	.999456409904201813127E	+0
0.025	.100057492280688296645E	1	.999421991371961501435E	+0
0.026	.100060966286499322053E	1	.999386865984114549495E	+0
0.027	.100064506613903303750E	1	.999351046564331501767E	+0
0.028	.100068111937465816540E	1	.999314545180641194974E	+0
0.029	.100071781000501851382E	1	.999277373213411415300E	+0
0.030	.100075512609105458103E	1	.999239541415039847896E	+0
0.031	.100079305626882745151E	1	.999201059962610989880E	+0
0.032	.100083158970285163933E	1	.999161938504549750181E	+0
0.033	.100087071604457886893E	1	.999122186202123643599E	+0
0.034	.100091042539532416238E	1	.999081811766502208867E	+0
0.035	.100095070827304121897E	1	.999040823491966663440E	+0
0.036	.100099155558244806806E	1	.998999229285768812875E	+0
0.037	.100103295858808082171E	1	.998957036695061386631E	+0
0.038	.100107490888991661411E	1	.998914252931258711955E	+0
0.039	.100111739840125916437E	1	.998870884892134287400E	+0
0.040	.100116041932862394704E	1	.998826939181918270059E	+0

Table B-8 (Cont'd)

THE AUXILIARY FUNCTIONS W(z) AND X(z) FOR $0.001 < \frac{1}{x} < 0.100$

$$z = \frac{3}{2} x^{-3/2}$$

1/x	-W(z)		X(z)	
0.041	.100120396415339638908E	1	.998782422129621455839E	+0
0.042	.100124802561506711929E	1	.998737339805781724922E	+0
0.043	.100129259669587413282E	1	.998691698037803087532E	+0
0.044	.100133767060670364548E	1	.998645502424035553434E	+0
0.045	.100138324077412005782E	1	.998598758346725402663E	+0
0.046	.100142930082841138970E	1	.998551470983949494440E	+0
0.047	.100147584459255021698E	1	.998503645320633580165E	+0
0.048	.100152286607198191074E	1	.998455286158742817448E	+0
0.049	.100157035944516214688E	1	.998406398126722514442E	+0
0.050	.100161831905477446404E	1	.998356985688258323803E	+0
0.051	.100166673939956630278E	1	.998307053150417450084E	+0
0.052	.100171561512674863814E	1	.998256604671225755511E	+0
0.053	.100176494102491015642E	1	.998205644266729809768E	+0
0.054	.100181471201740203703E	1	.998154175817587819602E	+0
0.055	.100186492315615390977E	1	.998102203075228864081E	+0
0.056	.100191556961588551099E	1	.998049729667615908557E	+0
0.057	.100196664668868207173E	1	.997996759104644559985E	+0
0.058	.100201814977890456678E	1	.997943294783206430654E	+0
0.059	.100207007439840870482E	1	.997889339991943226043E	+0
0.060	.100212241616204898918E	1	.997834897915715222424E	+0
0.061	.100217517078344635635E	1	.997779971639805622805E	+0
0.062	.100222833407099984751E	1	.997724564153880330859E	+0
0.063	.100228190192412450886E	1	.997668678355720942091E	+0
0.064	.100233587032969928113E	1	.997612317054747186691E	+0
0.065	.100239023535871003537E	1	.997555482975343661380E	+0
0.066	.100244499316307418008E	1	.997498178760004419678E	+0
0.067	.100250013997263439427E	1	.997440406972307860016E	+0
0.068	.100255567209231007240E	1	.997382170099733319749E	+0
0.069	.100261158589939599151E	1	.997323470556329858230E	+0
0.070	.100266787784099855207E	1	.997264310685246871075E	+0
0.071	.100272454443160070285E	1	.997204692761135418313E	+0
0.072	.100278158225074735537E	1	.997144618992428453833E	+0
0.073	.100283898794084371928E	1	.997084091523507517385E	+0
0.074	.100289675820505956235E	1	.997023112436762877955E	+0
0.075	.100295488980533292412E	1	.996961683754553591359E	+0
0.076	.100301337956046728406E	1	.996899607441073463186E	+0
0.077	.100307222434431662492E	1	.996837485404128467794E	+0
0.078	.100313142108405322892E	1	.996774719496830776928E	+0
0.079	.100319096675851340988E	1	.996711511519214185685E	+0
0.080	.100325085839661671898E	1	.996647863219775388840E	+0

Table B-8 (Cont'd)

THE AUXILIARY FUNCTIONS W(z) AND X(z) FOR $0.001 < \frac{1}{x} < 0.100$

$$z = \frac{3}{2} x^{-3/2}$$

1/x	-W(z)		X(z)	
0.081	.10033110930758544 7332E	1	.99658377629694524817 1E	+0
0.082	.10033716679208437 3326E	1	.99651925240049391507 1E	+0
0.083	.10034325801019431 2199E	1	.99645429313287340409 4E	+0
0.084	.10034938268339271 1620E	1	.99638890005050097781 4E	+0
0.085	.10035554053747156 5970E	1	.99632307466498647979 6E	+0
0.086	.10036173130241561 5695E	1	.99625681844430654688 5E	+0
0.087	.10036795471228550 9126E	1	.99619013281392844402 6E	+0
0.088	.10037421050510566 8855E	1	.99612301915788608789 5E	+0
0.089	.10038049842275662 0825E	1	.99605547881981066492 9E	+0
0.090	.10038681821087155 9301E	1	.99598751310391809833 1E	+0
0.091	.10039316961873693 4968E	1	.99591912327595547759 3E	+0
0.092	.10039955239919686 6033E	1	.99585031056410843717 7E	+0
0.093	.10040596630856118 4473E	1	.99578107615987134785 2E	+0
0.094	.10041241110651694 0665E	1	.99571142121888207253 5E	+0
0.095	.10041888655604319 9767E	1	.99564134686172293682 1E	+0
0.096	.10042539242332897 3409E	1	.99557085417468946154 6E	+0
0.097	.10043192847769413 8680E	1	.99549994421052832011 6E	+0
0.098	.10043849449151320 5318E	1	.99542861798914589338 1E	+0
0.099	.10044509024014179 9453E	1	.99535687649828871947 0E	+0
0.100	.10045171550184573 9878E	1	.99528472069419705933 6E	+0

Table B-9

THE AIRY FUNCTIONS Ai(-x) AND Bi(-x) FOR 0 < x < 10

x	Ai(-x)	Bi(-x)
0.0	.355028053887817239256E +0	.614926627446000735147E +0
0.1	.380848668120121511779E +0	.569999043002954860436E +0
0.2	.406284187444801410362E +0	.524509032818485512824E +0
0.3	.430903095285580858271E +0	.477977840109892946757E +0
0.4	.454225613888667398390E +0	.430020939948503375191E +0
0.5	.475728091610539588798E +0	.380352659751053850170E +0
0.6	.494849525431149688296E +0	.328791840760869438206E +0
0.7	.511000397575010149392E +0	.275268011987879698653E +0
0.8	.523573949705774008352E +0	.219827510897581810416E +0
0.9	.531959945610973913548E +0	.162638948925966249326E +0
1.0	.535560883292352118801E +0	.103997389496944611887E +0
1.1	.533810510430502159809E +0	.443265867408266030104E -1
1.2	.526194374802120073867E +0	-.158213701846320836897E -1
1.3	.512272006041030930879E +0	-.757696441503407035344E -1
1.4	.491700181061290728603E +0	-.134724060952794752050E +0
1.5	.464256577748869406479E +0	-.191784861157041220017E +0
1.6	.429862976769135179190E +0	-.245963200211373485937E +0
1.7	.388607037396328737307E +0	-.296202657610495735319E +0
1.8	.340761559124213953313E +0	-.341405831830134963927E +0
1.9	.286800562794908440067E +0	-.380465877453648821742E +0
2.0	.227407428201685575994E +0	-.412302587956398488088E +0
2.1	.163484512999292737342E +0	-.435902348230726745418E +0
2.2	.961453780076690018041E -1	-.450360984168207312412E +0
2.3	.267063330573568456604E -1	-.454928234394364981659E +0
2.4	-.433341404403095142049E -1	-.449052276282107513898E +0
2.5	-.112325067692966089190E +0	-.432422471840705293030E +0
2.6	-.178502428936403343801E +0	-.405008278130032558032E +0
2.7	-.240038109742457254467E +0	-.367092111821007818944E +0
2.8	-.295097592999208665413E +0	-.319293888938312099924E +0
2.9	-.341905095672983040965E +0	-.262584998164697014933E +0
3.0	-.378814293677658074350E +0	-.198289626374926543211E +0
3.1	-.404382222390978324288E +0	-.128071651289754540310E +0
3.2	-.417443420564151365179E +0	-.539057556305392836512E -1
3.3	-.417180937374550128771E +0	.219679998977745437773E -1
3.4	-.403190484245899727402E +0	.971061905486607192532E -1
3.5	-.375533823140431911932E +0	.168939837481058611851E +0
3.6	-.334777477474821899786E +0	.234866306247767034975E +0
3.7	-.282013061841931501738E +0	.292352610071451994721E +0
3.8	-.218855975618855783816E +0	.339046470756664082991E +0
3.9	-.147419905640744195021E +0	.372890578319395619318E +0
4.0	-.702655329492895150957E -1	.392234705706999289558E +0

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Table B-9 (Cont'd)

THE AIRY FUNCTIONS Ai(-x) AND Bi(-x) FOR 0 < x < 10

x	Ai(-x)	Bi(-x)
4.1	.967697951871433283138E -2	.395939740740138921462E +0
4.2	.892107632394505790145E -1	.383467361270944661885E +0
4.3	.164997809272772304379E +0	.354949062610930446158E +0
4.4	.233703258073163129975E +0	.311228599006395781249E +0
4.5	.292152781055359466887E +0	.253872657696932636790E +0
4.6	.337495975489462863187E +0	.185145757947212677030E +0
4.7	.367367482877331330869E +0	.107946945548553655293E +0
4.8	.380036676642792938416E +0	.257077933650826925314E -1
4.9	.374536354705838799451E +0	-.577465531404353120349E -1
5.0	.350761007024114319779E +0	-.138369134901600576375E +0
5.1	.309525996287317515927E +0	-.212089131569036475936E +0
5.2	.252580338104744734695E +0	-.275027044189640774021E +0
5.3	.182567931068339499387E +0	-.323716076748792403992E +0
5.4	.102934596110134652261E +0	-.355317080080467545504E +0
5.5	.177615412765749755965E -1	-.367813453915711991096E +0
5.6	-.683306996861678639290E -1	-.360172225437713344244E +0
5.7	-.150620157978535293745E +0	-.332458253187901684319E +0
5.8	-.224351916334958875935E +0	-.285890211049324246850E +0
5.9	-.285122779555179903034E +0	-.222829692947813693366E +0
6.0	-.329145173629823105240E +0	-.146698376670557037848E +0
6.1	-.353511676120964874673E +0	-.618225481962808902104E -1
6.2	-.356421073668961431057E +0	.267908089653358262769E -1
6.3	-.337347649216135062331E +0	.113737009008197976896E +0
6.4	-.297137622136627796395E +0	.193541360615780175119E +0
6.5	-.238020301997115803576E +0	.261012657636483951842E +0
6.6	-.163526462727729552083E +0	.311599945811209548296E +0
6.7	-.783124718012559191819E -1	.341727738606750260127E +0
6.8	.121045242773650381496E -1	.349084179040394786340E +0
6.9	.101687397739764515792E +0	.332837842983428733087E +0
7.0	.184280835250505637301E +0	.293762071854414020110E +0
7.1	.254036328561978364411E +0	.234250878329856827500E +0
7.2	.305851523368626571561E +0	.158217390090497692915E +0
7.3	.335770370515147308967E +0	.708741137698963122107E -1
7.4	.341323752232338641094E +0	-.215965185718835998981E -1
7.5	.321775716380647875261E +0	-.112463485076490806408E +0
7.6	.278250234880197330050E +0	-.194933756473876470951E +0
7.7	.213720373789192980439E +0	-.262670068805394676388E +0
7.8	.132851544626067326912E +0	-.310300566147412082315E +0
7.9	.417018836173870433440E -1	-.333878563003046910549E +0
8.0	-.527050503563862026505E -1	-.331251580751137859963E +0

Table B-9 (Cont'd)

THE AIRY FUNCTIONS Ai(-x) AND Bi(-x) FOR 0 < x < 10

x	Ai(-x)	Bi(-x)
8.1	-.142908147093581424377E +0	-.302303309060702033165E +0
8.2	-.221599454803604415167E +0	-.249040191297943596885E +0
8.3	-.282231759958830619293E +0	-.175505563096446039117E +0
8.4	-.319597189726107978383E +0	-.875179820004110091821E -1
8.5	-.330290237630203879022E +0	.775443644765840446080E -2
8.6	-.313112452617262463685E +0	.102356470012673830991E +0
8.7	-.269204540700509324562E +0	.188203626247790888932E +0
8.8	-.202054447376745762278E +0	.257782401703263694214E +0
8.9	-.117266306371752132940E +0	.304832413364962959881E +0
9.0	-.221337215473414036634E -1	.324947323455244917922E +0
9.1	.749598872735548019466E -1	.316034712393298914417E +0
9.2	.165268004651479631477E +0	.278584254357115239686E +0
9.3	.240473796853185974428E +0	.215708345763898821874F [illegible]
9.4	.293477556112067757598E +0	.132938761140427214287E [illegible]
9.5	.319103247719128201379E +0	.377854324894665022643E -[illegible]
9.6	.314651583311693258632E +0	-.609129273601140532029E -1
9.7	.280237501916297438275E +0	-.153794208777252970928E +0
9.8	.218867432663286421209E +0	-.231863308817480920680E +0
9.9	.136235026447979751373E +0	-.287383557724355170119E +0
10.0	.402412384864431906390E -1	-.314679829643838633171E +0

Table B-10

AIRY FUNCTION DERIVATIVES Ai'(-x) AND Bi'(-x) FOR 0 < x < 10

x	Ai'(-x)	Bi'(-x)
0.0	-.258819403792806798405E +0	.448288357353826357914E +0
0.1	-.256958112323646174612E +0	.451213362293461242095E +0
0.2	-.251032674005547812657E +0	.459385294586834096588E +0
0.3	-.240545127258154608736E +0	.471880216300647919995E +0
0.4	-.225031409302415031574E +0	.487734864049147533881E +0
0.5	-.204081670339547386146E +0	.505933713623847166571E +0
0.6	-.177362598696566032997E +0	.525401152299153565907E +0
0.7	-.144641285643321027402E +0	.544999120069181906657E +0
0.8	-.105809991187968867529E +0	.563530594596331154361E +0
0.9	-.609109980021988362488E -1	.579749261265281444695E +0
1.0	-.101605671166452093956E -1	.592375626422792350818E +0
1.1	.460291547106078656159E -1	.600119696489910873156E +0
1.2	.107031569272280793684E +0	.601710157437464408877E +0
1.3	.171991806753774033071E +0	.595929749409011227163E +0
1.4	.239819119936296593973E +0	.581656243921938517930E +0
1.5	.309186967202410420421E +0	.557908103021897354135E +0
1.6	.378542191951880827041E +0	.523893541446409847927E +0
1.7	.446124554636075052690E +0	.479061338473447817394E +0
1.8	.509997627719706648394E +0	.423151373738689289482E +0
1.9	.568091716812299215807E +0	.356242512634247698662E +0
2.0	.618259020741691041414E +0	.278795166921169522688E +0
2.1	.658340692814343420645E +0	.191685632328429868240E +0
2.2	.686244824909001709826E +0	.962291859385645307700E -1
2.3	.700033662876576002359E +0	-.581105930705154359886E -2
2.4	.698017601544441867077E +0	-.112232370963516179458E +0
2.5	.678952734264794363371E +0	-.220420154874629587690E +0
2.6	.641637987111061794889E +0	-.327397168707100529084E +0
2.7	.586007200144331444102E +0	-.429895343082015062996E +0
2.8	.512210981543479350035E +0	-.524450397432683101777E +0
2.9	.421182811603648366855E +0	-.607518287971096867234E +0
3.0	.314583769216598813646E +0	-.675611222685258537678E +0
3.1	.194820446003978903684E +0	-.725449571508164802997E +0
3.2	.650311469952631513669E -1	-.754124553110841361057E +0
3.3	-.709636171778361286629E -1	-.759265117504794454976BE +0
3.4	-.208749049752733243212E +0	-.739201625105518170704E +0
3.5	-.343443433454048146305E +0	-.693116284907288801749E +0
3.6	-.469863966303251126647E +0	-.621172826642320206678E +0
3.7	-.582727803652957978040E +0	-.524613614909683491062E +0
3.8	-.676882571408873634068E +0	-.405815920628965598951E +0
3.9	-.747558085535477471988E +0	-.268298362892145678982E +0
4.0	-.790628575368581380301E +0	-.116670567438340893664E +0

Table B-10 (Cont'd)

AIRY FUNCTION DERIVATIVES Ai′(-x) AND Bi′(-x) FOR 0 < x < 10

x	Ai′(-x)	Bi′(-x)
4.1	-.80287253541821499O189E +0	.43478717931659297743lE -1
4.2	-.78221560786245196399 8E +0	.20575691122112262390OE +0
4.3	-.72794081383881804657OE +0	.36320468095665562906 8E +0
4.4	-.64085018328756365186 9E +0	.50858932154467567508 3E +0
4.5	-.52336253231574770069 7E +0	.63474476777366370975 4E +0
4.6	-.37953391433584533571 5E +0	.73494444367130626644 8E +0
4.7	-.21499018008957243134 1E +0	.80328925850719965966OE +0
4.8	-.36765104312346077472 5E -1	.83508975839556434895 3E +0
4.9	.14695742731095607499OE +0	.82721903336068038276 4E +0
5.0	.32719281855444313684 4E +0	.77841177300189924607 4E +0
5.1	.49458599838493705510OE +0	.68948512842205131003 2E +0
5.2	.63990516690128384335 6E +0	.56345897957517367729 4E +0
5.3	.75457541994701104103 6E +0	.40555694088331552818 2E +0
5.4	.83122307200334625868OE +0	.22307496421783366878OE +0
5.5	.86419721777139839077 9E +0	.25111583073630925961 6E -1
5.6	.85003256004893156261 6E +0	-.17783759557946047209 7E +0
5.7	.78781722463519971378 1E +0	-.37440903400372219022 6E +0
5.8	.67943151996650276407 2E +0	-.55300203169101126131 3E +0
5.9	.52962857256300237831 4E +0	-.70247952231549159994 6E +0
6.0	.34593548728134289487 5E +0	-.81289878510506700045 2E +0
6.1	.13836393725271684847 2E +0	-.87622530153251547901 6E +0
6.2	-.81068556196304158032 7E -1	-.88697896206682812795OE +0
6.3	-.29899160898473956420 8E +0	-.84276109680897025718 9E +0
6.4	-.50147985025496865641 8E +0	-.74461386630573286775 8E +0
6.5	-.67495249251320217304 6E +0	-.59717066629162201692OE +0
6.6	-.80711924947739184717 9E +0	-.40856733931004221565 7E +0
6.7	-.88790796525555354459 3E +0	-.19009877716374965990 5E +0
6.8	-.91030400515880440593 5E +0	.44376784534242448915 7E -1
6.9	-.87103105868638765790 4E +0	.27926390596966473839 5E +0
7.0	-.77100816841012654770 2E +0	.49824459005811348880 5E +0
7.1	-.61552878754022817213OE +0	.68542057764269389711 1E +0
7.2	-.41412428115703515910 5E +0	.82650634027200478553OE +0
7.3	-.18009580448329322433 7E +0	.90998427043632467365 1E +0
7.4	.70276323643265952354 2E -1	.92812809007040669312OE +0
7.5	.31880950669855459626 2E +0	.87780228154576092235 2E +0
7.6	.54671881905734806952OE +0	.76095509188391053863 6E +0
7.7	.73605241760464206731 6E +0	.58474044748990395682OE +0
7.8	.87115540424658935835 3E +0	.36122930432440057268 7E +0
7.9	.94004299802628012645 4E +0	.10670215481213814453 4E +0
8.0	.93556093819830655101 4E +0	-.15945049781298138945 1E +0

Table B-10 (Cont'd)

AIRY FUNCTION DERIVATIVES Ai'(-x) and Bi'(-x) FOR 0 < x < 10

x	Ai'(-x)	Bi'(-x)
8.1	.85621858632862456239OE +0	-.41615663954012758140 4E +0
8.2	.70659869786280530337 7E +0	-.64232293090842293392 6E +0
8.3	.49727679025320957682 5E +0	-.81860044074879846489 5E +0
8.4	.24422089414528412452 9E +0	-.92910958374875022901 5E +0
8.5	-.32313348284639135941 3E -1	-.96296916512017479814 4E +0
8.6	-.30933027241563230748 1E +0	-.91547918019618345270 6E +0
8.7	-.56297684950185465967 3E +0	-.78882622507556771054 8E +0
8.8	-.77061300974804222200 0E +0	-.59221370922880893161 4E +0
8.9	-.91289275742524983188 1E +0	-.34136475372178074790 4E +0
9.0	-.97566398092633159473 0E +0	-.57400513843669254343 0E -1
9.1	-.95149681545191769749 9E +0	.23484378658496913225 7E +0
9.2	-.84067107380380081818 7E +0	.50894401554578444920 8E +0
9.3	-.65149240789560037831 7E +0	.73928028302987294986 1E +0
9.4	-.39986236638984482277 6E +0	.90348537067684153705 9E +0
9.5	-.10809531881187123901 1E +0	.98471407000211970393 3E +0
9.6	.19695044232125911406 8E +0	.97349917954711307131 0E +0
9.7	.48628629123926820905 6E +0	.86898387659821058071 7E +0
9.8	.73154486362832362255 7E +0	.67936774213113309118 7E +0
9.9	.90781333153715043912 6E +0	.42147208918383337612 4E +0
10.0	.99626504413279005593 7E +0	.11941411339990923809 3E +0

Table B-11

THE AUXILIARY FUNCTIONS F(-x) AND χ(-x) FOR 0 < x < 10

x	F(-x)	χ(-x)
0.0	.710056107775634478512E +0	.523598775598298873077E +0
0.1	.685525066670180080129E +0	.589033441900681614322E +0
0.2	.663458036710586271510E +0	.659059510580702280188E +0
0.3	.643537328492149523686E +0	.733650221730043295678E +0
0.4	.625490940867036941095E +0	.812764547904918292785E +0
0.5	.609085677821361095074E +0	.896350013442875280481E +0
0.6	.594121306948635690701E +0	.984345226848848781909E +0
0.7	.580425606555722108979E +0	.107668209984391649132E 1
0.8	.567850169814125194669E +0	.117328775043571503178E 1
0.9	.556266852726435050947E +0	.127408610380105735195E 1
1.0	.545564768597701525318E +0	.137899921506759781326E 1
1.1	.535647745573670494003E +0	.148794834359307037018E 1
1.2	.526432175904848336978E +0	-.154073784323803616388E 1
1.3	.517845196123292515510E +0	-.142395198393784815224E 1
1.4	.509823146449156858298E +0	-.130336342908260226106E 1
1.5	.502310265624860191582E +0	-.117904724135731159251E 1
1.6	.495257584146918691863E +0	-.105107656988233757919E 1
1.7	.488621984656310986247E +0	-.919522457028157744873E +0
1.8	.482365403179365215416E +0	-.784453714098194577901E +0
1.9	.476454149093587008897E +0	-.645936851458430438161E +0
2.0	.470858325228351985482E +0	-.504036051170643780640E +0
2.1	.465551332490497421645E +0	-.358813172339182537158E +0
2.2	.460509445910932254409E +0	-.210327781219446132354E +0
2.3	.455711451111934158422E +0	-.586371996839580786613E -1
2.4	.451138331957995658548E +0	.962034330697882460879E -1
2.5	.446773001629524359927E +0	.254141089623340423406E +0
2.6	.442600070594267240281E +0	.415124777438138059409E +0
2.7	.438605645985022929667E +0	.579105458432873516396E +0
2.8	.434777157757025902729E +0	.746035966948005235782E +0
2.9	.431103207722124959443E +0	.915870927551718309747E +0
3.0	.427573438162862196197E +0	.108856667377772722900E 1
3.1	.424178417237311989360E +0	.126408116859677727289E 1
3.2	.420909538811367838258E +0	.144237392719625234522E 1
3.3	.417758934712663181748E +0	-.151818671112566769918E 1
3.4	.414719397700801934256E +0	-.133445304015490477230E 1
3.5	.411784313701476116241E +0	-.114805397674165331223E 1
3.6	.408947602065197072678E +0	-.959024511701029668519E +0
3.7	.406203662791279537130E +0	-.767398425473925377930E +0
3.8	.403547329809813845690E +0	-.573208347048091910950E +0
3.9	.400973829543150160472E +0	-.376485809801573574456E +0
4.0	.398478744077653446360E +0	-.177261304364624520849E +0

Table B-11 (Cont'd)

THE AUXILIARY FUNCTIONS F(-x) AND χ(-x) FOR 0 < x < 10

x	F(-x)	χ(-x)
4.1	.396057978369296382848E +0	.244356713875101403574E -1
4.2	.393707730985646885583E +0	.228576565088177621198E +0
4.3	.391424467954153791204E +0	.435133725250592521629E +0
4.4	.389204899344157008727E +0	.644080359125642339235E +0
4.5	.387045958259268376954E +0	.855390492915135155137E +0
4.6	.384944731958960945037E +0	.106903893421136745707E 1
4.7	.382898694864436350938E +0	.128500123653722583673E 1
4.8	.380905193235010317135E +0	.150325366586533351868E 1
4.9	.378961931328120366931E +0	-.141781948459009825537E 1
5.0	.377066708879254078016E +0	-.119505530988115750886E 1
5.1	.375217459758157524567E +0	-.970068243086399816426E +0
5.2	.373412241675065848442E +0	-.742879467627442658344E +0
5.3	.371649226825785307056E +0	-.513509596083290558236E +0
5.4	.369926693377574382492E +0	-.281978694052541206621E +0
5.5	.368243917709197207595E +0	-.483063027368412350818E -1
5.6	.366596667328490950681E +0	.187488539675966737614E +0
5.7	.364986194399500362183E +0	.425387277755472592994E +0
5.8	.363410229818858493547E +0	.665371819630237683763E +0
5.9	.361867477787778881352E +0	.907424518782799862764E +0
6.0	.360356710831896444993E +0	.115152815677787177236E 1
6.1	.358876765226359340483E +0	.139766592686738502200E 1
6.2	.357426536738124267050E +0	-.149577123517007524297E 1
6.3	.356004977001422578536E +0	-.124561405146589531702E 1
6.4	.354611089445912374368E +0	-.993470837666995155459E +0
6.5	.353243926500171540619E +0	-.739356901957424034469E +0
6.6	.351902586295969483520E +0	-.483287211383032657170E +0
6.7	.350586209901225201809E +0	-.225276403977977863785E +0
6.8	.349293978711754666645E +0	.346611996820764209914E -1
6.9	.348025112033864465506E +0	.296511585467276782088E +0
7.0	.346778864841590233409E +0	.560261035190751971535E +0
7.1	.345554525693932724621E +0	.825896116586953381768E +0
7.2	.344351414798833436348E +0	.109340367374431943777E 1
7.3	.343168882211874670130E +0	.136277081796713253701E 1
7.4	.342006306158802590470E +0	-.150760773454676114225E 1
7.5	.340863091471970915822E +0	-.123455905669566127076E -1
7.6	.339738668131700251165E +0	-.359687939998720468133E +0
7.7	.338632489904355071462E +0	-.683006287879166402595E +0
7.8	.337544033069666956045E +0	-.404525776571120930567E +0
7.9	.336472795230487629845E +0	-.124257862128323433879E +0
8.0	.335418294198746428144E +0	.157786212859422630064E +0

Table B-11 (Cont'd)

THE AUXILIARY FUNCTIONS F(-x) AND χ(-x) FOR 0 < x < 10

x	F(-x)	χ(-x)
8.1	.334380066951920800187E +0	.441595412827024424980E +0
8.2	.333357668654311445829E +0	.727158903057944906811E +0
8.3	.332350671741850987115E +0	.101446604375781170029E 1
8.4	.331358665055571453111E +0	.130350638436527055439E 1
8.5	.330381253037215509795E +0	-.154732299550149366855E 1
8.6	.329418054965803033545E +0	-.125484687693490608596E 1
8.7	.328468704242261632688E +0	-.960667828322174910235E +0
8.8	.327532847715500026932E +0	-.664795595837479370929E +0
8.9	.326610145047547439190E +0	-.367239751698365639565E +0
9.0	.325700268115122694818E +0	-.680097486563611605510E -1
9.1	.324802900445145664357E +0	.232885175680120763276E +0
9.2	.323917736682002911457E +0	.535435894682767491443E +0
9.3	.323044482084410603647E +0	.839633438692856989493E +0
9.4	.322182852049986421994E +0	.114546898115626742348E 1
9.5	.321332571665729722940E +0	.145293383489740139911E 1
9.6	.320493375282756780376E +0	-.137957320506401805321E 1
9.7	.319665006113756663655E +0	-.106887525062043646225E 1
9.8	.318647215851744174331E +0	-.756573245460743396231E +0
9.9	.318039764308783158990E +0	-.442675353940778404488E +0
10.0	.317242419073487239172E +0	-.127189617791998181797E +0

Table B-12

THE AUXILIARY FUNCTIONS G(-x) AND ψ(-x) FOR 0 < x < 10

x	G(-x)	ψ(-x)
0.0	.517638807585613596810E +0	-.523598775598298873077E + 0
0.1	.519250392201201784980E +0	-.517677843855539476125E +0
0.2	.523500002197715358154E +0	-.500116246203854068369E +0
0.3	.529653562985833280705E +0	-.471424193749893501067E +0
0.4	.537144703763959186183E +0	-.432277636840310791171E +0
0.5	.545543995246759811110E +0	-.383413589183332985647E +0
0.6	.554530307786398170518E +0	-.325563704954037801780E +0
0.7	.563866245122666783363E +0	-.259416908319300552585E +0
0.8	.573378134638296601507E +0	-.185601656401773296386E +0
0.9	.582940267622046705637E +0	-.104680337395444232536E +0
1.0	.592462758242176345396E +0	-.171505552034521286178E -1
1.1	.601882325042458626803E +0	.765500796962704030308E -1
1.2	.611155356995506706059E +0	.176037693586464271948E +0
1.3	.620252728991267291261E +0	.280975584462165394387E +0
1.4	.629155939636905478001E +0	.391067956794663567734E +0
1.5	.637854240485486189114E +0	.506054238325406035601E +0
1.6	.646342505067549880317E +0	.625704076356197306008E +0
1.7	.654619648551132055584E +0	.749813012067943376767E +0
1.8	.662687456782357092190E +0	.878198782860658286282E +0
1.9	.670549719647027888228E +0	.101069818274392944219E 1
2.0	.678211590749581284533E +0	.114716440635687399315E 1
2.1	.685679115517178111347E +0	.128746480563520227379E 1
2.2	.692958884740491107291E +0	.143147899513866873621E 1
2.3	.700057781594252213366E +0	-.156249540336276606356E 1
2.4	.706982798346570468249E +0	-.141137350320389900607E 1
2.5	.713740905016479495246E +0	-.125684022612690892377E 1
2.6	.720338956728956002358E +0	-.109898066844442225630E 1
2.7	.726783629854581938742E +0	-.937873591938026194276E +0
2.8	.733081379507782752109E +0	-.773592141218890982868E +0
2.9	.739238412834240462340E +0	-.606204460426888167581E +0
3.0	.745260673907319695682E +0	-.435774225253500334846E +0
3.1	.751153837095018224937E +0	-.262361103546275637749E +0
3.2	.756923306541551498564E +0	-.860211555044935655995E -1
3.3	.762574219996682262494E +0	.931928173752049113862E -1
3.4	.768111455669883972095E +0	.275230966968223987282E +0
3.5	.773539641121505252507E +0	.460046130426667686354E +0
3.6	.778863163456221430885E +0	.647593593974665488167E +0
3.7	.784086180275361076924E +0	.837830884104925885533E +0
3.8	.789212630989281507216E +0	.103071758239480720588E 1
3.9	.794246248200187437180E +0	.122621516074719524869E 1
4.0	.799190568948193368565E +0	.142428683434947841484E 1

Table B-12 (Cont'd)

THE AUXILIARY FUNCTIONS G(-x) AND ψ(-x) FOR 0 < x < 10

x	G(-x)		ψ(-x)	
4.1	.80404894567548167l192E	+0	-.151669522353918070788E	1
4.2	.808824556809993349956E	+0	-.131357938539370748616E	1
4.3	.813520416904915027116E	+0	-.110799059734451453425E	1
4.4	.818139386296111178574E	+0	-.899959860854939646721E	+0
4.5	.822684180258739585962E	+0	-.689516973236561002359E	+0
4.6	.827157377658214842839E	+0	-.476690606528444655191E	+0
4.7	.831561429100696942054E	+0	-.261508379250849194564E	+0
4.8	.835898664595332228468E	+0	-.439969218317589530947E	-1
4.9	.840171300745290503232E	+0	.175818063599578120140E	+0
5.0	.844381447487781477634E	+0	.397911748446898110853E	+0
5.1	.848531114405121591294E	+0	.622260127379910565288E	+0
5.2	.852622216629883459693E	+0	.848839972102365533434E	+0
5.3	.856656580367442898789E	+0	.107762878874780091706E	1
5.4	.860635948059031479432E	+0	.130860477854412135033E	1
5.5	.864561983207849384927E	+0	.154174680143015464324E	1
5.6	.868436274890002612235E	+0	-.136455831124604539644E	1
5.7	.872260341971082121080E	+0	-.112714517365717285347E	1
5.8	.876035637048161681424E	+0	-.8876257961230[illegible]8307746E	+0
5.9	.879763550135901422812E	+0	-.6460189978344[illegible]656487E	+0
6.0	.883445412114338873721E	+0	-.402343085697844260908E	+0
6.1	.887082497954852809149E	+0	-.156615877919021551990E	+0
6.2	.890676029739714650430E	+0	.911452739427772037477E	-1
6.3	.894227179489609303784E	+0	.340923464323467164194E	+0
6.4	.897737071812519953795E	+0	.592702214291307838158E	+0
6.5	.901206786386434187105E	+0	.846465453369565681894E	+0
6.6	.904637360287444389125E	+0	.110219750245156439426E	1
6.7	.908029790173984451370E	+0	.135988305772384984232E	1
6.8	.911385034337167058519E	+0	-.152208547806873814761E	1
6.9	.914704014626459944950E	+0	-.126053739554673339458E	1
7.0	.917987618259263681754E	+0	-.997079613718492100342E	+0
7.1	.921236699522325546220E	+0	-.731726078363095322299E	+0
7.2	.924452081372341367564E	+0	-.464490427673114399531E	+0
7.3	.927634556942557341738E	+0	-.195386003666475301416E	+0
7.4	.930784890961684025294E	+0	.755741369988520503742E	-1
7.5	.933903821090972437148E	+0	.348377212767067968128E	+0
7.6	.936992059184874858238E	+0	.623010707446694083374E	+0
7.7	.940050292480318058296E	+0	.899462360958682585056E	+0
7.8	.943079184719251940367E	+0	.117772016053921019430E	1
7.9	.946079377208799746445E	+0	.145777233236833661661E	1
8.0	.949051489823024909124E	+0	-.140198531999185832399E	1

Table B-12 (Cont'd)

THE AUXILIARY FUNCTIONS G(-x) AND ψ(-x) FOR 0 < x < 10

x	G(-x)		ψ(-x)	
8.1	.951996121950042394963E	+0	-.111837880883546617289E	1
8.2	.954913853387937124668E	+0	-.833011891096890479914E	+0
8.3	.957805245192707063521E	+0	-.545895460906111592807E	+0
8.4	.960670840481222261835E	+0	-.257040209781478049651E	+0
8.5	.963511165191982020787E	+0	.335433670606351568079E	-1
8.6	.966326728806259102996E	+0	.325844964672693814381E	+0
8.7	.969118025032041244338E	+0	.619854462603760803016E	+0
8.8	.971885532453014998480E	+0	.915561919373585913265E	+0
8.9	.974629715144684091648E	+0	.121295756717946652210E	1
9.0	.977351023259573002225E	+0	.151203180682322962149E	1
9.1	.980049893583335507418E	+0	-.132881745074326679870E	1
9.2	.982726750063466627604E	+0	-.102641417472638746652E	1
9.3	.985382004312203986993E	+0	-.722360140509946134394E	+0
9.4	.988016056085100398461E	+0	-.416664319599198291283E	+0
9.5	.990629293736652827408E	+0	-.109335537345546935347E	+0
9.6	.993222094654283204799E	+0	.199617523078314935871E	+0
9.7	.995794825671883301877E	+0	.510186316855072149989E	+0
9.8	.998347843464058550124E	+0	.822362433843791040576E	+0
9.9	.100088149492213383375E	1	.113613759503823486187E	1
10.0	.100339611751291747805E	1	.145150364915588514051E	1

Table B-13

THE AIRY FUNCTIONS Ai(-x) AND Bi(-x) FOR x = 10.0(0.1)25.0

x	Ai(-x)	Bi(-x)
10.0	.402412384864431906450E -1	-.314679829643838633175E +0
10.1	-.597268111334541563644E -1	-.310767512287939160859E +0
10.2	-.153696782607083167946E +0	-.275734229100683682474E +0
10.3	-.232108018854829267138E +0	-.212822518390315659636E +0
10.4	-.286805455630829621003E +0	-.128189973445723471846E +0
10.5	-.311926035051050600856E +0	-.303561232640210131669E -1
10.6	-.304578464997366788682E +0	.706206434055528471920E -1
10.7	-.265233672673468532137E +0	.164161186485161247964E +0
10.8	-.197769992056371797856E +0	.240278305780461363805E +0
10.9	-.109151792225158841412E +0	.290667456575535745931E +0
11.0	-.875958925570238132406E -2	.309654767426781886332E +0
11.1	.925704336464582262528E -1	.294890786833811241463E +0
11.2	.183703674051253539814E +0	.247700786558615510150E +0
11.3	.254439825115047288320E +0	.173036490435860838240E +0
11.4	.296684512423485059168E +0	.790166652187811602628E -1
11.5	.305422970043592656397E +0	-.239092723559457575783E -1
11.6	.279374548683323686800E +0	-.124094835265161019075E +0
11.7	.221237808565027051397E +0	-.210008380453832472436E +0
11.8	.137479553885071913538E +0	-.271577189243467759620E +0
11.9	.376730243393585238784E -1	-.301406091377845716626E +0
12.0	-.665551750543731295416E -1	-.295719912078073056714E +0
12.1	-.162859453505113731063E +0	-.254904692434865255286E +0
12.2	-.239641096279134769780E +0	-.183565176617004729919E +0
12.3	-.287472080256441363314E +0	-.900713135085551519727E -1
12.4	-.300299844535014257854E +0	.143715982005231298790E -1
12.5	-.276274561381160248219E +0	.117033367257392776617E +0
12.6	-.218080099918026314406E +0	.205201680530603469518E +0
12.7	-.132706918893897185726E +0	.267772254484722057110E +0
12.8	-.306733238462919117877E -1	.296686446729433108868E +0
12.9	.752272785815361915793E -1	.288026418024190884104E +0
13.0	.171510439370537044641E +0	.242613229092627199327E +0
13.1	.245723407147919197080E +0	.166010776721386871807E +0
13.2	.288086678032392882024E +0	.679119720515721922348E -1
13.3	.292835007407974528706E +0	-.390357992138569857288E -1
13.4	.259068469943414170550E +0	-.140831544556963389366E +0
13.5	.190981243296220292664E +0	-.223950103580022791491E +0
13.6	.974138903842128641854E -1	-.277162527805999445007E +0
13.7	-.923544696837094871932E -2	-.293100296168828183611E +0
13.8	-.114616074462635171716E +0	-.269340321937080265384E +0
13.9	-.204344332205273782068E +0	-.208843947385235140607E +0
14.0	-.265983482784077798396E +0	-.119665552797624523100E +0

Table B-13 (Cont'd)

THE AIRY FUNCTIONS Ai(-x) AND Bi(-x) FOR x = 10.0(0.1)25.0

x	Ai(-x)	Bi(-x)
14.1	-.29081008514996654240OE +0	-.139449179819057303158E -1
14.2	-.275111233707217394091E +0	.937010056063833635907E -1
14.3	-.220812567684753037941E +0	.188181084943215964558E +0
14.4	-.135323956376930263935E +0	.256056757239035563509E +0
14.5	-.305974189395514228158E -1	.287492243517527755240E +0
14.6	.784944011048661970258E -1	.277740807135961818392E +0
14.7	.176247385514908148444E +0	.227935185937784278396E +0
14.8	.248393528798947757662E +0	.145042705101175547154E +0
14.9	.284219508153073098728E +0	.409630871062547004781E -1
15.0	.278217490870828929532E +0	-.691265945310100611765E -1
15.1	.231006381313208420753E +0	-.168958580952192226378E +0
15.2	.149360880167607559296E +0	-.243582556265014671221E +0
15.3	.453142726642754678562E -1	-.281639100357291913367E +0
15.4	-.655593766657786769160E -1	-.277148644512263496030E +0
15.5	-.166447954090419767435E +0	-.230526530754712210696E +0
15.6	-.241853776309456335635E +0	-.148642777434945160149E +0
15.7	-.280008778231409792267E +0	-.438863503404062751514E -1
15.8	-.274771784867787423562E +0	.676523425806300522417E -1
15.9	-.226693950298588365530E +0	.168624065964169058925E +0
16.0	-.143057931669099697712E +0	.243123151428227216730E +0
16.1	-.368522003634394557182E -1	.279229116187111048782E +0
16.2	.751891724478489316167E -1	.270976772753752360411E +0
16.3	.175195586927892081738E +0	.219420661690464175667E +0
16.4	.247016437066724377416E +0	.132593822101696703640E +0
16.5	.278868480560550838314E +0	.243335984326956926555E -1
16.6	.265334889965864006144E +0	-.878681881139096494492E -1
16.7	.208367580336450845224E +0	-.185666818653931467286E +0
16.8	.117096950227064733015E +0	-.252874291645393359750E +0
16.9	.644492332728703505772E -2	-.278182521778200078603E +0
17.0	-.105262300290952390249E +0	-.257135921002343182134E +0
17.1	-.199309661928061248022E +0	-.192999160312283731216E +0
17.2	-.259744953337002906108E +0	-.963407131723165460050E -1
17.3	-.276134329617757533339E +0	.166345106620348942834E -1
17.4	-.245438411334952268887E +0	.126754403954158017658E +0
17.5	-.172660590662226267760E +0	.215120245578695335030E +0
17.6	-.701200288287906168874E -1	.266374079965369105099E +0
17.7	.445633965857197437876E -1	.271424902922468651251E +0
17.8	.151456073406667976582E +0	.229141108518447431837E +0
17.9	.231771307055489357663E +0	.146682144363863919525E +0
18.0	.271204540804414221595E +0	.383724885083839979915E -1

Table B-13 (Cont'd)

THE AIRY FUNCTIONS Ai(-x) AND Bi(-x) FOR x = 10.0(0.1)25.0

x	Ai(-x)	Bi(-x)
18.1	.262546229752412494995E +0	-.767223854747841476899E -1
18.2	.207084918263242858069E +0	-.178120442694505485837E +0
18.3	.114514803358369449170E +0	-.247574861603869420837E +0
18.4	.132347636384990236688E -2	-.272401841160857962017E +0
18.5	-.112088539775540476113E +0	-.247870690720432925206E +0
18.6	-.205064480270311008121E +0	-.178194010413059284270E +0
18.7	-.260473450197573740704E +0	-.758983119620659835698E -1
18.8	-.267922901767988366698E +0	.403521287652813175947E -1
18.9	-.225789964287493626666E +0	.149115486345776058365E +0
19.0	-.141661276880422656284E +0	.230121090094588314723E +0
19.1	-.310568561078791882496E -1	.268081552604911008342E +0
19.2	.853664439744650937246E -1	.255646098695170245720E +0
19.3	.185641664849651817410E +0	.194913102364998585773E +0
19.4	.250646640265866088606E +0	.971764005853246836007E -1
19.5	.267800272102583945751E +0	-.190916422575766242140E -1
19.6	.233577383245297582873E +0	-.131677595542802977208E +0
19.7	.154325796369767516332E +0	-.218856668625046315174E +0
19.8	.451837712582207590615E -1	-.263613001791182791587E +0
19.9	-.727388201110130226840E -1	-.257026292386847116144E +0
20.0	-.176406127077984689695E +0	-.200139309322651349186E +0
20.1	-.245361739252998172806E +0	-.103900094453851058503E +0
20.2	-.265812642308826248378E +0	.128575785114412207673E -1
20.3	-.233468179608765823777E +0	.127041937713412395259E +0
20.4	-.154535417377747590940E +0	.215852962935146300754E +0
20.5	-.446256803970119096398E -1	.261362137869230292146E +0
20.6	.742577711056999709623E -1	.254197910553626351414E +0
20.7	.178086104916828730764E +0	.195567247818799204658E +0
20.8	.245667656304481558079E +0	.971609533605542754950E -1
20.9	.263022748613744069096E +0	-.210886329961638689517E -1
21.0	.226358493678988966159E +0	-.134987305578277476120E +0
21.1	.142995162777952029897E +0	-.221014362106683341358E +0
21.2	.300032546952012581099E -1	-.261210951193556689911E +0
21.3	-.892181325308606591682E -1	-.247000052408007773885E +0
21.4	-.189752403309930847825E +0	-.181112150417079536994E +0
21.5	-.250385042987495251055E +0	-.771668768753138113521E -1
21.6	-.258136730536138182288E +0	.430551913874110799173E -1
21.7	-.211130636648976960113E +0	.154123977943817141604E +0
21.8	-.119143469675093058140E +0	.232332290256699395867E +0
21.9	-.166917098241845910911E -2	.260796591341320072663E +0
22.0	.116144153760514129292E +0	.233181122304421416066E +0

Table B-13 (Cont'd)

THE AIRY FUNCTIONS Ai(-x) AND Bi(-x) FOR x = 10.0(0.1)25.0

22.1	.208852982732890023155E	+0	.155208360671721413919E	+0
22.2	.256237260684164219840E	+0	.435776625904232488430E	-1
22.3	.247778179525750627480E	+0	-.775299348873376605282E	-1
22.4	.185089790441932686802E	+0	-.181648213457231229791E	+0
22.5	.817235009040366456193E	-1	-.245816934556388009409E	+0
22.6	-.396693650193141301210E	-1	-.255699869409081844441E	+0
22.7	-.152244735413209501483E	+0	-.208877933322794891068E	+0
22.8	-.230893191464556195147E	+0	-.115542743277917843883E	+0
22.9	-.257881714707677996500E	+0	.362146809313581795924E	-2
23.0	-.226934053374082875109E	+0	.121951968900328556751E	+0
23.1	-.144793890460390343818E	+0	.212749314538662240304E	+0
23.2	-.298619426863361778777E	-1	.255329063760977416864E	+0
23.3	.918450352232953941836E	-1	.239806591701951222881E	+0
23.4	.192545590870573449857E	+0	.169492960428577318533E	+0
23.5	.249048926921211096860E	+0	.603005602288153187131E	-1
23.6	.248158255837413289596E	+0	-.627674638955167414957E	-1
23.7	.189850007580364689196E	+0	-.171291798133058584765E	+0
23.8	.874425637381373602537E	-1	-.240000432090737411562E	+0
23.9	-.353476431558870289455E	-1	-.252705997283569205111E	+0
24.0	-.149836590081886533396E	+0	-.206211046026896839591E	+0
24.1	-.229061832290777216468E	+0	-.111219352384403984311E	+0
24.2	-.254174141649923418851E	+0	.100238860098657826490E	-1
24.3	-.219007647648814962242E	+0	.128869532996228371242E	+0
24.4	-.131701657476820984512E	+0	.217011366862638827766E	+0
24.5	-.129260447032410925017E	-1	.253259832125682915364E	+0
24.6	.108931929645268814729E	+0	.228715168984665499798E	+0
24.7	.204519081643879730015E	+0	.149059393853194420619E	+0
24.8	.250609280392520862544E	+0	.333541190276506511687E	-1
24.9	.235819605845486604757E	+0	-.904331381588596179253E	-1
25.0	.163526578830429469433E	+0	-.192146815690378023824E	+0

Table B-14

THE AIRY DERIVATIVES Ai′(-x) AND Bi′(-x) FOR x = 10.0(0.1)25.0

x	Ai′(-x)		Bi′(-x)	
10.0	.99626504413279005589 2E	+0	.11941411339990923818 4E	+0
10.1	.98630683706012814809 5E	+0	-.19752877902133186015 4E	+0
10.2	.87698998000019466195 6E	+0	-.49769246188428512101 3E	+0
10.3	.67749282954091277893 5E	+0	-.75018586978857374913 0E	+0
10.4	.40656884037507168565 4E	+0	-.92812682641186776958 9E	+0
10.5	.90957487390681672806 5E	-1	-.10116140816303775186 5E	1
10.6	-.23713206714159534241 7E	+0	-.99010107964385990582 0E	+0
10.7	-.54324590031593679610 1E	+0	-.86387973414224358717 3E	+0
10.8	-.79430738718621806417 4E	+0	-.64446103069798284709 6E	+0
10.9	-.96226006822777324590 8E	+0	-.35374773790404555739 8E	+0
11.0	-.10273278736645794214 6E	1	-.22022995314464466625 3E	-1
11.1	-.98050614982505289903 8E	+0	.31508609193607221748 4E	+0
11.2	-.82495955004224843419 4E	+0	.62038365507859871608 0E	+0
11.3	-.57610761992286277470 0E	+0	.85922966430325768850 9E	+0
11.4	-.26031689811010133445 2E	+0	.10035593383790820954 0E	1
11.5	.87724154321784443111 5E	-1	.10353264046930834408 9E	1
11.6	.42871139792683259104 7E	+0	.94893760770695526378 0E	+0
11.7	.72313366159037479788 9E	+0	.75233866282229031482 9E	+0
11.8	.93589786065535150147 9E	+0	.46655210869869128816 5E	+0
11.9	.10406290259592336172 8E	1	.12364175631678618702 6E	+0
12.0	.10231104533679707298 5E	1	-.23673219783112331651 9E	+0
12.1	.88340231823096755046 0E	+0	-.57182121120131837294 3E	+0
12.2	.63631251174038143469 3E	+0	-.84086190019696732146 6E	+0
12.3	.31007878814201665186 9E	+0	-.10101178597436740115 8E	1
12.4	-.56663225663481390090 7E	-1	-.10572617696936247002 8E	1
12.5	-.41933133041950516453 2E	+0	-.97451653616717407210 2E	+0
12.6	-.73277547091736961993 0E	+0	-.77009836368870831516 0E	+0
12.7	-.95694539101927640950 7E	+0	-.46769574585194737509 0E	+0
12.8	-.10621354773225733819 3E	1	-.10395630703946273409 3E	+0
12.9	-.10331100184723800808 7E	1	.27578969397116391581 1E	+0
13.0	-.87151967787995336664 6E	+0	.62309724881928773364 2E	+0
13.1	-.59621238207884556652 5E	+0	.89259793395238440150 0E	+0
13.2	-.24129921481379101677 9E	+0	.10480275683464319996 1E	1
13.3	.14787197201463607509 2E	+0	.10672821816670515536 4E	1
13.4	.52039301825738595019 8E	+0	.94578137470553936217 6E	+0
13.5	.82643275142525423805 1E	+0	.69760874733408185728 4E	+0
13.6	.10239783333201546158 8E	1	.35417395574824247196 8E	+0
13.7	.10847644707773194267 9E	1	-.39532307075829851982 2E	-1
13.8	.99853865825849945960 6E	+0	-.43068271822482143234 5E	+0
13.9	.77499796688398813275 6E	+0	-.76565006660932960516 5E	+0
14.0	.44302487700284364111 9E	+0	-.99741181894933352408 9E	+0

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Table B-14 (Cont'd)

THE AIRY DERIVATIVES Ai'(-x) AND Bi'(-x) FOR x = 10.0(0.1)25.0

x	Ai'(-x)		Bi'(-x)	
14.1	.472115486355666765885E	-1	-.109229886180343662178E	1
14.2	-.357953614821164629413E	+0	-.103510594124872613670E	1
14.3	-.715510289995554659117E	+0	-.831765988005736218497E	+0
14.4	-.974066500772317867265E	+0	-.509101111919613196060E	+0
14.5	-.109532127288053921504E	1	-.111562222867033313330E	+0
14.6	-.105995567846932470435E	1	.304696133491382107292E	+0
14.7	-.870963119607232363692E	+0	.679651191540830963924E	+0
14.8	-.553822667571172753713E	+0	.958084332887527385500E	+0
14.9	-.153359664569547016905E	+0	.109784090090463602412E	1
15.0	.272374204308642020813E	+0	.107642975308437478672E	1
15.1	.660404728662501730364E	+0	.894905323392625020004E	+0
15.2	.952157783252348946377E	+0	.578336571628529231122E	+0
15.3	.110242563001320523032E	1	.172654730530824764425E	+0
15.4	.108659230079908491879E	1	-.261782581725203397075E	+0
15.5	.904937935430212199387E	+0	-.659050956680073412179E	+0
15.6	.583241164848367316261E	+0	-.957666665621878199572E	+0
15.7	.169441230102727826446E	+0	-.111022851126644042221E	1
15.8	-.273270009835393539571E	+0	-.109116927709903251333E	1
15.9	-.675974813766345733744E	+0	-.901322087898742796755E	+0
16.0	-.974764441621272718065E	+0	-.568455605976135372513E	+0
16.1	-.112101541663529422916E	1	-.143539384688662369616E	+0
16.2	-.108954045119886561724E	1	.306822514229106213165E	+0
16.3	-.883218176990026090747E	+0	.710711791061504269088E	+0
16.4	-.533218364669774117392E	+0	.100239655359665283110E	1
16.5	-.946225799635321398354E	-1	.113317710802271043737E	1
16.6	.362009625389870959214E	+0	.107977038510784074610E	1
16.7	.761883299930287241091E	+0	.848756977523135561474E	+0
16.8	.103825246668262429628E	1	.476208210497451835241E	+0
16.9	.114372980019870897500E	1	.223813572810349242070E	-1
17.0	.105868457664466007731E	1	-.437802065790987509681E	+0
17.1	.795204534241372456795E	+0	-.827035062947329574850E	+0
17.2	.395790302328915738055E	+0	-.107867029788725529483E	1
17.3	-.731801126304501338256E	-1	-.114832723355668948747E	1
17.4	-.532275998762611130794E	+0	-.102201427175637047270E	1
17.5	-.902404920480841690139E	+0	-.719239506839572813453E	+0
17.6	-.111852982783679174147E	1	-.290395379466557036689E	+0
17.7	-.114132453189384288830E	1	.191322622706109562398E	+0
17.8	-.964647324524293331949E	+0	.642229305996633783963E	+0
17.9	-.617368956302882529533E	+0	.982662119410130308701E	+0
18.0	-.159038915204968015408E	+0	.115118709410864179877E	1

Table B-14 (Cont'd)

THE AIRY DERIVATIVES Ai'(-x) AND Bi'(-x) FOR x = 10.0(0.1)25.0

x	Ai'(-x)		Bi'(-x)	
18.1	.330042796310298156555E	+0	.111594904950031600163E	1
18.2	.762751593691742777681E	+0	.881031009788712614086E	+0
18.3	.106067926153725079759E	1	.486507972495589007162E	+0
18.4	.116852099366926422614E	1	.197666784183106056865E	-2
18.5	.106464396227970844355E	1	-.485472038364935658718E	+0
18.6	.765772899597995396681E	+0	-.886812488925784222228E	+0
18.7	.324736911260775851879E	+0	-.112741971423411498413E	1
18.8	-.178529075579586810397E	+0	-.116117680080616277550E	1
18.9	-.651267828527478124343E	+0	-.979652784345695507255E	+0
19.0	-.100496112500513959378E	1	-.614473753956074056300E	+0
19.1	-.117204417550416675893E	1	-.132224069262729461444E	+0
19.2	-.111909840172227797042E	1	.377393554511028090387E	+0
19.3	-.853901896666906881238E	+0	.818098773850617369356E	+0
19.4	-.424797232694021062609E	+0	.110525965892911245455E	1
19.5	.877410883437571359200E	-1	.118235415605758114921E	1
19.6	.585952590826470816919E	+0	.103243325432938744527E	1
19.7	.973366417742721323216E	+0	.682207104202239023070E	+0
19.8	.117359784138861463627E	1	.197731089523875102414E	+0
19.9	.114568830901281222936E	1	-.327718649421177131132E	+0
20.0	.892862856736471238108E	+0	-.791429033839536479710E	+0
20.1	.462773086199508733390E	+0	-.110134416082828421772E	1
20.2	-.610781097382975813701E	-1	-.119454276829916135489E	1
20.3	-.575279661862273850171E	+0	-.105035831274577461519E	1
20.4	-.976840788479008551703E	+0	-.695348092353651747723E	+0
20.5	-.118393301970514749714E	1	-.198868028021685980226E	+0
20.6	-.115285324554661982917E	1	.340126017988364572981E	+0
20.7	-.887642345752699053538E	+0	.812618792718830391004E	+0
20.8	-.440177387396897004168E	+0	.112160402277393158327E	1
20.9	.995574183177934565472E	-1	.120221675879147821005E	1
21.0	.621294449908927073171E	+0	.103571559701365860222E	1
21.1	.101693461106264827929E	1	.654237038499290666034E	+0
21.2	.120307903204912785658E	1	.135067608577234284155E	+0
21.3	.113892410645316521610E	1	-.414664274533191169923E	+0
21.4	.835624066444336287217E	+0	-.879926744838514274014E	+0
21.5	.354902532239253279140E	+0	-.116190313408727225872E	1
21.6	-.203092964385294216800E	+0	-.119923142704175190159E	1
21.7	-.720402847451298325654E	+0	-.981754883622893096098E	+0
21.8	-.108615294099023436541E	1	-.553630727847345026278E	+0
21.9	-.122049837716963516289E	1	-.483454386758234593128E	-2
22.0	-.109241275127083367258E	1	.547421912891945699287E	+0

Table B-14 (Cont'd)

THE AIRY DERIVATIVES Ai′(-x) AND Bi′(-x) FOR x = 10.0(0.1)25.0

x	Ai′(-x)		Bi′(-x)	
22.1	-.727292565344394743859E	+0	.983600984289920216019E	+0
22.2	-.202441955964697765854E	+0	.120781785641272463487E	1
22.3	.368901353234816910706E	+0	.116922720492236174493E	1
22.4	.861793768757450812463E	+0	.873989793462842108274E	+0
22.5	.116693605500273035566E	1	.384922843144389914523E	+0
22.6	.121516096682896143123E	1	-.191416869186450255597E	+0
22.7	.993525544488586901473E	+0	-.727672608494007308447E	+0
22.8	.549184785031414531435E	+0	-.110378122428910880367E	1
22.9	-.201454616276740678855E	-1	-.123404224451517339746E	1
23.0	-.587335090044939794807E	+0	-.108702599667543368211E	1
23.1	-.102410540074090910953E	1	-.693621559870265166915E	+0
23.2	-.123016477641836705958E	1	-.141084777819383387183E	+0
23.3	-.115657708766664642159E	1	.445914976790118129053E	+0
23.4	-.817850595260867908332E	+0	.933233354200965089980E	+0
23.5	-.289672064320503993065E	+0	.120796544735886827644E	1
23.6	.307555511584434547970E	+0	.120489808290096909280E	1
23.7	.835906074856036809494E	+0	.922444164123728662057E	+0
23.8	.117177997337729840283E	1	.424074783180141153410E	+0
23.9	.123506405084388194212E	1	-.175451400786526707086E	+0
24.0	.100867440767719700927E	1	-.736202561843073287367E	+0
24.1	.543625365777346149685E	+0	-.112567083954423304591E	1
24.2	-.519371349724973511594E	-1	-.125028168562025354430E	1
24.3	-.637522685656670707786E	+0	-.107828488157359570765E	1
24.4	-.107331791882811907766E	1	-.648341859750441227180E	+0
24.5	-.125371741875871908787E	1	-.613972173339283373111E	-1
24.6	-.113329528944890046902E	1	.542614665327625647722E	+0
24.7	-.738749468116937325952E	+0	.101796045915865478914E	1
24.8	-.163577726350710106699E	+0	.124837314379900104686E	1
24.9	.453632542451851464298E	+0	.117584146918034836769E	1
25.0	.962378851387697410445E	+0	.815719715754605857322E	+0

Table B-15

THE AUXILIARY FUNCTIONS F(-x) AND $\chi(-x)$ FOR x = 10.0(0.1)25.0

x	F(-x)		$\chi(-x)$	
10.0	.317242419073487239176E	+0	.218639589573365544874E	2
10.1	.316454955186051274927E	+0	.221810246159942349940E	2
10.2	.315677154829926359588E	+0	.224996623917411067827E	2
10.3	.314908807038974420817E	+0	.228198645930435390283E	2
10.4	.314149707419286111290E	+0	.231416236395329788726E	2
10.5	.313399657884768096617E	+0	.234649320593742025394E	2
10.6	.312658466405703526920E	+0	.237897824867191340115E	2
10.7	.311925946769537810973E	+0	.241161676592427289904E	2
10.8	.311201918353191159856E	+0	.244440804157575930526E	2
10.9	.310486205906245059783E	+0	.247735136939041644589E	2
11.0	.309778639344392170640E	+0	.251044605279134445996E	2
11.1	.309079053552578400371E	+0	.254369140464394031080E	2
11.2	.308387288197302324092E	+0	.257708674704583207945E	2
11.3	.307703187547570927291E	+0	.261063141112324622307E	2
11.4	.307026600304042061289E	+0	.264432473683355914904E	2
11.5	.306357379435913195325E	+0	.267816607277379596868E	2
11.6	.305695382025143205520E	+0	.271215477599485019040E	2
11.7	.305040469117619214633E	+0	.274629021182120842944E	2
11.8	.304392505580904032339E	+0	.278057175367597398357E	2
11.9	.303751359968221676086E	+0	.281499878291099238610E	2
12.0	.303116904388358898790E	+0	.284957068864189082462E	2
12.1	.302489014381179722955E	+0	.288428686758785164118E	2
12.2	.301867568798467783437E	+0	.291914672391594802700E	2
12.3	.301252449689827906777E	+0	.295414966908987752116E	2
12.4	.300643542193393890124E	+0	.298929512172293603974E	2
12.5	.300040734431103966534E	+0	.302458250743508192149E	2
12.6	.299443917408319028796E	+0	.306001125871394589859E	2
12.7	.298852984917571398311E	+0	.309558081477964900456E	2
12.8	.298267833446243830631E	+0	.313129062145329623670E	2
12.9	.297688362087989602406E	+0	.316714013102901930916E	2
13.0	.297114472457714978226E	+0	.320312880214944708673E	2
13.1	.296546068609955158930E	+0	.323925609968448728755E	2
13.2	.295983056960484010235E	+0	.327552149461330780388E	2
13.3	.295425346211006503697E	+0	.331192446390941052316E	2
13.4	.294872847276790909252E	+0	.334846449042869485113E	2
13.5	.294325473217105395695E	+0	.338514106280041225635E	2
13.6	.293783139168330855130E	+0	.342195367532091707907E	2
13.7	.293245762279628500269E	+0	.345890182785012259231E	2
13.8	.292713261651047117722E	+0	.349598502571057487284E	2
13.9	.292185558273960822254E	+0	.353320277958906044825E	2
14.0	.291662574973733770772E	+0	.357055460544066693947E	2

Table B-15 (Cont'd)

THE AUXILIARY FUNCTIONS F(-x) AND χ(-x) FOR x = 10.0(0.1)25.0

x	F(-x)		χ(-x)	
14.1	.291144236354513583034E	+0	.360804002439521902410E	2
14.2	.290630468746060199807E	+0	.364565856266601501424E	2
14.3	.290121200152521607838E	+0	.368340975146079217640E	2
14.4	.289616360203072292703E	+0	.372129312689485163145E	2
14.5	.289115880104334462204E	+0	.375930822990627626320E	2
14.6	.288619692594506030177E	+0	.379745460617317754181E	2
14.7	.288127731899123077938E	+0	.383573180603290953677E	2
14.8	.287639933688388031791E	+0	.387413938440319066418E	2
14.9	.287156235035998122645E	+0	.391267690070507588131E	2
15.0	.286676574379411839734E	+0	.395134391878772412064E	2
15.1	.286200891481494065713E	+0	.399014000685490774521E	2
15.2	.285729127393483395255E	+0	.402906473739321271326E	2
15.3	.285261224419227803452E	+0	.406811768710187996739E	2
15.4	.284797126080637352563E	+0	.410729843682424031242E	2
15.5	.284336777084305014675E	+0	.414660657148069672663E	2
15.6	.283880123289248951148E	+0	.418604168000320965874E	2
15.7	.283427111675731734923E	+0	.422560335527124240694E	2
15.8	.282977690315114035561E	+0	.426529119404912515833E	2
15.9	.282531808340702215800E	+0	.430510479692479768647E	2
16.0	.282089415919551118409E	+0	.434504376824989207114E	2
16.1	.281650464225185058858E	+0	.438510771608111811284E	2
16.2	.281214905411201688123E	+0	.442529625212291537282E	2
16.3	.280782692585724955671E	+0	.446560899167133697836E	2
16.4	.280353779786674890120E	+0	.450604555355913149307E	2
16.5	.279928121957823328428E	+0	.454660556010199026933E	2
16.6	.279505674925606068004E	+0	.458728863704592877061E	2
16.7	.279086395376663193588E	+0	.462809441351577138482E	2
16.8	.278670240836080545720E	+0	.466902252196471023969E	2
16.9	.278257169646306453637E	+0	.471007259812490948712E	2
17.0	.277847140946718955535E	+0	.475124428095912743964E	2
17.1	.277440114653819776414E	+0	.479253721261332982698E	2
17.2	.277036051442032330971E	+0	.483395103837026828991E	2
17.3	.276634912725081968936E	+0	.487548540660399904765E	2
17.4	.276236660637937585255E	+0	.491713996873531746409E	2
17.5	.275841258019294580082E	+0	.495891437918808499609E	2
17.6	.275448668394579975819E	+0	.500080829534642573932E	2
17.7	.275058855959461282526E	+0	.504282137751277049250E	2
17.8	.274671785563841450908E	+0	.508495328886672693904E	2
17.9	.274287422696322965609E	+0	.512720369542475520067E	2
18.0	.273905733469124812522E	+0	.516957226600062864940E	2

Table B-15 (Cont'd)

THE AUXILIARY FUNCTIONS F(-x) AND χ(-x) FOR x = 10.0(0.1)25.0

x	F(-x)	χ(-x)
18.1	.27352668460343670381 2E +0	.52120586721666604715 5E 2
18.2	.27315024341519556500 9E +0	.52546625882156770664 3E 2
18.3	.27277637780126988131 2E +0	.52973836911237199282 1E 2
18.4	.27240505622603806648 8E +0	.53402216605134582058 3E 2
18.5	.27203624770834755885 6E +0	.53831761786182946636 3E 2
18.6	.27166992180884186603 1E +0	.54262469302471482754 4E 2
18.7	.27130604861764327452 8E +0	.54694336027498971746 9E 2
18.8	.27094459874237941306 2E +0	.55127358859834661601 4E 2
18.9	.27058554329654231069 4E +0	.55561534722785434136 7E 2
19.0	.27022885388816902350 0E +0	.55996860564069115299 6E 2
19.1	.26987450260883331748 1E +0	.56433333355493783865 8E 2
19.2	.26952246202293829172 4E +0	.56870950092642937954 0E 2
19.3	.26917270515730020508 3E +0	.57309707794566382767 1E 2
19.4	.26882520549101413302 8E +0	.57749603503476706835 6E 2
19.5	.26847993694559242916 1E +0	.58190634284451217775 5E 2
19.6	.26813687387536729927 7E +0	.58632797225139212178 6E 2
19.7	.26779599105814911522 9E +0	.59076089435474457757 6E 2
19.8	.26745726368613240196 6E +0	.59520508047392769238 6E 2
19.9	.26712066735704172461 4E +0	.59966050214554562767 5E 2
20.0	.26678617806550998381 3E +0	.60412713112072276755 6E 2
20.1	.26645377219468189739 4E +0	.60860493936242550161 3E 2
20.2	.26612342650803570535 0E +0	.61309389904283052159 0E 2
20.3	.26579511814141638341 6E +0	.61759398254073860014 1E 2
20.4	.26546882459527388887 7E +0	.62210516243903284776 7E 2
20.5	.26514452372710019104 5E +0	.62662741152218047075 2E 2
20.6	.26482219374405905841 5E +0	.63116070277377707907 5E 2
20.7	.26450181319580278542 4E +0	.63570500937413261864 9E 2
20.8	.26418336096747024425 7E +0	.64026030469789802634 0E 2
20.9	.26386681627286084171 2E +0	.64482656231173173031 7E 2
21.0	.26355215864777914804 3E +0	.64940375597200514099 6E 2
21.1	.26323936794354514434 0E +0	.65399185962254630010 7E 2
21.2	.26292842432066520765 6E +0	.65859084739242087721 6E 2
21.3	.26261930824265911905 3E +0	.66320069359374972360 1E 2
21.4	.26231200047003853934 2E +0	.66782137271956221402 7E 2
21.5	.26200648205443255072 8E +0	.67245285944168462638 8E 2
21.6	.26170273433285601023 8E +0	.67709512860866282841 0E 2
21.7	.26140073892211660278 9E +0	.68174815524371855903 0E 2
21.8	.26110047771335661838 8E +0	.68641191454273861016 8E 2
21.9	.26080193286672560951 3E +0	.69108638187229623181 1E 2
22.0	.26050508680618021126 0E +0	.69577153276770410056 7E 2

Table B-15 (Cont'd)

THE AUXILIARY FUNCTIONS $F(-x)$ AND $\chi(-x)$ FOR $x = 10.0(0.1)25.0$

x	F(-x)		χ(-x)	
22.1	.26020992221440752871 2E	+0	.70046734293109820794 1E	2
22.2	.25991642202786861334 5E	+0	.70517378822955204074 0E	2
22.3	.25962456943195866327 5E	+0	.70989084469322044152 3E	2
22.4	.25933434785628069099 0E	+0	.71461848851351255170 3E	2
22.5	.25904574097002950709 3E	+0	.71935669604129325509 6E	2
22.6	.25875873267748296960 5E	+0	.72410544378511255322 3E	2
22.7	.25847330711359754574 4E	+0	.72886470840946231800 2E	2
22.8	.25818944863970532696 3E	+0	.73363446673305988050 9E	2
22.9	.25790714183930972847 5E	+0	.73841469572715792758 1E	2
23.0	.25762637151397719175 9E	+0	.74320537251388019078 1E	2
23.1	.25734712267932229263 5E	+0	.74800647436458242429 6E	2
23.2	.25706938056108373862 1E	+0	.75281797869823818050 6E	2
23.3	.25679313059128881757 7E	+0	.75763986307984890342 5E	2
23.4	.25651835840450393511 3E	+0	.76247210521887787151 3E	2
23.5	.25624504983416895110 8E	+0	.76731468296770753240 8E	2
23.6	.25597319090901309599 9E	+0	.77216757432011978270 2E	2
23.7	.25570276784955031536 2E	+0	.77703075740979875622 7E	2
23.8	.25543376706465195681 3E	+0	.78190421050885569464 7E	2
23.9	.25516617514819477654 3E	+0	.78678791202637548363 5E	2
24.0	.25489997887578230383 2E	+0	.79168184050698444770 3E	2
24.1	.25463516520153766094 0E	+0	.79658597462943900597 2E	2
24.2	.25437172125496599270 6E	+0	.80150029320523480024 4E	2
24.3	.25410963433788471527 8E	+0	.80642477517723591544 0E	2
24.4	.25384889192141984654 0E	+0	.81135939961832382122 2E	2
24.5	.25358948164306673226 2E	+0	.81630414573006567171 8E	2
24.6	.25333139130381353162 4E	+0	.82125899284140160869 2E	2
24.7	.25307460886532587380 3E	+0	.82622392040735072104 8E	2
24.8	.25281912244719114377 3E	+0	.83119890800773532163 9E	2
24.9	.25256492032422090024 5E	+0	.83618393534592320951 0E	2
25.0	.25231199092380997226 6E	+0	.84117898224758759327 9E	2

Table B-16

THE AUXILIARY FUNCTIONS G(-x) AND ψ(-x) FOR x = 10.0(0.1)25.0

x	G(-x)		ψ(-x)	
10.0	.100339611751291747802E	1	.203010595706946445711E	2
10.1	.100589203962115751983E	1	.206180084908902114959E	2
10.2	.100836958087656652156E	1	.209365323715877783206E	2
10.3	.101082905246623735291E	1	.212566234254260617475E	2
10.4	.101327075743322221272E	1	.215782739803115843489E	2
10.5	.101569499096200042515E	1	.219014764765748480523E	2
10.6	.101810204065151696259E	1	.222262234642248712084E	2
10.7	.102049218677643299186E	1	.225525076002977072504E	2
10.8	.102286570253719179367E	1	.228803216462948018234E	2
10.9	.102522285429946797114E	1	.232096584657072687406E	2
11.0	.102756390182353475873E	1	.235405110216223743671E	2
11.1	.102988909848405329980E	1	.238728723744087160665E	2
11.2	.103219869148075882505E	1	.242067356794767641496E	2
11.3	.103449292204049159127E	1	.245420941851116092074E	2
11.4	.103677202561099509889E	1	.248789412303749185984E	2
11.5	.103903623204688037564E	1	.252172702430732579469E	2
11.6	.104128576578813287919E	1	.255570747377900764703E	2
11.7	.104352084603151773089E	1	.258983483139787894209E	2
11.8	.104574168689521944827E	1	.262410846541145174860E	2
11.9	.104794849757703400731E	1	.265852775219021621820E	2
12.0	.105014148250641385490E	1	.269309207605386085660E	2
12.1	.105232084149065032933E	1	.272780082910269524896E	2
12.2	.105448676985546276389E	1	.276265341105407494925E	2
12.3	.105663945858024927768E	1	.279764922908363767147E	2
12.4	.105877909442824083973E	1	.283278769767116882375E	2
12.5	.106090586007178757061E	1	.286806823845092283928E	2
12.6	.106301993421299436670E	1	.290349028006623471076E	2
12.7	.106512149169991174945E	1	.293905325802826365722E	2
12.8	.106721070363847730857E	1	.297475661457871797095E	2
12.9	.106928773750039318322E	1	.301059979855641683183E	2
13.0	.107135275722711566954E	1	.304658226526755126053E	2
13.1	.107340592333012422051E	1	.308270347635951243252E	2
13.2	.107544739298762878086E	1	.311896289969816131134E	2
13.3	.107747732013780654461E	1	.315536000924841900052E	2
13.4	.107949585556913180681E	1	.319189428495806237853E	2
13.5	.108150314700667557588E	1	.322856521264461448288E	2
13.6	.108349933919660499514E	1	.326537228388522376644E	2
13.7	.108548457398690636558E	1	.330231499590943077476E	2
13.8	.108745899040570964705E	1	.333939285149472499879E	2
13.9	.108942272473690671937E	1	.337660535886479865825E	2
14.0	.109137591059323038994E	1	.341395203159040797740E	2

Table B-16 (Cont'd)

THE AUXILIARY FUNCTIONS G(-x) AND ψ(-x) FOR x = 10.0(0.1)25.0

x	G(-x)		ψ(-x)	
14.1	.10933186789868961 2256E	1	.34514323884927561 3696E	2
14.2	.10952511583979037 1568E	1	.34890459535493155 3561E	2
14.3	.10971734748400916 6263E	1	.35267922558020102 8046E	2
14.4	.10990857519250326 6559E	1	.35646708292676829 5549E	2
14.5	.11009881109238547 3662E	1	.36026812128507727 0283E	2
14.6	.11028806708270684 8919E	1	.36408229502581344 9704E	2
14.7	.11047635484024775 9101E	1	.36790955899159322 0666E	2
14.8	.11066368582512459 0153E	1	.37174986848885406 2969E	2
14.9	.11085007128621915 4565E	1	.37560317927993941 6137E	2
15.0	.11103552226643750 6782E	1	.37946944757537221 1521E	2
15.1	.11122004960780458 6002E	1	.38334863002631129 7373E	2
15.2	.11140366395640082 5273E	1	.38724068371718520 0172E	2
15.3	.11158637576714659 9346E	1	.39114556615849787 1593E	2
15.4	.11176819530844013 0321E	1	.39506323527980126 7437E	2
15.5	.11194913266665422 9204E	1	.39899364942282979 3517E	2
15.6	.11212919775049702 2228E	1	.40293676733479183 3723E	2
15.7	.11230840029524159 2674E	1	.40689254816181374 8116E	2
15.8	.11248674986682926 1271E	1	.41086095144253189 4284E	2
15.9	.11266425586585103 0527E	1	.41484193710182838 3267E	2
16.0	.11284092753141153 0059E	1	.41883546544470643 3223E	2
16.1	.11301677394487962 0541E	1	.42284149715030132 9131E	2
16.2	.11319180403352964 2930E	1	.42685999326602313 6175E	2
16.3	.11336602657407713 6637E	1	.43089091520182744 8022E	2
16.4	.11353945019611269 4867E	1	.43493422472461057 9176E	2
16.5	.11371208338543747 7149E	1	.43898988395272573 3639E	2
16.6	.11388393448730375 7604E	1	.44305785535061679 9848E	2
16.7	.11405501170956375 2564E	1	.44713810172356653 5133E	2
16.8	.11422532312572984 2284E	1	.45123058621255601 1564E	2
16.9	.11439487667794917 8491E	1	.45533527228923229 9519E	2
17.0	.11456368017989555 1988E	1	.45945212375098146 5328E	2
17.1	.11473174131958128 2310E	1	.46358110471610405 6009E	2
17.2	.11489906766209178 4149E	1	.46772217961909033 6357E	2
17.3	.11506566665224536 2729E	1	.47187531320599263 2919E	2
17.4	.11523154561718069 2321E	1	.47604047052989222 4834E	2
17.5	.11539671176887433 8320E	1	.48021761694645830 3916E	2
17.6	.11556117220659059 3651E	1	.48440671810959660 5457E	2
17.7	.11572493391926581 4487E	1	.48860773996718538 7581E	2
17.8	.11588800378782935 8156E	1	.49282064875689651 0221E	2
17.9	.11605038858746314 7527E	1	.49704541100209943 5428E	2
18.0	.11621209498980181 0896E	1	.50128199350784603 8860E	2

Table B-16 (Cont'd)

THE AUXILIARY FUNCTIONS G(-x) AND ψ(-x) FOR x = 10.0(0.1)25.0

x	G(-x)		ψ(-x)	
18.1	.116373129565075274389E	1	.505530363356934187516E	2
18.2	.116533498784195614778E	1	.509790487906048102232E	2
18.3	.116693209020789914564E	1	.514062334781973583964E	2
18.4	.116852266553180797714E	1	.518345871877886241696E	2
18.5	.117010677566316263735E	1	.522641067349710916125E	2
18.6	.117168448153650379484E	1	.526947889612550547970E	2
18.7	.117325584318976332265E	1	.531266307337182792017E	2
18.8	.117482091978213294163E	1	.535596289446622728956E	2
18.9	.117637976961148496169E	1	.539937805112750075773E	2
19.0	.117793245013135861307E	1	.544290823752999342688E	2
19.1	.117947901796752498622E	1	.548655315027111430253E	2
19.2	.118101952893414314437E	1	.553031248833945204057E	2
19.3	.118255403804951953627E	1	.557418595308347627023E	2
19.4	.118408259955148241741E	1	.561817324818081070264E	2
19.5	.118560526691238258536E	1	.566227407960806463081E	2
19.6	.118712209285373134790E	1	.570648815561120980834E	2
19.7	.118863312936048627053E	1	.575081518667649006598E	2
19.8	.119013842769499489330E	1	.579525488550185138064E	2
19.9	.119163803841060626229E	1	.583980696696888045747E	2
20.0	.119313201136495979141E	1	.588447114811524021978E	2
20.1	.119462039573296065218E	1	.592924714810759092442E	2
20.2	.119610324001945058348E	1	.597413468821498593283E	2
20.3	.119758059207158271928E	1	.601913349178273146848E	2
20.4	.119905249909090874911E	1	.606424328420669998760E	2
20.5	.120051900764518645418E	1	.610946379290808706770E	2
20.6	.120198016367991539915E	1	.615479474730860199648E	2
20.7	.120343601252960830778E	1	.620023587880608250794E	2
20.8	.120488659892880540711E	1	.624578692075052436600E	2
20.9	.120633196702283879092E	1	.629144760842051674945E	2
21.0	.120777216037835362741E	1	.633721767900007462841E	2
21.1	.120920722199359281913E	1	.638309687155585955730E	2
21.2	.121063719430845151307E	1	.642908492701478053635E	2
21.3	.121206211921430765770E	1	.647518158814196680926E	2
21.4	.121348203806363460845E	1	.652138659951910468030E	2
21.5	.121489699167940159606E	1	.656769970752313063686E	2
21.6	.121630702036426769087E	1	.661412066030527326449E	2
21.7	.121771216390957472169E	1	.666064920777043663321E	2
21.8	.121911246160414443989E	1	.670728510155691802308E	2
21.9	.122050795224288505619E	1	.675402809501645303590E	2
22.0	.122189867413521212178E	1	.680087794319458131944E	2

Table B-16 (Cont'd)

THE AUXILIARY FUNCTIONS G(-x) AND ψ(-x) FOR x = 10.0(0.1)25.0

x	G(-x)		ψ(-x)	
22.1	.122328466511328857329E	1	.684783440281132629844E	2
22.2	.122466596254008861577E	1	.689489723224218247463E	2
22.3	.122604260331728997647E	1	.694206619149940401842E	2
22.4	.122741462389299892615E	1	.698934104221358852944E	2
22.5	.122878206026931233349E	1	.703672154761554999842E	2
22.6	.123014494800972089066E	1	.708420747251847514566E	2
22.7	.123150332224635752605E	1	.713179858330035745801E	2
22.8	.123285721768709490114E	1	.717949464788670338268E	2
22.9	.123420666862249577438E	1	.722729543573350527162E	2
23.0	.123555170893261990400E	1	.727520071781047580235E	2
23.1	.123689237209369105480E	1	.732321026658453872577E	2
23.2	.123822869118462757055E	1	.737132385600357091755E	2
23.3	.123956069889343987372E	1	.741954126148039082848E	2
23.4	.124088842752349815723E	1	.746786225987698854543E	2
23.5	.124221190899967343997E	1	.751628662948899278993E	2
23.6	.124353117487435506674E	1	.756481415003037028876E	2
23.7	.124484625633334764613E	1	.761344460261835306057E	2
23.8	.124615718420165033525E	1	.766217776975858926595E	2
23.9	.124746398894912129780E	1	.771101343533051336906E	2
24.0	.124876670069603008336E	1	.775995138457293145838E	2
24.1	.125006534921850059827E	1	.780899140406981766927E	2
24.2	.125135996395384726477E	1	.785813328173631774538E	2
24.3	.125265057400580689279E	1	.790737680680495586509E	2
24.4	.125393720814966871910E	1	.795672176981204094891E	2
24.5	.125521989483730500125E	1	.800616796258426874860E	2
24.6	.125649866220210448814E	1	.805571517822551610373E	2
24.7	.125777353806381102586E	1	.810536321110382382998E	2
24.8	.125904454993326949609E	1	.815511185683856478720E	2
24.9	.126031172501708122497E	1	.820496091228779374756E	2
25.0	.126157509022217094241E	1	.825491017553577576259E	2

Table B-17

THE FUNCTIONS $\exp(\pm\xi)$ WITH $\xi = \frac{2}{3} x^{3/2}$

x	ξ	exp(−ξ)	exp(+ξ)
5.0	0.74535593E 01	0.57937576E-03	0.17259955E 04
5.5	0.85990947E 01	0.18427251E-03	0.54267451E 04
6.0	0.97979584E 01	0.55564916E-04	0.17996967E 05
6.5	0.11047874E 02	0.15920949E-04	0.62810324E 05
7.0	0.12346838E 02	0.43434658E-05	0.23023088E 06
7.5	0.13693062E 02	0.11302602E-05	0.88475205E 06
8.0	0.15084942E 02	0.28099122E-06	0.35588299E 07
8.5	0.16521028E 02	0.66835672E-07	0.14962070E 08
9.0	0.17999999E 02	0.15229988E-07	0.65659930E 08
9.5	0.19520643E 02	0.33288345E-08	0.30040544E 09
10.0	0.21081848E 02	0.69866552E-09	0.14313000E 10
10.5	0.22682589E 02	0.14095370E-09	0.70945282E 10
11.0	0.24321912E 02	0.27360721E-10	0.36548743E 11
11.5	0.25998929E 02	0.51145606E-11	0.19552021E 12
12.0	0.27712811E 02	0.92146545E-12	0.10852279E 13
12.5	0.29462780E 02	0.16013157E-12	0.62448647E 13
13.0	0.31248108E 02	0.26860816E-13	0.37228949E 14
13.5	0.33068108E 02	0.43521416E-14	0.22977193E 15
14.0	0.34922130E 02	0.68157127E-15	0.14671979E 16
14.5	0.36809564E 02	0.10323075E-15	0.96870354E 16
15.0	0.38729829E 02	0.15130341E-16	0.66092360E 17
15.5	0.40682369E 02	0.21471936E-17	0.46572418E 18
16.0	0.42666659E 02	0.29519239E-18	0.33876210E 19
16.5	0.44682205E 02	0.39333694E-19	0.25423495E 20
17.0	0.46728522E 02	0.50823000E-20	0.19676130E 21
17.5	0.48805162E 02	0.63707001E-21	0.15696862E 22
18.0	0.50911681E 02	0.77506386E-22	0.12902162E 23
18.5	0.53047666E 02	0.91556933E-23	0.10922165E 24
19.0	0.55212715E 02	0.10505636E-23	0.95186996E 24
19.5	0.57406438E 02	0.11713866E-24	0.85368905E 25
20.0	0.59628470E 02	0.12696499E-25	0.78761863E 26
20.5	0.61878453E 02	0.13382249E-26	0.74725857E 27
21.0	0.64156047E 02	0.13720888E-27	0.72881578E 28
21.5	0.66460923E 02	0.13689499E-28	0.73048689E 29
22.0	0.68792757E 02	0.13294885E-29	0.75216895E 30
22.5	0.71151241E 02	0.12572113E-30	0.79541118E 31
23.0	0.73536075E 02	0.11579449E-31	0.86359887E 32
23.5	0.75946978E 02	0.10390730E-32	0.96239626E 33
24.0	0.78383660E 02	0.90867511E-34	0.11005033E 35
24.5	0.80845866E 02	0.77461553E-35	0.12909630E 36
25.0	0.83333325E 02	0.64386735E-36	0.15531149E 37
25.5	0.85845779E 02	0.52197713E-37	0.19157927E 38
26.0	0.88382991E 02	10^{-38}	10^{+38}

Table B-18

THE FUNCTIONS $\cos(\xi + \pi/4)$ AND $\sin(\xi + \pi/4)$ WITH $\xi = \frac{2}{3} x^{3/2}$

x	ξ	$\cos(\xi + \pi/4)$	$\sin(\xi + \pi/4)$
5.0	0.74535593E 01	-0.37553661E-00	0.92680752E 00
5.5	0.85990947E 01	-0.99918865E 00	0.40274188E-01
6.0	0.97979584E 01	-0.40064228E-00	-0.91623458E 00
6.5	0.11047874E 02	0.74310495E 00	-0.66917483E 00
7.0	0.12346838E 02	0.84412497E 00	0.53614651E 00
7.5	0.13693062E 02	-0.33470652E-00	0.94232239E 00
8.0	0.15084942E 02	-0.98684579E 00	-0.16166458E-00
8.5	0.16521028E 02	0.27663564E-01	-0.99961729E 00
9.0	0.17999999E 02	0.99794253E 00	-0.64114621E-01
9.5	0.19520643E 02	0.11406205E-00	0.99347363E 00
10.0	0.21081848E 02	-0.99233399E 00	0.12358493E-00
10.5	0.22682589E 02	-0.93819257E-01	-0.99558925E 00
11.0	0.24321912E 02	0.99967665E 00	-0.25427952E-01
11.5	0.25998929E 02	-0.80702017E-01	0.99673826E 00
12.0	0.27712811E 02	-0.97504447E 00	-0.22200969E-00
12.5	0.29462780E 02	0.39222410E-00	-0.91986965E 00
13.0	0.31248108E 02	0.81528299E 00	0.57906298E 00
13.5	0.33068108E 02	-0.76225052E 00	0.64728188E 00
14.0	0.34922130E 02	-0.40847914E-00	-0.91276778E 00
14.5	0.36809564E 02	0.99458133E 00	-0.10396106E-00
15.0	0.38729829E 02	-0.24286562E-00	0.97005986E 00
15.5	0.40682369E 02	-0.80975439E 00	-0.58676922E 00
16.0	0.42666659E 02	0.86268566E 00	-0.50574021E 00
16.5	0.44682205E 02	0.85386481E-01	0.99634793E 00
17.0	0.46728522E 02	-0.92489740E 00	-0.38021707E-00
17.5	0.48805162E 02	0.78075548E 00	-0.62483643E 00
18.0	0.50911681E 02	0.13875034E-00	0.99032743E 00
18.5	0.53047666E 02	-0.91063064E 00	-0.41322157E-00
19.0	0.55212715E 02	0.85223433E 00	-0.52315996E 00
19.5	0.57406438E 02	-0.72309267E-01	0.99738223E 00
20.0	0.59628470E 02	-0.74942156E 00	-0.66209325E 00
20.5	0.61878453E 02	0.98592072E 00	-0.16721262E-00
21.0	0.64156047E 02	-0.51310262E 00	0.85832710E 00
21.5	0.66460923E 02	-0.29353403E-00	-0.95594871E 00
22.0	0.68792757E 02	0.89466443E 00	0.44673906E-00
22.5	0.71151241E 02	-0.94923777E 00	0.31455915E-00
23.0	0.73536075E 02	0.47419146E-00	-0.88042159E 00
23.5	0.75946978E 02	0.23444170E-00	0.97213025E 00
24.0	0.78383660E 02	-0.80847432E 00	-0.58853170E 00
24.5	0.80845866E 02	0.99874297E 00	-0.50124244E-01
25.0	0.83333325E 02	-0.76207855E 00	0.64748432E 00

Table B-19

THE COEFFICIENTS $A_n(x)$ IN THE CLASSICAL ASYMPTOTIC EXPANSIONS

n	$A_n(4.0)$	$A_n(5.0)$	$A_n(6.0)$
1	0.01302 08333 33333	0.00931 69499 06249 123	0.00708 76439 31664 288
2	0.00130 54741 75347	0.00066 84027 77777 778	0.00038 68071 63065 844
3	0.00025 04425 28430	0.00009 17514 67913 566	0.00004 03921 91245 878
4	0.00007 12522 03727	0.00001 86783 37693 705	0.00000 62553 37501 375
5	0.00002 69051 29011	0.00000 50467 28962 103	0.00000 12857 32541 582
6	0.00001 26701 67091	0.00000 17005 61040 314	0.00000 03295 80122 467
7	0.00000 71505 37008	0.00000 06867 26050 060	0.00000 01012 46597 827
8	0.00000 47041 78155	0.00000 03232 68661 295	0.00000 00362 56731 648
9	0.00000 35349 39430	0.00000 01738 18821 499	0.00000 00148 30301 469
10	0.00000 29872 07930	0.00000 01051 03035 298	0.00000 00068 21762 064
11	0.00000 28040 43428	0.00000 00705 94281 263	0.00000 00034 85611 350
12	0.00000 28947 12367	0.00000 00521 46502 152	0.00000 00019 58676 684
13	0.00000 32594 50764	0.00000 00420 14497 292	0.00000 00012 00507 489
14	0.00000 39754 87102	0.00000 00366 67396 572	0.00000 00007 97028 606
15	0.00000 52212 77765	0.00000 00344 58907 563	0.00000 00005 69801 344
16	0.00000 73466 70944	0.00000 00346 93672 628	0.00000 00004 36415 705
17	0.00001 10256 33462	0.00000 00372 56171 056	0.00000 00003 56513 882
18	0.00001 75800 79049	0.00000 00425 06022 946	0.00000 00003 09426 018
19	0.00002 96784 31147	0.00000 00513 45876 150	0.00000 00002 84341 373
20	0.00005 28840 27375	0.00000 00654 67254 927	0.00000 00002 75795 229
21	0.00009 91903 41525	0.00000 00878 62584 873	0.00000 00002 81575 410
22	0.00019 53396 91281	0.00000 01238 11238 451	0.00000 00003 01841 498
23	0.00040 29986 99596	0.00000 01827 71382 114	0.00000 00003 38965 273
24	0.00086 91845 86792	0.00000 02820 66524 968	0.00000 00003 97948 334
25	0.00195 61180 20588	0.00000 04542 23465 271	0.00000 00004 87498 001
26	0.00458 56312 37166	0.00000 07619 18677 348	0.00000 00006 22071 098
27	0.01117 96875 75410	0.00000 13291 50442 553	0.00000 00008 25531 549
28	0.02830 37830 62708	0.00000 24078 19573 349	0.00000 00011 37657 689
29	0.07431 01387 75537	0.00000 45233 74094 742	0.00000 00016 25841 869
30	0.20206 29424 60391	0.00000 88010 82409 911	0.00000 00024 06467 581

Table B-19 (Cont'd)

THE COEFFICIENTS $A_n(x)$ IN THE CLASSICAL ASYMPTOTIC EXPANSIONS

n	$A_n(7.0)$	$A_n(8.0)$
1	0.00562 44713 24542 071	0.00460 35597 73349 918
2	0.00024 35870 18140 590	0.00016 31842 71918 403
3	0.00002 01853 76792 154	0.00001 10681 00634 412
4	0.00000 24806 75793 795	0.00000 11133 15683 228
5	0.00000 04046 22206 163	0.00000 01486 31243 536
6	0.00000 00823 07593 516	0.00000 00247 46420 099
7	0.00000 00200 64999 234	0.00000 00049 37688 679
8	0.00000 00057 01999 523	0.00000 00011 48480 995
9	0.00000 00018 50837 624	0.00000 00003 05124 468
10	0.00000 00006 75607 906	0.00000 00000 91162 351
11	0.00000 00002 73940 934	0.00000 00000 30254 488
12	0.00000 00001 22157 614	0.00000 00000 11042 451
13	0.00000 00000 59415 872	0.00000 00000 04396 018
14	0.00000 00000 31303 387	0.00000 00000 01895 660
15	0.00000 00000 17759 090	0.00000 00000 00880 242
16	0.00000 00000 10793 876	0.00000 00000 00437 896
17	0.00000 00000 06997 345	0.00000 00000 00232 348
18	0.00000 00000 04619 406	0.00000 00000 00130 982
19	0.00000 00000 03514 443	0.00000 00000 00078 178
20	0.00000 00000 02705 098	0.00000 00000 00049 252
21	0.00000 00000 02191 648	0.00000 00000 00032 661
22	0.00000 00000 01864 382	0.00000 00000 00022 741
23	0.00000 00000 01661 464	0.00000 00000 00016 587
24	0.00000 00000 01547 898	0.00000 00000 00012 648
25	0.00000 00000 01504 763	0.00000 00000 00010 064
26	0.00000 00000 01523 755	0.00000 00000 00008 341
27	0.00000 00000 01604 681	0.00000 00000 00007 190
28	0.00000 00000 01754 876	0.00000 00000 00006 436
29	0.00000 00000 01990 183	0.00000 00000 00005 974
30	0.00000 00000 02337 623	0.00000 00000 00005 743

Table B-20

THE COEFFICIENTS $A_n(x)$ IN THE CLASSICAL ASYMPTOTIC EXPANSIONS

n	$A_n(1.0)$	n	$A_n(3.0)$
1	.1041666666666666+00	1	.2004688434686201-01
2	.8355034722222217-01	2	.3094457304526749-02
3	.1282265745563271+00	3	.9139709547822118-03
4	.2918490264641404+00	4	.4003416000879845-03
5	.8816272674437575+00	5	.2327424509298436-03
6	.3321408281862767+01	6	.1687450227029808-03
7	.1499576298686259+02	7	.1466207352735687-03
8	.7892301301158699+02	8	.1485075728285682-03
9	.4744515388679998+03	9	.1718125647581276-03
10	.3207490090999999+04	10	.2235354993240953-03
11	.2408654959999999+05	11	.3230530821478668-03
12	.1989231199999999+06	12	.5134553428329405-03
13	.1791901999999999+07	13	.8901225166930805-03
14	.1748437699999999+08	14	.1671490117081857-02

n	$A_n(2.0)$	n	$A_n(4.0)$
1	.3682847818679937-01	1	.1302083333333333-01
2	.1044379340277778-01	2	.1305474175347221-02
3	.5666867524818843-02	3	.2504425284303265-03
4	.4560141038502203-02	4	.7125220372659678-04
5	.4870348588191003-02	5	.2690512901134514-04
6	.6487125550513236-02	6	.1267016709084612-04
7	.1035508368855157-01	7	.7150537007743164-05
8	.1926831372353205-01	8	.4704178155159174-05
9	.4095311285131311-01	9	.3534939429670571-05
10	.9788482943725630-01	10	.2987207929603755-05
11	.2598840722281831+00	11	.2804043423384427-05
12	.7588314819335980+00	12	.2894712379202246-05
13	.2416736708491792+01	13	.3259450750192627-05
14	.8337200641632135+01	14	.3975487061325111-05

Table B-20 (Cont'd)

n	$A_n(5.0)$	n	$A_n(7.0)$
1	.9316949906249129-02	1	.5624471324542075-02
2	.6684027777777783-03	2	.2435870181405896-03
3	.9175146791356576-04	3	.2018537679215353-04
4	.1867833769370503-04	4	.2480675793794599-05
5	.5046728962103385-05	5	.4046222061628177-06
6	.1700561040313743-05	6	.8230759351605851-07
7	.6867260500602920-06	7	.2006499923361713-07
8	.3232686612954618-06	8	.5701999522550561-08
9	.1738188214990007-06	9	.1850837624446468-08
10	.1051030353018885-06	10	.6756079061392759-09
11	.7059428114303642-07	11	.2739409335901954-09
12	.5214650236928036-07	12	.1221576141683263-09
13	.4201449710962982-07	13	.5941587181038829-10
14	.3666739619430430-07	14	.3130338639011568-10

n	$A_n(6.0)$	n	$A_n(8.0)$
1	.7087643931664294-02	1	.4603559773349922-02
2	.3868071630658438-03	2	.1631842719184028-03
3	.4039219124587786-04	3	.1106810063441180-04
4	.6255337501374764-05	4	.1113315683228076-05
5	.1285732541581796-05	5	.1486312435361024-06
6	.3295801224667600-06	6	.2474642009930891-07
7	.1012465978266584-06	7	.4937688679004468-08
8	.3625673164759974-07	8	.1148480994911912-08
9	.1483030146893619-07	9	.3051244679936253-09
10	.6821762064333982-08	10	.9116235136730252-10
11	.3485611344451478-08	11	.3025448790613831-10
12	.1958676692325371-08	12	.1104245139771371-10
13	.1200507483750372-08	13	.4396018463906861-11
14	.7970285973939246-09	14	.1895659952795571-11

Table B-20 (Cont'd)

n	$A_n(9.0)$	n	$A_n(11.0)$
1	.3858024691358024-02	1	.2855221066077309-02
2	.1146095297972869-03	2	.6277261248852160-04
3	.6514584898456896-05	3	.2640650700435702-05
4	.5491654322194565-06	4	.1647411669505825-06
5	.6144212011714596-07	5	.1364080105949640-07
6	.8573135330131605-08	6	.1408601341980650-08
7	.1433580940896128-08	7	.1743192599377849-09
8	.2794431984520721-09	8	.2514730417204718-10
9	.6221827729592102-10	9	.4143721988720650-11
10	.1557857328952611-10	10	.7678476262157784-12
11	.4332845327303985-11	11	.1580502862083735-12
12	.1325317987590837-11	12	.3577807164816551-13
13	.4421659748825643-12	13	.8833989802828937-14
14	.1597928946225705-12	14	.2362673846383553-14

n	$A_n(10.0)$	n	$A_n(12.0)$
1	.3294039229342062-02	1	.2505860543357751-02
2	.8355034722222220-04	2	.4835089538323045-04
3	.4054880321593885-05	3	.1785099521059007-05
4	.2918490264641404-06	4	.9773964845898059-07
5	.2787950212432688-07	5	.7102735929255482-08
6	.3321408281862769-08	6	.6437111766928895-09
7	.4742076629053547-09	7	.6991421474150121-10
8	.7892301301158706-10	8	.8851741124902261-11
9	.1500347502194787-10	9	.1280103361294624-11
10	.3207490091000004-11	10	.2081836567484729-12
11	.7616835771061771-12	11	.3760832852542712-13
12	.1989231200000002-12	12	.7471758622456988-14
13	.5666491663811048-13	13	.1619123425722283-14
14	.1748437700000002-13	14	.3800528513879401-15

Table B-20 (Cont'd)

n	$A_n(13.0)$
1	.2222356555389540-02
2	.3802928867647803-04
3	.1245182347741390-05
4	.6046417549651142-07
5	.3896815746645977-08
6	.3132074570459569-09
7	.3016917042901795-10
8	.3387532176858264-11
9	.4344668082341731-12
10	.6266355634582820-13
11	.1003943514868797-13
12	.1768906675025272-14
13	.3399530048116087-15
14	.7076846306335666-16

n	$A_n(15.0)$
1	.1793047845466397-02
2	.2475565843621399-04
3	.6539843789930920-06
4	.2562186240563100-07
5	.1332295530314341-08
6	.8639745162392619-10
7	.6714448506525714-11
8	.6082870183058153-12
9	.6294480063492231-13
10	.7324811241851952-14
11	.9468230395801812-15
12	.1345992373915983-15
13	.2087058885185549-16
14	.3505368842018450-17

n	$A_n(14.0)$
1	.1988550907086491-02
2	.3044837726757369-04
3	.8920760506585824-06
4	.3876055927804059-07
5	.2235243014034292-08
6	.1607570185860516-09
7	.1385556349862650-10
8	.1392089727185193-11
9	.1597582806544551-12
10	.2061791705747298-13
11	.2955711239413460-14
12	.4659943167431874-15
13	.8013413594672413-16
14	.1492661780839711-16

n	$A_n(16.0)$
1	.1627604166666666-02
2	.2039803398980033-04
3	.4891455633404814-06
4	.1739555755043866-07
5	.8210793765669292-09
6	.4833285175646258-10
7	.3409641746398528-11
8	.2803908678984149-12
9	.2633735112600454-13
10	.2782054179910341-14
11	.3264336175499981-15
12	.4212360915265520-16
13	.5928906375979980-17
14	.9039211048105041-18

Table B-20 (Cont'd)

n	$A_n(17.0)$	n	$A_n(19.0)$
1	.1486125153408904-02	1	.1257761698851765-02
2	.1700597338127871-04	2	.1218112658145827-04
3	.3723556040551098-06	3	.2257287290885441-06
4	.1209106958799956-07	4	.6203497952650543-08
5	.5210964367439643-09	5	.2262731415887266-09
6	.2800799187888995-10	6	.1029294982338471-10
7	.1804076266354483-11	7	.5611200117439291-12
8	.1354617245418725-12	8	.3565828447263459-13
9	.1161800130671895-13	9	.2588321660295779-14
10	.1120550677967813-14	10	.2112814209833057-15
11	.1200514442213222-15	11	.1915755918869904-16
12	.1414505620569984-16	12	.1910383001496502-17
13	.1817859387797468-17	13	.2077873171033199-18
14	.2530595976950671-18	14	.2448074004256393-19

n	$A_n(18.0)$	n	$A_n(20.0)$
1	.1364017710622199-02	1	.1164618738281141-02
2	.1432619122466088-04	2	.1044379340277778-04
3	.2879066973946474-06	3	.1792020857686831-06
4	.8580709878429038-08	4	.4560141038502205-08
5	.3394229670727538-09	5	.1540139453766902-09
6	.1674440494166337-10	6	.6487125550513241-11
7	.9899363327025878-12	7	.3274564981748065-12
8	.6822343712208841-13	8	.1926831372353206-13
9	.5370479222374401-14	9	.1295051138840621-14
10	.4754203274391557-15	10	.9788482943725640-16
11	.4674963856154057-16	11	.8218255958407700-17
12	.5055686903346446-17	12	.7588314819335988-18
13	.5963488755215175-18	13	.7642392503772465-19
14	.7619518977287890-19	14	.8337200641632148-20

Appendix C

TABLES TO FACILITATE HAND COMPUTATION OF THE ROOT t_1 FOR THE CASE OF A REAL EARTH

In this section we present some tables which permit one to rather easily compute (with a desk calculator) the numerical value of the first root t_1 of the equation

$$w_1'(t_1) - q\, w_1(t_1) = 0 \tag{C-1}$$

when the impedance parameter q has a phase which is approximately 45°. This case occurs very frequently in the case of the propagation of radio waves around the earth's surface.

Let us first review the theoretical solution. We consider a sphere of radius a. We assume that the time dependance is of the form $\exp(-i\omega t)$ and that the propagation constants are of the form

$$k^2 = \omega^2 \varepsilon_o \mu_o \quad , \quad r > a \tag{C-2a}$$

$$k_1^2 = \omega^2 \varepsilon_1 \mu_o + i\omega\mu_o\sigma \quad , \quad r < a \tag{C-2b}$$

For the propagation of vertically polarized waves, the theory leads to an expression for the impedance parameter q which is of the form

$$q = i(ka/2)^{\frac{1}{3}}(k/k_1)\sqrt{1 - (k/k_1)^2} \tag{C-3a}$$

Therefore, if $\omega\varepsilon_1 << \sigma$, and $\sigma \to \infty$, we see that

$$q \approx \sqrt{i}\ (ka/2)^{\frac{1}{3}}\sqrt{\omega\varepsilon_o/\sigma} \tag{C-3b}$$

Eq. (C-3b) shows us that in the limiting case $\arg(q) = 45^\circ$; but if we retain the full form given in Eq. (C-3a) we find that the actual value of arg(q) is approximately 45°.

The theory of the propagation of waves around a convex surface has been developed (See Section 1) in terms of the diffraction function

$$D(x,y,y_o,q) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(ixt)\left|w_1(t-y_>)v(t-y_<) - \frac{v'(t)-q\,v(t)}{w_1'(t)-q\,w_1(t)}\,w_1(t-y_o)w_1(t-y)\right| dt$$

$$= -i\sum_{s=1}^{\infty} \frac{\exp(ixt_s)}{t_s-q^2}\,\frac{w_1(t_s-y)}{w_1(t_s)}\,\frac{w_1(t_s-y_o)}{w_1(t_s)} \qquad \text{(C-4)}$$

Fig. 2-2 reveals that the root t_1 has the smallest imaginary part when $30^\circ < \arg(q) < 210^\circ$. Therefore, when $x > \sqrt{y} + \sqrt{y_o}$ and $x \to \infty$, we find that we need only retain the first term in Eq. (C-4). Since the factor $\exp(ixt_1)$ then determines the manner in which the wave continues to propagate (when y and y_o are fixed), the ability to determine the numerical value of t_1 accurately will be required in many problems which involve the propagation of the ground wave to great distances.

Let us assume that we know the value of t_1 for some value q_o of the impedance parameter, and let us denote this value by $t_1(q_o)$. If we want to know the value of t_1 for some neighboring value of q, we can seek to express $t_1(q)$ in the form

$$t_1(q) \approx t_1(q_o) + A_1(q-q_o) + A_2(q-q_o)^2 + A_3(q-q_o)^3$$
$$+ A_4(q-q_o)^4 + \dots + A_N(q-q_o)^N \qquad \text{(C-5)}$$

We observe that the coefficients A_n will be functions of q_o. The coefficients can be determined by observing that $t(q)$ satisfies the Ricatti differential equation

$$[t(q) - q^2]\frac{dt(q)}{dq} = 1 \qquad \text{(C-6)}$$

If Eq. (C-5) is inserted in Eq. (C-6), we find that the first several values of A_n are given by

$$A_1 = (t_o - q_o^2)^{-1} \tag{C-7a}$$

$$A_2 = -(1/2)[1 - 2q_o(t_o - q_o^2)](t_o - q_o^2)^{-3} \tag{C-7b}$$

and further terms can be obtained by means of the recurrence relation

$$nA_n(t_o - q_o^2) + (n - 1)A_{n-1}(A_1 - 2q_o) + (n - 2)A_{n-2}(A_2 - 1)$$
$$+ (n - 3)A_{n-3}A_3 + \cdots + 2A_2A_{n-2} + A_1A_{n-1} = 0 \tag{C-8}$$

In Section 2 we showed that the roots for which the phase of q is zero (i.e., $q = Q$, where Q is real and positive) arise in a class of problems which are of practical importance. Therefore, in the tables that follow, we give the values of

$$t_o = t_1(q_o)$$

for the cases in which $q_o = Q\exp(i45^\circ)$ and $q_o = Q$. Table C-1 contains the data for the case $q_o = Q\exp(i45^\circ)$ for $Q = 0.00(0.05)4.00$, i.e., $Q = 0.00, 0.05, 0.10, 0.15, \ldots, 3.95$, and 4.00. The data for the case $q_o = Q$ is given in Table C-2 for $Q = 0.00(0.05)2.00$.

Table C-1

VALUES OF q_o, t_o, AND A_n

$$q_o = 0.00\exp(i45^\circ)$$

$$t_o = 0.50939649 + i0.88230059$$

n	Real A_n	Imag A_n
1	0.49077678E-00	-0.85005043E 00
2	-0.52716219E 00	0.
3	0.67202630E-01	0.11639834E-00
4	-0.35801253E-02	0.62010126E-02
5	0.98774254E-02	-0.23255247E-07
6	0.10064451E-02	0.17432125E-02
7	0.67314184E-03	-0.11658314E-02
8	0.56667980E-03	0.82246385E-07

$$q_o = 0.05\exp(i45^\circ)$$

$$t_o = 0.55678567 + i0.87077604$$

n	Real A_n	Imag A_n
1	0.45262659E-00	-0.88682587E 00
2	-0.53248200E 00	0.19428691E-01
3	0.65814348E-01	0.11701491E-00
4	-0.18961618E-02	0.79905694E-02
5	0.97795711E-02	0.62152158E-03
6	0.14634134E-02	0.16615978E-02
7	0.84904945E-03	-0.10119360E-02
8	0.60454502E-03	-0.74738907E-04

$$q_o = 0.10\exp(i45^\circ)$$

$$t_o = 0.60407773 + i0.86158898$$

n	Real A_n	Imag A_n
1	0.41272052E-00	-0.92261416E 00
2	-0.53804180E 00	0.38800108E-01
3	0.64394664E-01	0.11812087E-00
4	-0.34130665E-03	0.98899703E-02
5	0.97879910E-02	0.13325536E-02
6	0.19306374E-02	0.16650086E-02
7	0.10569446E-02	-0.86151452E-03
8	0.67524043E-03	-0.13223865E-03

Table C-1 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$$q_o = 0.15\exp(i45^\circ)$$

$$t_o = 0.65117579 + i0.85471159$$

n	Real A_n	Imag A_n
1	0.37104204E-00	-0.95743659E 00
2	-0.54389869E 00	0.58160997E-01
3	0.62907838E-01	0.11971661E-00
4	0.10900478E-02	0.11934237E-01
5	0.98866076E-02	0.21550881E-02
6	0.24161400E-02	0.17634061E-02
7	0.13074082E-02	-0.69951822E-03
8	0.78248135E-03	-0.16964135E-03

$$q_o = 0.20\exp(i45^\circ)$$

$$t_o = 0.69798303 + i0.85011455$$

n	Real A_n	Imag A_n
1	0.32756723E-00	-0.99131552E 00
2	-0.55011404E 00	0.77554256E-01
3	0.61312683E-01	0.12180702E-00
4	0.23884770E-02	0.14159878E-01
5	0.10058814E-01	0.31152634E-02
6	0.29251252E-02	0.19718236E-02
7	0.15990656E-02	-0.50853314E-03
8	0.93126450E-03	-0.18103705E-03

$$q_o = 0.25\exp(i45^\circ)$$

$$t_o = 0.74440248 + i0.84776678$$

n	Real A_n	Imag A_n
1	0.28226469E-00	-0.10242750E 01
2	-0.55675475E 00	0.97018810E-01
3	0.59561034E-01	0.12440058E-00
4	0.35361777E-02	0.16605039E-01
5	0.10283797E-01	0.42436861E-02
6	0.34591338E-02	0.23111593E-02
7	0.19377197E-02	-0.26731351E-03
8	0.11271954E-02	-0.15615424E-03

Table C-1 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$$q_o = 0.30\exp(i45^\circ)$$

$$t_o = 0.79033678 + i0.84763502$$

n	Real A_n	Imag A_n
1	0.23507548E-00	-0.10563412E 01
2	-0.56369424E 00	0.11658908E-00
3	0.57596387E-01	0.12750845E-00
4	0.45135615E-02	0.19309778E-01
5	0.10536160E-01	0.55760220E-02
6	0.40143593E-02	0.28089224E-02
7	0.23251572E-02	0.50682307E-04
8	0.13759191E-02	-0.78781296E-04

$$q_o = 0.35\exp(i45^\circ)$$

$$t_o = 0.83568808 + i0.84968355$$

n	Real A_n	Imag A_n
1	0.18601296E-00	-0.10875440E 01
2	-0.57161380E 00	0.13629427E-00
3	0.55351876E-01	0.13114310E-00
4	0.52770384E-02	0.22315605E-01
5	0.10782291E-01	0.71528966E-02
6	0.45797289E-02	0.35000101E-02
7	0.27589741E-02	0.47896879E-03
8	0.16813800E-02	0.74375224E-04

$$q_o = 0.40\exp(i45^\circ)$$

$$t_o = 0.88035785 + i0.85387373$$

n	Real A_n	Imag A_n
1	0.13496263E-00	-0.11179169E 01
2	-0.58000366E 00	0.15615704E-00
3	0.52748542E-01	0.13531664E-00
4	0.57765931E-02	0.25664920E-01
5	0.10976752E-01	0.90198285E-02
6	0.51334415E-02	0.44271453E-02
7	0.32294668E-02	0.10594568E-02
8	0.20436674E-02	0.33615415E-03

Table C-1 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$$q_o = 0.45\exp(i45^\circ)$$

$$t_o = 0.92424677 + i0.86016357$$

n	Real A_n	Imag A_n
1	0.81881676E-01	-0.11474991E 01
2	-0.58916447E 00	0.17619231E-00
3	0.49692807E-01	0.14003857E-00
4	0.57422909E-02	0.29398875E-01
5	0.11058241E-01	0.11225963E-01
6	0.56383029E-02	0.56409825E-02
7	0.37150677E-02	0.18441891E-02
8	0.24541350E-02	0.75138412E-03

$$q_o = 0.50\exp(i45^\circ)$$

$$t_o = 0.96725465 + i0.86850721$$

n	Real A_n	Imag A_n
1	0.26679860E-01	-0.11763366E 01
2	-0.59920827E 00	0.19640484E-00
3	0.46074087E-01	0.14531258E-00
4	0.56827610E-02	0.33555016E-01
5	0.10943777E-01	0.13621948E-01
6	0.60355474E-02	0.71987110E-02
7	0.41752204E-02	0.26950638E-02
8	0.28693176E-02	0.13779397E-02

$$q_o = 0.55\exp(i45^\circ)$$

$$t_o = 1.0092804 + i0.87885439$$

n	Real A_n	Imag A_n
1	-0.30067451E-01	-0.12044838E 01
2	-0.61025987E 00	0.21678644E-00
3	0.41762462E-01	0.15113252E-00
4	0.48808203E-02	0.36162646E-01
5	0.10522406E-01	0.16855521E-01
6	0.62363749E-02	0.91607014E-02
7	0.45412576E-02	0.42827047E-02
8	0.33004466E-02	0.22875719E-02

Table C-1 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$$q_o = 0.60\exp(i45^\circ)$$

$$t_o = 1.0502221 + i\ 0.89114979$$

n	Real A_n	Imag A_n
1	-0.90292547E-01	-0.12320057E 01
2	-0.62245751E 00	0.23731278E-00
3	0.36606105E-01	0.15747670E-00
4	0.33889300E-02	0.43235994E-01
5	0.96477230E-02	0.20364205E-01
6	0.61130758E-02	0.11583455E-01
7	0.47035185E-02	0.60801339E-02
8	0.35994176E-02	0.35616199E-02

$$q_o = 0.65\exp(i45^\circ)$$

$$t_o = 1.0899773 + i0.90533239$$

n	Real A_n	Imag A_n
1	-0.15228532E-00	-0.12589797E 01
2	-0.63595340E 00	0.25793859E-00
3	0.30429444E-01	0.16430058E-00
4	0.10251596E-02	0.48764682E-01
5	0.81305329E-02	0.24363540E-01
6	0.54857755E-02	0.14507200E-01
7	0.44963928E-02	0.83514688E-02
8	0.36369886E-02	0.52810881E-02

$$q_o = 0.70\exp(i45^\circ)$$

$$t_o = 1.1284428 + i0.92133459$$

n	Real A_n	Imag A_n
1	-0.21674313E-00	-0.12854989E 01
2	-0.65091322E 00	0.27859244E-00
3	0.23032676E-01	0.17152733E-00
4	-0.24298231E-02	0.54700182E-01
5	0.57319522E-02	0.28829713E-01
6	0.41132031E-02	0.17934771E-01
7	0.36816528E-02	0.11128113E-01
8	0.31917967E-02	0.75018863E-02

Table C-1 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$$q_o = 0.75\exp(i45^\circ)$$

$$t_o = 1.1655153 + i0.93908155$$

n	Real A_n	Imag A_n
1	-0.28377413E-00	-0.13116743E 01
2	-0.66751487E 00	0.27917005E-00
3	0.14192763E-01	0.17903575E-00
4	-0.72326457E-02	0.60937604E-01
5	0.21608219E-02	0.33674400E-01
6	0.16846272E-02	0.21799766E-01
7	0.19352160E-02	0.14370657E-01
8	0.17286288E-02	0.10211703E-01

$$q_o = 0.80\exp(i45^\circ)$$

$$t_o = 1.2010921 + i0.95849018$$

n	Real A_n	Imag A_n
1	-0.35349092E-00	-0.13376389E 01
2	-0.68594492E 00	0.31952658E-00
3	0.30683625E-02	0.18664596E-00
4	-0.13672118E-01	0.67292419E-01
5	-0.29219291E-02	0.38710553E-01
6	-0.21777098E-02	0.25920190E-01
7	-0.11544094E-02	0.17910162E-01
8	-0.58899035E-03	0.13257854E-01

$$q_o = 0.85\exp(i45^\circ)$$

$$t_o = 1.2350714 + i0.97946823$$

n	Real A_n	Imag A_n
1	-0.42600704E-00	-0.13635510E 01
2	-0.70639335E 00	0.35946810E-00
3	-0.87910885E-02	0.19410221E-00
4	-0.22055050E-01	0.73472234E-01
5	-0.98845738E-02	0.43609564E-01
6	-0.78955507E-02	0.29937501E-01
7	-0.60601790E-02	0.21365411E-01
8	-0.48798396E-02	0.16240927E-01

Table C-1 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$$q_o = 0.90\exp(i45^{\circ})$$

$$t_o = 1.2673537 + i1.0019134$$

n	Real A_n	Imag A_n
1	-0.50143421E 00	-0.13895977E 01
2	-0.72904442E 00	0.35874235E-00
3	-0.23428267E-01	0.20105492E-00
4	-0.32678955E-01	0.79045778E-01
5	-0.19084015E-01	0.47852303E-01
6	-0.15877599E-01	0.33244967E-01
7	-0.13239773E-01	0.24044226E-01
8	-0.11446172E-01	0.18384216E-01

$$q_o = 0.95\exp(i45^{\circ})$$

$$t_o = 1.2978426 + i1.0257123$$

n	Real A_n	Imag A_n
1	-0.57967604E 00	-0.14159986E 01
2	-0.75406393E 00	0.37702996E-00
3	-0.40452259E-01	0.20704291E-00
4	-0.45786110E-01	0.83412436E-01
5	-0.30791914E-01	0.50682299E-01
6	-0.26409216E-01	0.34919862E-01
7	-0.22978704E-01	0.24850582E-01
8	-0.20557209E-01	0.18410957E-01

$$q_o = 1.00\exp(i45^{\circ})$$

$$t_o = 1.3264471 + i1.0507400$$

n	Real A_n	Imag A_n
1	-0.66142152E 00	-0.14430096E 01
2	-0.78158081E 00	0.39393754E-00
3	-0.60000297E-01	0.21148045E-00
4	-0.61494213E-01	0.85782430E-01
5	-0.45079466E-01	0.51078147E-01
6	-0.39472765E-01	0.33690391E-01
7	-0.35126768E-01	0.22249578E-01
8	-0.31879149E-01	0.14515670E-01

Table C-1 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$$q_o = 1.05\exp(i45^\circ)$$

$$t_o = 1.3530825 + i1.0768585$$

n	Real A_n	Imag A_n
1	-0.74613634E 00	-0.14709239E 01
2	-0.81166378E 00	0.40899394E-00
3	-0.82086617E-01	0.21365441E-00
4	-0.79701199E-01	0.85179453E-01
5	-0.61657120E-01	0.47772381E-01
6	-0.54497832E-01	0.27987981E-01
7	-0.48729718E-01	0.14381056E-01
8	-0.43959461E-01	0.45776843E-02

$$q_o = 1.10\exp(i45^\circ)$$

$$t_o = 1.3776740 + i1.1039177$$

n	Real A_n	Imag A_n
1	-0.83405215E 00	-0.15000723E 01
2	-0.84429192E 00	0.42165411E-00
3	-0.10654067E-00	0.21273929E-00
4	-0.99968848E-01	0.80485356E-01
5	-0.79682529E-01	0.39355525E-01
6	-0.70071500E-01	0.16157454E-01
7	-0.61626483E-01	-0.55768762E-03
8	-0.53708897E-01	-0.13193293E-01

$$q_o = 1.15\exp(i45^\circ)$$

$$t_o = 1.4001585 + i1.1317546$$

n	Real A_n	Imag A_n
1	-0.92515410E 00	-0.15308213E 01
2	-0.87932143E 00	0.43131225E-00
3	-0.13293539E-00	0.20784044E-00
4	-0.12140156E-00	0.70547698E-01
5	-0.97578061E-01	0.24505138E-01
6	-0.83701950E-01	-0.31087198E-02
7	-0.70198639E-01	-0.23558985E-01
8	-0.56226173E-01	-0.39048227E-01

Table C-1 (Cont'd)

VALUES OF t_o, q_o, AND A_n

$$q_o = 1.20\exp(i45^{\circ})$$

$$t_o = 1.4204878 + i1.1601938$$

n	Real A_n	Imag A_n
1	-0.10193676E 01	-0.15635658E 01
2	-0.91645177E 00	0.43732899E-00
3	-0.16052481E-00	0.19807306E-00
4	-0.14255226E-00	0.54366705E-01
5	-0.11293600E-00	0.23640746E-02
6	-0.91802556E-01	-0.30091969E-01
7	-0.69592178E-01	-0.53707222E-01
8	-0.45527863E-01	-0.70082074E-01

$$q_o = 1.25\exp(i45^{\circ})$$

$$t_o = 1.4386325 + i1.1890502$$

n	Real A_n	Imag A_n
1	-0.11165450E 01	-0.15987185E 01
2	-0.95519596E 00	0.43907478E-00
3	-0.18820219E-00	0.18268245E-00
4	-0.16140678E-00	0.31361737E-01
5	-0.12262517E-00	-0.26959103E-01
6	-0.90114665E-01	-0.63235075E-01
7	-0.54779880E-01	-0.87042430E-01
8	-0.16715474E-01	-0.98916637E-01

$$q_o = 1.30\exp(i45^{\circ})$$

$$t_o = 1.4545836 + i1.2181302$$

n	Real A_n	Imag A_n
1	-0.12164553E 01	-0.16366923E 01
2	-0.99486415E 00	0.43599030E-00
3	-0.21450788E-00	0.16119861E-00
4	-0.17550460E-00	0.16844275E-02
5	-0.12321142E-00	-0.61880367E-01
6	-0.74737786E-01	-0.98498941E-01
7	-0.22621813E-01	-0.11606693E-00
8	0.30495062E-01	-0.11402132E-00

Table C-1 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$$q_o = 1.35\exp(i45^\circ)$$

$$t_o = 1.4683571 + i1.2472358$$

n	Real A_n	Imag A_n
1	-0.13187756E 01	-0.16778798E 01
2	-0.10345690E 01	0.42765906E-00
3	-0.23770989E-00	0.13360475E-00
4	-0.18224201E-00	-0.33499251E-01
5	-0.11173595E-00	-0.99034086E-01
6	-0.43705740E-01	-0.12936444E-00
7	0.25519520E-01	-0.13078818E-00
8	0.88272503E-01	-0.10302761E-00

$$q_o = 1.40\exp(i45^\circ)$$

$$t_o = 1.4799945 + i1.2761691$$

n	Real A_n	Imag A_n
1	-0.14230932E 01	-0.17226270E 01
2	-0.10732608E 01	0.41388478E-00
3	-0.25597150E-00	0.10048096E-00
4	-0.17935414E-00	-0.71871694E-01
5	-0.86737260E-01	-0.13343360E-00
6	0.13580442E-02	-0.14794017E-00
7	0.81885668E-01	-0.12182394E-00
8	0.13995598E-00	-0.59056594E-01

$$q_o = 1.45\exp(i45^\circ)$$

$$t_o = 1.4895648 + i1.3047369$$

n	Real A_n	Imag A_n
1	-0.15289126E 01	-0.17712054E 01
2	-0.11097967E 01	0.39475746E-00
3	-0.26759717E-00	0.63073153E-01
4	-0.16549239E-00	-0.11009146E-00
5	-0.49195597E-01	-0.15928116E-00
6	0.54357100E-01	-0.14723893E-00
7	0.13288063E-00	-0.84913188E-01
8	0.16474293E-00	0.12669551E-01

Table C-1 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$$q_o = 1.50\exp(i45^{o})$$

$$t_o = 1.4971633 + i1.3327571$$

n	Real A_n	Imag A_n
1	-0.16356749E 01	-0.18237878E 01
2	-0.11430400E 01	0.37069287E-00
3	-0.27131661E-00	0.23242419E-01
4	-0.14073069E-00	-0.14424573E-00
5	-0.29466183E-02	-0.17136026E-00
6	0.10547882E-00	-0.12397455E-00
7	0.16306426E-00	-0.24660904E-01
8	0.14741138E-00	0.92863109E-01

$$q_o = 1.55\exp(i45^{o})$$

$$t_o = 1.5029100 + i1.3600639$$

n	Real A_n	Imag A_n
1	-0.17427856E 01	-0.18804292E 01
2	-0.11719757E 01	0.34242941E-00
3	-0.26653893E-00	-0.16725001E-01
4	-0.10679856E-00	-0.17056987E-00
5	0.45805302E-01	-0.16660023E-00
6	0.14381885E-00	-0.80574372E-01
7	0.16147916E-00	0.45217568E-01
8	0.88044179E-01	0.15450577E-00

$$q_o = 1.60\exp(i45^{o})$$

$$t_o = 1.5069460 + i1.3865130$$

n	Real A_n	Imag A_n
1	-0.18496485E 01	-0.19410583E 01
2	-0.11958211E 01	0.31097712E-00
3	-0.25350136E-00	-0.54421844E-01
4	-0.66899695E-01	-0.18624154E-00
5	0.89887080E-01	-0.14517745E-00
6	0.16101225E-00	-0.25134958E-01
7	0.12704954E-00	0.10636009E-00
8	0.46215098E-02	0.17554051E-00

Table C-1 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$$q_o = 1.65\exp(i45^\circ)$$

$$t_o = 1.5094280 + i1.4119853$$

n	Real A_n	Imag A_n
1	-0.19557007E 01	-0.20054810E 01
2	-0.12141093E 01	0.27752244E-00
3	-0.23326014E-00	-0.87670457E-01
4	-0.25119858E-01	-0.18998946E-00
5	0.12301879E-00	-0.11061875E-00
6	0.15418245E-00	0.31000870E-01
7	0.69618856E-01	0.14309153E-00
8	-0.74772467E-01	0.15021475E-00

$$q_o = 1.70\exp(i45^\circ)$$

$$t_o = 1.5105226 + i1.4363894$$

n	Real A_n	Imag A_n
1	-0.20604455E 01	-0.20733957E 01
2	-0.12267265E 01	0.24330847E-00
3	-0.20751836E-00	-0.11483032E-00
4	0.14399479E-01	-0.18230417E-00
5	0.14140411E-00	-0.65806430E-01
6	0.12672086E-00	0.77010261E-01
7	0.57148401E-02	0.14857285E-00
8	-0.12565487E-00	0.91309742E-01

$$q_o = 1.75\exp(i45^\circ)$$

$$t_o = 1.5104005 + i1.4596621$$

n	Real A_n	Imag A_n
1	-0.21634762E 01	-0.21444190E 01
2	-0.12338984E 01	0.20951215E-00
3	-0.17833886E-00	-0.13498454E-00
4	0.48251616E-01	-0.16519434E-00
5	0.14443447E-00	-0.26305988E-01
6	0.80574337E-01	0.10588993E-00
7	-0.48447195E-01	0.12640835E-00
8	-0.13778460E-00	0.22142768E-01

Table C-1 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$$q_o = 1.80\exp(i45^\circ)$$

$$t_o = 1.5092304 + i1.4817674$$

n	Real A_n	Imag A_n
1	-0.22644915E 01	-0.22181155E 01
2	-0.12361307E 01	0.17714363E-00
3	-0.14782319E-00	-0.14797209E-00
4	0.74281961E-01	-0.14160273E-00
5	0.13433974E-00	0.11285232E-01
6	0.43226732E-01	0.11587257E-00
7	-0.82717918E-01	0.87439504E-01
8	-0.11672077E-00	-0.35463447E-01

$$q_o = 1.85\exp(i45^\circ)$$

$$t_o = 1.5071735 + i1.5026938$$

n	Real A_n	Imag A_n
1	-0.23632956E 01	-0.22940300E 01
2	-0.12341227E 01	0.14698199E-00
3	-0.11783593E-00	-0.15428083E-00
4	0.91723242E-01	-0.11475278E-00
5	0.11509898E-00	0.40386754E-01
6	0.45425406E-02	0.10972737E-00
7	-0.94947357E-01	0.44297522E-01
8	-0.77406165E-01	-0.69537286E-01

$$q_o = 1.90\exp(i45^\circ)$$

$$t_o = 1.5043806 + i1.5224510$$

n	Real A_n	Imag A_n
1	-0.24597927E 01	-0.23717146E 01
2	-0.12266731E 01	0.11955151E-00
3	-0.89826740E-01	-0.15485409E-00
4	0.10100264E-00	-0.87495010E-01
5	0.91154896E-01	0.59676795E-01
6	-0.24694620E-01	0.92771936E-01
7	-0.89262489E-01	0.68570913E-02
8	-0.35528116E-01	-0.79280217E-01

Table C-1 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$$q_o = 1.95\exp(i45^\circ)$$

$$t_o = 1.5009886 + i1.5410661$$

n	Real A_n	Imag A_n
1	-0.25539726E 01	-0.24507305E 01
2	-0.12205941E 01	0.95133723E-01
3	-0.64762509E-01	-0.15087279E-00
4	0.10335550E-00	-0.62020496E-01
5	0.66394028E-01	0.69689599E-01
6	-0.43041542E-01	0.70744311E-01
7	-0.72656005E-01	-0.19745617E-01
8	-0.17330296E-02	-0.71207707E-01

$$q_o = 2.00\exp(i45^\circ)$$

$$t_o = 1.4971197 + i1.5585788$$

n	Real A_n	Imag A_n
1	-0.26458935E 01	-0.25307620E 01
2	-0.12106464E 01	0.73803359E-01
3	-0.43150991E-01	-0.14356328E-00
4	0.10040439E-00	-0.39712039E-01
5	0.43603485E-01	0.72107168E-01
6	-0.51524753E-01	0.48328004E-01
7	-0.51968273E-01	-0.34721857E-01
8	0.19796682E-01	-0.54013134E-01

$$q_o = 2.05\exp(i45^\circ)$$

$$t_o = 1.4928805 + i1.5750388$$

n	Real A_n	Imag A_n
1	-0.27356638E 01	-0.26114253E 01
2	-0.11994978E 01	0.55476008E-01
3	-0.25122064E-01	-0.13405856E-00
4	0.93811707E-01	-0.21220735E-01
5	0.24373738E-01	0.69057363E-01
6	-0.52466226E-01	0.28546988E-01
7	-0.32178292E-01	-0.40031955E-01
8	0.29761702E-01	-0.34945822E-01

Table C-1 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$$q_o = 2.10\exp(i45^\circ)$$

$$t_o = 1.4883628 + i1.5905017$$

n	Real A_n	Imag A_n
1	-0.28234249E 01	-0.26924697E 01
2	-0.11877027E 01	0.39957666E-01
3	-0.10531931E-01	-0.12331774E-00
4	0.85052217E-01	-0.66337522E-02
5	0.92893421E-02	0.62596819E-01
6	-0.48484058E-01	0.12833690E-01
7	-0.15935236E-01	-0.38701627E-01
8	0.31260084E-01	-0.18419635E-01

$$q_o = 2.15\exp(i45^\circ)$$

$$t_o = 1.4836442 + i1.6050261$$

n	Real A_n	Imag A_n
1	-0.29093366E 01	-0.27736759E 01
2	-0.11756996E 01	0.26988509E-01
3	0.93543349E-03	-0.11209558E-00
4	0.75302979E-01	0.43331879E-02
5	-0.17636636E-02	0.54423767E-01
6	-0.41871601E-01	0.14392892E-02
7	-0.41272982E-02	-0.33647855E-01
8	0.27792244E-01	-0.62236485E-02

$$q_o = 2.20\exp(i45^\circ)$$

$$t_o = 1.4787898 + i1.6186719$$

n	Real A_n	Imag A_n
1	-0.29935668E 01	-0.28548717E 01
2	-0.11638186E 01	0.16278045E-01
3	0.96834677E-02	-0.10094688E-00
4	0.65421522E-01	0.12165319E-01
5	-0.92962272E-02	0.45780734E-01
6	-0.34336335E-01	-0.60884871E-02
7	0.35238722E-02	-0.27113503E-01
8	0.22169391E-01	0.15994097E-02

Table C-1 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$$q_o = 2.25\exp(i45^\circ)$$

$$t_o = 1.4738531 + i1.6314986$$

n	Real A_n	Imag A_n
1	-0.30762823E 01	-0.29359249E 01
2	-0.11522961E 01	0.75314543E-02
3	0.16140974E-01	-0.90251160E-01
4	0.55977318E-01	0.17420936E-01
5	-0.13981827E-01	0.37478443E-01
6	-0.26983359E-01	-0.10512557E-01
7	0.78323085E-02	-0.20551095E-01
8	0.16302145E-01	0.58631638E-02

$$q_o = 2.30\exp(i45^\circ)$$

$$t_o = 1.4688782 + i1.6435635$$

n	Real A_n	Imag A_n
1	-0.31576431E 01	-0.30167385E 01
2	-0.11412913E 01	0.46611596E-03
3	0.20721167E-01	-0.80245519E-01
4	0.47307265E-01	0.20647744E-01
5	-0.16502746E-01	0.29977524E-01
6	-0.20421964E-01	-0.12645830E-01
7	0.97333679E-02	-0.14743836E-01
8	0.11112744E-01	0.76129670E-02

$$q_o = 2.35\exp(i45^\circ)$$

$$t_o = 1.4639002 + i1.6549218$$

n	Real A_n	Imag A_n
1	-0.32377992E 01	-0.30972430E 01
2	-0.11309019E 01	-0.51776654E-02
3	0.23798249E-01	-0.71058545E-01
4	0.39575886E-01	0.22336286E-01
5	-0.17466106E-01	0.23484930E-01
6	-0.14908195E-01	-0.13210613E-01
7	0.10061622E-01	-0.10003487E-01
8	0.69951870E-02	0.77914037E-02

Table C-1 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$$q_o = 2.40\exp(i45^\circ)$$

$$t_o = 1.4589476 + i1.6656250$$

n	Real A_n	Imag A_n
1	-0.33168883E 01	-0.31773922E 01
2	-0.11211799E 01	-0.96317171E-02
3	0.25696705E-01	-0.62740476E-01
4	0.32829142E-01	0.22899927E-01
5	-0.17369481E-01	0.18040799E-01
6	-0.10475364E-01	-0.12783258E-01
7	0.94661792E-02	-0.63559423E-02
8	0.39676845E-02	0.71139441E-02

$$q_o = 2.45\exp(i45^\circ)$$

$$t_o = 1.4540429 + i1.6757223$$

n	Real A_n	Imag A_n
1	-0.33950352E 01	-0.32571570E 01
2	-0.11121429E 01	-0.13098456E-01
3	0.26688969E-01	-0.55288094E-01
4	0.27037656E-01	0.22671692E-01
5	-0.16576942E-01	0.13586625E-01
6	-0.70342458E-02	-0.11789446E-01
7	0.84051536E-02	-0.36807614E-02
8	0.18790521E-02	0.60625041E-02

$$q_o = 2.50\exp(i45^\circ)$$

$$t_o = 1.4492032 + i1.6852585$$

n	Real A_n	Imag A_n
1	-0.34723518E 01	-0.33365223E 01
2	-0.11037847E 01	-0.15751884E-01
3	0.26998235E-01	-0.48663935E-01
4	0.22128706E-01	0.21910591E-01
5	-0.15430962E-01	0.10013570E-01
6	-0.44409981E-02	-0.10524012E-01
7	0.71761011E-02	-0.18016712E-02
8	0.52418395E-03	0.49292143E-02

Table C-1 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$$q_o = 2.55\exp(i45^{\circ})$$

$$t_o = 1.4444420 + i1.6942755$$

n	Real A_n	Imag A_n
1	-0.35489374E 01	-0.34154832E 01
2	-0.10960830E 01	-0.17739509E-01
3	0.26803680E-01	-0.42810526E-01
4	0.18008585E-01	0.20812096E-01
5	-0.14070429E-01	0.71941244E-02
6	-0.25387698E-02	-0.91789643E-02
7	0.59563193E-02	-0.53809933E-03
8	-0.29505823E-03	0.38711811E-02

$$q_o = 2.60\exp(i45^{\circ})$$

$$t_o = 1.4397693 + i1.7028119$$

n	Real A_n	Imag A_n
1	-0.36248801E 01	-0.34940425E 01
2	-0.10890052E 01	-0.19185089E-01
3	0.26246816E-01	-0.37660400E-01
4	0.14577132E-01	0.19519807E-01
5	-0.12649424E-01	0.50010202E-02
6	-0.11802283E-02	-0.78709296E-02
7	0.48402283E-02	0.26992033E-03
8	-0.74419327E-03	0.29585057E-02

$$q_o = 2.65\exp(i45^{\circ})$$

$$t_o = 1.4351924 + i1.7109028$$

n	Real A_n	Imag A_n
1	-0.37002569E 01	-0.35722084E 01
2	-0.10825128E 01	-0.20191938E-01
3	0.25437778E-01	-0.33142857E-01
4	0.11736647E-01	0.18136401E-01
5	-0.11253928E-01	0.33174241E-02
6	-0.23781255E-03	-0.66642665E-02
7	0.36690072E-02	0.75289337E-03
8	-0.95010809E-03	0.22100975E-02

Table C-1 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$$q_o = 2.70\exp(i45^\circ)$$

$$t_o = 1.4307162 + 1.7185812$$

n	Real A_n	Imag A_n
1	-0.37751360E 01	-0.36499930E 01
2	-0.10765644E 01	-0.20845375E-01
3	0.24461174E-01	-0.29188336E-01
4	0.93968058E-02	0.16733225E-01
5	-0.99355357E-02	0.20414288E-02
6	0.39336304E-03	-0.55888654E-02
7	0.30519950E-02	0.10116339E-02
8	-0.10044345E-02	0.16177900E-02

$$q_o = 2.75\exp(i45^\circ)$$

$$t_o = 1.4263443 + i1.7258766$$

n	Real A_n	Imag A_n
1	-0.38495766E 01	-0.37274108E 01
2	-0.10711180E 01	-0.21215896E-01
3	0.23381160E-01	-0.25731035E-01
4	0.74770088E-02	0.15358130E-01
5	-0.87220672E-02	0.10871763E-02
6	0.79646157E-03	-0.46530651E-02
7	0.23811133E-02	0.11209490E-02
8	-0.97028439E-03	0.11612948E-02

$$q_o = 2.80\exp(i45^\circ)$$

$$t_o = 1.4220782 + i1.7328160$$

n	Real A_n	Imag A_n
1	-0.39236308E 01	-0.38044778E 01
2	-0.10661326E 01	-0.21361263E-01
3	0.22245742E-01	-0.22710320E-01
4	0.59070366E-02	0.14041748E-01
5	-0.76254664E-02	0.38407662E-03
6	0.10356723E-02	-0.38525677E-02
7	0.18400040E-02	0.11346001E-02
8	-0.88934766E-03	0.81675809E-03

Table C-1 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$q_o = 2.85 \exp(i45^{\circ})$

$t_o = 1.4179189 + i1.7394242$

n	Real A_n	Imag A_n
1	-0.39973436E 01	-0.38812114E 01
2	-0.10615686E 01	-0.21328687E-01
3	0.21090315E-01	-0.20071374E-01
4	0.46267830E-02	0.12802324E-01
5	-0.66474937E-02	-0.12485612E-03
6	0.11595015E-02	-0.31763460E-02
7	0.14095004E-02	0.10901563E-02
8	-0.78803023E-03	0.56122144E-03

$q_o = 2.90 \exp(i45^{\circ})$

$t_o = 1.4138661 + i1.7457250$

n	Real A_n	Imag A_n
1	-0.40707552E 01	-0.39576285E 01
2	-0.10573889E 01	-0.21156660E-01
3	0.19940514E-01	-0.17765286E-01
4	0.35854494E-02	0.11649373E-01
5	-0.57837112E-02	-0.48498742E-03
6	0.12039574E-02	-0.26104024E-02
7	0.10706633E-02	0.10131820E-02
8	-0.68226423E-03	0.37459977E-03

$q_o = 2.95 \exp(i45^{\circ})$

$t_o = 1.4099192 + i1.7517384$

n	Real A_n	Imag A_n
1	-0.41438998E 01	-0.40337463E 01
2	-0.10535590E 01	-0.20876610E-01
3	0.18814492E-01	-0.15748823E-01
4	0.27404917E-02	0.10586388E-01
5	-0.50261766E-02	-0.73205028E-03
6	0.11953606E-02	-0.21400315E-02
7	0.80628451E-03	0.92059250E-03
8	-0.58105485E-03	0.24028598E-03

Table C-1 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$$q_o = 3.00\exp(i45^\circ)$$

$$t_o = 1.4060774 + i1.7574821$$

n	Real A_n	Imag A_n
1	-0.42168079E 01	-0.41695823E 01
2	-0.10500468E 01	-0.20513897E-01
3	0.17724701E-01	-0.13984065E-01
4	0.20565450E-02	0.96128322E-02
5	-0.43652582E-02	-0.69392930E-03
6	0.11526668E-02	-0.17511135E-02
7	0.60149744E-03	0.62321673E-03
8	-0.48897856E-03	0.14504905E-03

$$q_o = 3.05\exp(i45^\circ)$$

$$t_o = 1.4023372 + i1.7629775$$

n	Real A_n	Imag A_n
1	-0.42895061E 01	-0.41851520E 01
2	-0.10468232E 01	-0.20089266E-01
3	0.16679364E-01	-0.12437939E-01
4	0.15043601E-02	0.87256134E-02
5	-0.37908032E-02	-0.99224318E-03
6	0.10893353E-02	-0.14307850E-02
7	0.44387375E-03	0.72770145E-03
8	-0.40789157E-03	0.78018832E-04

$$q_o = 3.10\exp(i45^\circ)$$

$$t_o\ 1.3986976 + i1.7682386$$

n	Real A_n	Imag A_n
1	-0.43620173E 01	-0.42604715E 01
2	-0.10438616E 01	-0.19619147E-01
3	0.15683445E-01	-0.11081672E-01
4	0.10598423E-02	0.79200251E-02
5	-0.32928073E-02	-0.10436270E-02
6	0.10147535E-02	-0.11676928E-02
7	0.32325066E-03	0.63786249E-03
8	-0.33604566E-03	0.33175654E-04

Table C-1 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$$q_o = 3.15\exp(i45^\circ)$$

$$t_o = 1.3951562 + i1.7732802$$

n	Real A_n	Imag A_n
1	-0.44343621E 01	-0.43355558E 01
2	-0.10411378E 01	-0.19117019E-01
3	0.14739561E-01	-0.98903269E-02
4	0.[illegible]319847E-03	0.71905120E-02
5	-0.28618536E-02	-0.10608169E-02
6	0.93535078E-03	-0.95205159E-03
7	0.23145616E-03	0.55565957E-03
8	-0.27881856E-03	0.28548413E-05

$$q_o = 3.20\exp(i45^\circ)$$

$$t_o = 1.3917100 + i1.7781165$$

n	Real A_n	Imag A_n
1	-0.45065583E 01	-0.44104189E 01
2	-0.10386299E 01	-0.18593632E-01
3	0.13848622E-01	-0.88423163E-02
4	0.41820063E-03	0.65311504E-02
5	-0.24893192E-02	-0.10535108E-02
6	0.85542436E-03	-0.77555661E-03
7	0.16199705E-03	0.48186320E-03
8	-0.22916314E-03	-0.16690663E-04

$$q_o = 3.25\exp(i45^\circ)$$

$$t_o = 1.3883571 + i1.7827593$$

n	Real A_n	Imag A_n
1	-0.45786219E 01	-0.44850741E 01
2	-0.10363180E 01	-0.18057566E-01
3	0.13010284E-01	-0.79189835E-02
4	0.19157200E-03	0.59359587E-02
5	-0.21674543E-02	-0.10290437E-02
6	0.77774605E-03	-0.63124214E-03
7	0.10976191E-03	0.41650754E-03
8	-0.18787760E-03	-0.28649192E-04

Table C-1 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$$q_o = 3.30\exp(i45^{\circ})$$

$$t_o = 1.3850945 + i1.7872202$$

n	Real A_n	Imag A_n
1	-0.46505665E 01	-0.45595341E 01
2	-0.10341845E 01	-0.17515668E-01
3	0.12223350E-01	-0.71042042E-02
4	0.12450634E-04	0.53991216E-02
5	-0.18893922E-02	-0.99293625E-03
6	0.70401736E-03	-0.51331396E-03
7	0.70753825E-04	0.35919641E-03
8	-0.15376058E-03	-0.35339589E-04

$$q_o = 3.35\exp(i45^{\circ})$$

$$t_o = 1.3819194 + i1.7915103$$

n	Real A_n	Imag A_n
1	-0.47224044E 01	-0.46338108E 01
2	-0.10322130E 01	-0.16973265E-01
3	0.11486032E-01	-0.63840523E-02
4	-0.12803816E-03	0.49151176E-02
5	-0.16491113E-02	-0.94931187E-03
6	0.63519484E-03	-0.41698273E-03
7	0.41865315E-04	0.30930389E-03
8	-0.12569413E-03	-0.38433459E-04

$$q_o = 3.40\exp(i45^{\circ})$$

$$t_o = 1.3788290 + i1.7956392$$

n	Real A_n	Imag A_n
1	-0.47941465E 01	-0.47079150E 01
2	-0.10303891E 01	-0.16434585E-01
3	0.10796162E-01	-0.57464899E-02
4	-0.23714352E-03	0.44787870E-02
5	-0.14413653E-02	-0.90122464E-03
6	0.57172528E-03	-0.33830739E-03
7	0.20690443E-04	0.26610129E-03
8	-0.10268013E-03	-0.39125869E-04

Table C-1 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$$q_o = 3.45\exp(i45^{\circ})$$

$$t_o = 1.3758211 + i1.7996162$$

n	Real A_n	Imag A_n
1	-0.48658023E 01	-0.47818574E 01
2	-0.10286996E 01	-0.15903011E-01
3	0.10151343E-01	-0.51811139E-02
4	-0.32078498E-03	0.40853693E-02
5	-0.12616122E-02	-0.85091316E-03
6	0.51371504E-03	-0.27405603E-03
7	0.53735118E-05	0.22083779E-03
8	-0.83852606E-04	-0.38263702E-04

$$q_o = 3.50\exp(i45^{\circ})$$

$$t_o = 1.3728925 + i1.8034494$$

n	Real A_n	Imag A_n
1	-0.49373803E 01	-0.48556474E 01
2	-0.10271328E 01	-0.15381002E-01
3	0.95490669E-02	-0.46789175E-02
4	-0.38379899E-03	0.37305136E-02
5	-0.11059324E-02	-0.79999913E-03
6	0.46105070E-03	-0.22158436E-03
7	-0.55131951E-05	0.19678722E-03
8	-0.68473867E-04	-0.36440902E-04

$$q_o = 3.55\exp(i45^{\circ})$$

$$t_o = 1.3700408 + i1.8071472$$

n	Real A_n	Imag A_n
1	-0.50088878E 01	-0.49292940E 01
2	-0.10256760E 01	-0.14870457E-01
3	0.89667938E-02	-0.42321042E-02
4	-0.43013439E-03	0.34102689E-02
5	-0.97095511E-03	-0.74963500E-03
6	0.41346172E-03	-0.17873184E-03
7	-0.13064817E-04	0.16927404E-03
8	-0.55924583E-04	-0.34069068E-04

Table C-1 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$$q_o = 3.60\exp(i45^\circ)$$

$$t_o\ 1.3672636 + i1.8107165$$

n	Real A_n	Imag A_n
1	-0.50803321E 01	-0.50028058E 01
2	-0.10243256E 01	-0.14372756E-01
3	0.84620083E-02	-0.38339126E-02
4	-0.46301585E-03	0.31210649E-02
5	-0.85378578E-03	-0.70062093E-03
6	0.37067942E-03	-0.14373558E-03
7	-0.18119431E-04	0.14568627E-03
8	-0.45689525E-04	-0.31429048E-04

$$q_o = 3.65\exp(i45^\circ)$$

$$t_o = 1.3645581 + i1.8141641$$

n	Real A_n	Imag A_n
1	-0.51517186E 01	-0.50761906E 01
2	-0.10230669E 01	-0.13888909E-01
3	0.79722615E-02	-0.34784739E-02
4	-0.48507719E-03	0.28596894E-02
5	-0.75194304E-03	-0.65349318E-03
6	0.33227621E-03	-0.11515796E-03
7	-0.21316995E-04	0.12547878E-03
8	-0.37343342E-04	-0.28708592E-04

$$q_o = 3.70\exp(i45^\circ)$$

$$t_o = 1.3619222 + i1.8174963$$

n	Real A_n	Imag A_n
1	-0.52230531E 01	-0.51494560E 01
2	-0.10218941E 01	-0.13419677E-01
3	0.75151992E-02	-0.31006881E-02
4	-0.49846887E-03	0.26232601E-02
5	-0.66330115E-03	-0.60859183E-03
6	0.29789285E-03	-0.91626419E-04
7	-0.23144849E-04	0.10817268E-03
8	-0.30536673E-04	-0.26029773E-04

Table C-1 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$$q_o = 3.75\exp(i45^\circ)$$

$$t_o = 1.3593533 + i1.8207189$$

n	Real A_n	Imag A_n
1	-0.52943401E 01	-0.52226083E 01
2	-0.10207999E 01	-0.12965300E-01
3	0.70885750E-02	-0.26761138E-02
4	-0.50494686E-03	0.24091948E-02
5	-0.58603879E-03	-0.56611179E-03
6	0.26715484E-03	-0.72784453E-04
7	-0.23973297E-04	0.93350790E-04
8	-0.24983640E-04	-0.23468540E-04

$$q_o = 3.80\exp(i45^\circ)$$

$$t_o = 1.3568492 + i1.8238375$$

n	Real A_n	Imag A_n
1	-0.53655838E 01	-0.52956545E 01
2	-0.10197762E 01	-0.12526073E-01
3	0.66902680E-02	-0.26208824E-02
4	-0.50594430E-03	0.22151868E-02
5	-0.51659682E-03	-0.52614284E-03
6	0.23970441E-03	-0.57252096E-04
7	-0.24082687E-04	0.80652606E-04
8	-0.20451079E-04	-0.21069042E-04

$$q_o = 3.85\exp(i45^\circ)$$

$$t_o = 1.3544079 + i1.8268576$$

n	Real A_n	Imag A_n
1	-0.54367881E 01	-0.53686001E 01
2	-0.10188229E 01	-0.12102067E-01
3	0.63182786E-02	-0.23916164E-02
4	-0.50262973E-03	0.20391754E-02
5	-0.45963939E-03	-0.48869811E-03
6	0.21520581E-03	-0.44592585E-04
7	-0.23684528E-04	0.69767899E-04
8	-0.16749181E-04	-0.18653638E-04

Table C-1 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$$q_o = 3.90 \exp(i45^\circ)$$

$$t_o = 1.3520270 + i1.8297833$$

n	Real A_n	Imag A_n
1	-0.55079567E 01	-0.54414516E 01
2	-0.10179289E 01	-0.11693042E-01
3	0.59707344E-02	-0.21653626E-02
4	-0.49595626E-03	0.18793207E-02
5	-0.40602091E-03	-0.45373727E-03
6	0.19334895E-03	-0.34285807E-04
7	-0.22937772E-04	0.60430849E-04
8	-0.13723475E-04	-0.16630120E-04

$$q_o = 3.95 \exp(i45^\circ)$$

$$t_o = 1.3497046 + i1.8326191$$

n	Real A_n	Imag A_n
1	-0.55790924E 01	-0.55142124E 01
2	-0.10170912E 01	-0.11298958E-01
3	0.56458866E-02	-0.19995363E-02
4	-0.48669912E-03	0.17339818E-02
5	-0.36275933E-03	-0.42118280E-03
6	0.17385051E-03	-0.25906769E-04
7	-0.21961065E-04	0.52414380E-04
8	-0.11248565E-04	-0.14996757E-04

$$q_o = 4.00 \exp(i45^\circ)$$

$$t_o = 1.3474386 + i1.8353696$$

n	Real A_n	Imag A_n
1	-0.56501981E 01	-0.55868888E 01
2	-0.10163056E 01	-0.10919458E-01
3	0.53421079E-02	-0.18318711E-02
4	-0.47548915E-03	0.16016949E-02
5	-0.32301106E-03	-0.39093304E-03
6	0.15645374E-03	-0.19107663E-04
7	-0.20842511E-04	0.45524789E-04
8	-0.92225168E-05	-0.13345870E-04

Table C-2

VALUES OF q_o, t_o, AND A_n

$q_o = 0.00$

$t_o = 0.50939649 + i\ 0.88230059$

n	Real A_n	Imag A_n
1	0.49077678E-00	-0.85005043E 00
2	-0.52716219E 00	0.
3	0.67202630E-01	0.11639834E-00
4	-0.35801253E-02	0.62010126E-02
5	0.98774254E-02	-0.23255247E-07
6	0.10064451E-02	0.17432125E-02
7	0.67314184E-03	-0.11658314E-02
8	0.56667980E-03	0.82246385E-07

$q_o = 0.05$

$t_o = 0.53512579 + i0.83981265$

n	Real A_n	Imag A_n
1	0.43856310E-00	-0.84917433E 00
2	-0.51712305E 00	0.17552909E-01
3	0.66736226E-01	0.11764263E-00
4	-0.10698236E-02	0.62612829E-02
5	0.10218605E-01	0.40162159E-03
6	0.12807611E-02	0.13332575E-02
7	0.89227157E-03	-0.11828364E-02
8	0.53176835E-03	-0.90049768E-04

$q_o = 0.10$

$t_o = 0.56326947 + i0.79741256$

n	Real A_n	Tmag A_n
1	0.38735110E-00	-0.84653358E 00
2	-0.50711574E 00	0.35293913E-01
3	0.66781138E-01	0.11890952E-00
4	0.15369735E-02	0.64214895E-02
5	0.10653329E-01	0.79369623E-03
6	0.16296961E-02	0.91028214E-03
7	0.11004362E-02	-0.12421864E-02
8	0.51116494E-03	-0.21209446E-03

Table C-2 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$$q_o = 0.15$$

$$t_o = 0.59387758 + i0.75518902$$

n	Real A_n	Imag A_n
1	0.33714148E-00	-0.84210916E 00
2	-0.49706203E-00	0.53227700E-01
3	0.67359254E-01	0.12021576E-00
4	0.42665052E-02	0.66498225E-02
5	0.11203521E-01	0.10048656E-02
6	0.20502731E-02	0.45719365E-03
7	0.13024642E-02	-0.13573345E-02
8	0.50004933E-03	-0.30996638E-03

$$q_o = 0.20$$

$$t_o = 0.62700043 + i0.71323170$$

n	Real A_n	Imag A_n
1	0.28794296E-00	-0.83558138E 00
2	-0.46687997E-00	0.71361089E-01
3	0.68498044E-01	0.12157167E-00
4	0.71562034E-02	0.69120374E-02
5	0.11890515E-01	0.10078371E-02
6	0.25407451E-02	-0.48147774E-04
7	0.15008060E-02	-0.15438503E-02
8	0.49126036E-03	-0.57670641E-03

$$q_o = 0.25$$

$$t_o = 0.66268898 + i0.67163127$$

n	Real A_n	Imag A_n
1	0.23977263E-00	-0.82785004E 00
2	-0.47648284E-00	0.89701863E-01
3	0.70232038E-01	0.12298039E-00
4	0.10224310E-01	0.71701970E-02
5	0.12734948E-01	0.76787202E-03
6	0.31003409E-02	-0.63410822E-03
7	0.16945567E-02	-0.18209343E-02
8	0.47468224E-03	-0.82429291E-03

Table C-2 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$q_o = 0.30$

$t_o = 0.70099524 + i0.63047944$

n	Real A_n	Imag A_n
1	0.19265667E-00	-0.81793389E 00
2	-0.46577841E-00	0.10825760E-00
3	0.72603515E-01	0.12443665E-00
4	0.13532405E-01	0.73800662E-02
5	0.13757250E-01	0.67571130E-03
6	0.37259462E-02	-0.13362344E-02
7	0.18776381E-02	-0.22121786E-02
8	0.43509024E-03	-0.11439832E-02

$q_o = 0.35$

$t_o = 0.74197277 + i0.58986899$

n	Real A_n	Imag A_n
1	0.14663053E-00	-0.80617117E 00
2	-0.45466733E-00	0.12703449E-00
3	0.75663691E-01	0.12592572E-00
4	0.17119835E-01	0.74887159E-02
5	0.14976501E-01	0.15008233E-03
6	0.44118442E-02	-0.21994417E-02
7	0.20366920E-02	-0.27471110E-02
8	0.35001275E-03	-0.15455974E-02

$q_o = 0.40$

$t_o = 0.78567719 + i0.54989380$

n	Real A_n	Imag A_n
1	0.10174035E-00	-0.79251954E 00
2	-0.44304179E-00	0.14603568E-00
3	0.79473514E-01	0.12742114E-00
4	0.21038392E-01	0.74310044E-02
5	0.16409188E-01	-0.66570904E-03
6	0.51459903E-02	-0.32802539E-02
7	0.21472513E-02	-0.34621124E-02
8	0.18598482E-03	-0.20475867E-02

Table C-2 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$$q_o = 0.45$$

$$t_o = 0.83216666 + i0.51064889$$

n	Real A_n	Imag A_n
1	0.58043281E-01	-0.77695664E 00
2	-0.43078417E-00	0.16525917E-00
3	0.84104737E-01	0.12688175E-00
4	0.25343055E-01	0.71254536E-02
5	0.16066501E-01	-0.18469491E-02
6	0.57046667E-02	-0.46470838E-02
7	0.21685036E-02	-0.44010455E-02
8	-0.10658202E-03	-0.26663819E-02

$$q_o = 0.50$$

$$t_o = 0.88150253 + i0.47223036$$

n	Real A_n	Imag A_n
1	0.15608648E-01	-0.75946065E 00
2	-0.41776510E-00	0.18469547E-00
3	0.89640354E-01	0.13024806E-00
4	0.30090535E-01	0.64690531E-02
5	0.17950216E-01	-0.34926626E-02
6	0.66460062E-02	-0.63731966E-02
7	0.20355204E-02	-0.56146751E-02
8	-0.60030641E-03	-0.34224112E-02

$$q_o = 0.55$$

$$t_o = 0.93374994 + i0.43473539$$

n	Real A_n	Imag A_n
1	-0.25479670E-01	-0.74001116E 00
2	-0.40384207E-00	0.20432473E-00
3	0.96174251E-01	0.13143735E-00
4	0.35335816E-01	0.53300074E-02
5	0.22045446E-01	-0.57309012E-02
6	0.72999721E-02	-0.86179763E-02
7	0.16477215E-02	-0.71572965E-02
8	-0.14006944E-02	-0.43123186E-02

Table C-2 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$q_o = 0.60$

$t_o = 0.98897858 + i0.39826210$

n	Real A_n	Imag A_n
1	-0.65124152E-01	-0.71859045E 00
2	-0.38885761E-00	0.22411231E-00
3	0.10381107E-00	0.13233694E-00
4	0.41127338E-01	0.35407408E-02
5	0.24310289E-01	-0.67240490E-02
6	0.77525707E-02	-0.11447660E-01
7	0.85454329E-03	-0.90793680E-02
8	-0.26562860E-02	-0.53127281E-02

$q_o = 0.65$

$t_o = 1.0472634 + i.0.36290942$

n	Real A_n	Imag A_n
1	-0.10320996E-00	-0.69518525E 00
2	-0.37263785E-00	0.24400392E-00
3	0.11266380E-00	0.13279637E-00
4	0.47497964E-01	0.88842097E-03
5	0.26659375E-01	-0.12674160E-01
6	0.78256704E-02	-0.15021605E-01
7	-0.56468973E-03	-0.11411824E-01
8	-0.45704381E-02	-0.63441502E-02

$q_o = 0.70$

$t_o = 1.1086857 + i0.32877678$

n	Real A_n	Imag A_n
1	-0.13960421E-00	-0.66978885E 00
2	-0.35499176E-00	0.26391937E-00
3	0.12284925E-00	0.13261701E-00
4	0.54431506E-01	-0.28962810E-02
5	0.28940794E-01	-0.17625857E-01
6	0.72480750E-02	-0.19482108E-01
7	-0.29245988E-02	-0.14135721E-01
8	-0.74080730E-02	-0.72278067E-02

Table C-2 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$q_o = 0.75$

$t_o = 1.1733337 + i0.29596369$

n	Real A_n	Imag A_n
1	-0.17415380E-00	-0.64240444E 00
2	-0.33571065E-00	0.28374435E-00
3	0.13448029E-00	0.13154036E-00
4	0.61942057E-01	-0.81485390E-02
5	0.30903779E-01	-0.24464583E-01
6	0.56192616E-02	-0.24948687E-01
7	-0.66568513E-02	-0.17128546E-01
8	-0.11488824E-01	-0.76135847E-02

$q_o = 0.80$

$t_o = 1.2413039 + i0.26456921$

n	Real A_n	Imag A_n
1	-0.20668434E-00	-0.61304841E 00
2	-0.31457024E-00	0.30332016E-00
3	0.14765367E-00	0.12923298E-00
4	0.69843848E-01	-0.15278535E-01
5	0.32150637E-01	-0.32901528E-01
6	0.23635466E-02	-0.31468273E-01
7	-0.12325263E-01	-0.20077866E-01
8	-0.17140146E-01	-0.66750199E-02

$q_o = 0.85$

$t_o = 1.3127021 + i0.23469114$

n	Real A_n	Imag A_n
1	-0.23679792E-00	-0.56175585E 00
2	-0.29133429E-00	0.32243183E-00
3	0.16242892E-00	0.12527142E-00
4	0.77908345E-01	-0.24774824E-01
5	0.32080834E-01	-0.43440334E-01
6	-0.33133578E-02	-0.35925108E-01
7	-0.20609067E-01	-0.22341965E-01
8	-0.24579314E-01	-0.39673666E-02

Table C-2 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$q_o = 0.90$

$t_o = 1.3876447 + i0.20642491$

n	Real A_n	Imag A_n
1	-0.26487317E-00	-0.54658702E 00
2	-0.26576160E-00	0.34079287E-00
3	0.17879927E-00	0.11912633E-00
4	0.85702459E-01	-0.37193755E-01
5	0.29817777E-01	-0.56309552E-01
6	-0.12450532E-01	-0.46893570E-01
7	-0.32210658E-01	-0.22749615E-01
8	-0.33650915E-01	0.27294014E-02

$q_o = 0.95$

$t_o = 1.4662595 + i0.17986219$

n	Real A_n	Imag A_n
1	-0.29006452E-00	-0.51363472E 00
2	-0.23762028E-00	0.35802938E-00
3	0.19664627E-00	0.11015970E-00
4	0.92533557E-01	-0.55127269E-01
5	0.24139163E-01	-0.71534265E-01
6	-0.26307254E-01	-0.54409514E-01
7	-0.47618759E-01	-0.19351963E-01
8	-0.45330926E-01	0.15492537E-01

$q_o = 1.00$

$t_o = 1.5486874 + i0.15508894$

n	Real A_n	Imag A_n
1	-0.31230473E-00	-0.47703449E-00
2	-0.20670792E-00	0.37366179E-00
3	0.21567942E-00	0.97605970E-01
4	0.97353712E-01	-0.73127528E-01
5	0.13428456E-01	-0.88732112E-01
6	-0.46207459E-01	-0.59653530E-01
7	-0.66608068E-01	-0.91785008E-02
8	-0.50908878E-01	0.37077615E-01

Table C-2 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$q_o = 1.05$

$t_o = 1.6350828 + i0.13218309$

n	Real A_n	Imag A_n
1	-0.33150674E-00	-0.43897590E-00
2	-0.17258365E-00	0.36708962E-00
3	0.23535423E-00	0.80671722E-01
4	0.96664297E-01	-0.97569833E-01
5	-0.42917822E-02	-0.10679898E-00
6	-0.73146694E-01	-0.59573591E-01
7	-0.87348204E-01	0.11813474E-01
8	-0.50859480E-01	0.69666044E-01

$q_o = 1.10$

$t_o = 1.7256153 + i0.11121146$

n	Real A_n	Imag A_n
1	-0.34678288E-00	-0.39971452E-00
2	-0.13611348E-00	0.39757879E-00
3	0.25477710E-00	0.56282639E-01
4	0.94444534E-01	-0.12641752E-00
5	-0.31160440E-01	-0.12349015E-00
6	-0.10701422E-00	-0.49581347E-01
7	-0.10504445E-00	0.48218970E-01
8	-0.33781846E-01	0.11373176E-00

$q_o = 1.15$

$t_o = 1.8204682 + i0.09222614$

n	Real A_n	Imag A_n
1	-0.35843727E-00	-0.35958664E-00
2	-0.96530037E-01	0.40426635E-00
3	0.27259596E-00	0.29800404E-01
4	0.82170694E-01	-0.15883438E-00
5	-0.68952637E-01	-0.13495134E-00
6	-0.14525838E-00	-0.23607950E-01
7	-0.11035906E-00	0.10343695E-00
8	0.13424838E-01	0.16197757E-00

Table C-2 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$q_o = 1.20$

$t_o = 1.9198395 + i0.07526016$

n	Real A_n	Imag A_n
1	-0.36600704E-00	-0.31902035E-00
2	-0.54508518E-01	0.40618338E-00
3	0.28691886E-00	-0.53838246E-02
4	0.59015419E-01	-0.19298114E-00
5	-0.11809609E-00	-0.13539547E-00
6	-0.18111543E-00	0.24882649E-01
7	-0.88518978E-01	0.17593172E-00
8	0.10370239E-00	0.19563024E-00

$q_o = 1.25$

$t_o = 2.0239391 + i0.06032268$

n	Real A_n	Imag A_n
1	-0.36928052E-00	-0.27854304E-00
2	-0.10750655E-01	0.40231539E-00
3	0.29529958E-00	-0.47260826E-01
4	0.22368342E-01	-0.22504088E-00
5	-0.17613634E-00	-0.11731674E-00
6	-0.20191094E-00	0.10018373E-00
7	-0.21264985E-01	0.25331524E-00
8	0.23899319E-00	0.17913409E-00

$q_o = 1.30$

$t_o = 2.1329851 + i0.04739397$

n	Real A_n	Imag A_n
1	-0.36813535E-00	-0.23678201E-00
2	0.33640496E-01	0.39171547E-00
3	0.29486471E-00	-0.94893631E-01
4	-0.29211505E-01	-0.24943322E-00
5	-0.23588288E-00	-0.72644435E-01
6	-0.18901705E-00	0.19942410E-00
7	0.10518846E-00	0.30624695E-00
8	0.39143325E-00	0.65604385E-01

Table C-2 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$q_o = 1.35$

$t_o = 2.2471991 + i0.03642072$

n	Real A_n	Imag A_n
1	-0.36258218E-00	-0.20044874E-00
2	0.77120438E-01	0.37366951E-00
3	0.26266301E-00	-0.14603504E-00
4	-0.94627773E-01	-0.25[illegible]1489E-00
5	-0.28416622E-00	0.31643937E-02
6	-0.12196006E-00	0.30631079E-00
7	0.28428629E-00	0.28636159E-00
8	0.48534942E-00	-0.17648890E-00

$q_o = 1.40$

$t_o = 2.3667974 + i0.02731259$

n	Real A_n	Imag A_n
1	-0.35286616E-00	-0.16430580E-00
2	0.11773645E-00	0.34791677E-00
3	0.25642068E-00	-0.19689482E-00
4	-0.16877199E-00	-0.24537781E-00
5	-0.30259745E-00	0.10840094E-00
6	0.10973007E-01	0.38732324E-00
7	0.47026827E-00	0.15202834E-00
8	0.40593039E-00	-0.51228032E 00

$q_o = 1.45$

$t_o = 2.4919823 + i0.01994098$

n	Real A_n	Imag A_n
1	-0.33920253E-00	-0.13110943E-00
2	0.15329533E-00	0.31467478E-00
3	0.21524344E-00	-0.24227031E-00
4	-0.24174544E-00	-0.20334130E-00
5	-0.27203128E-00	0.22842067E-00
6	0.17763565E-00	0.39716548E-00
7	0.57369076E 00	-0.11495102E-00
8	0.66568863E-01	-0.79608690E 00

Table C-2 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$q_o = 1.50$

$t_o = 2.6229306 + i0.01414122$

n	Real A_n	Imag A_n
1	-0.32238712E-00	-0.10153283E-00
2	0.18163661E-00	0.27580611E-00
3	0.16071396E-00	-0.27627596E-00
4	-0.29984519E-00	-0.13220273E-00
5	-0.18304741E-00	0.35575100E-00
6	0.39108021E-00	0.29879971E-00
7	0.49517125E-00	-0.44792168E-00
8	-0.47707365E-00	-0.80556014E 00

$q_o = 1.55$

$t_o = 2.7597834 + i0.00971885$

n	Real A_n	Imag A_n
1	-0.30317284E-00	-0.76079421E-01
2	0.20105654E-00	0.23282619E-00
3	0.97245142E-01	-0.29368742E-00
4	-0.32903231E-00	-0.39348166E-01
5	-0.44033110E-01	0.39672526E-00
6	0.51831686E 00	0.92667054E-01
7	0.19892638E-00	-0.70283797E 00
8	-0.96562304E 00	-0.40065500E-00

$q_o = 1.60$

$t_o = 2.9026376 + i0.00646011$

n	Real A_n	Imag A_n
1	-0.28250285E-00	-0.55006620E-01
2	0.21070257E-00	0.18868265E-00
3	0.31658978E-01	-0.29156594E-00
4	-0.32017936E-00	0.60113462E-01
5	0.11470491E-00	0.38595626E-00
6	0.51500867E 00	-0.16602948E-00
7	-0.22480080E-00	-0.72881055E 00
8	-0.10676555E 01	0.29171895E-00

Table C-2 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$$q_o = 1.65$$

$$t_o = 3.0515411 + i0.00414553$$

n	Real A_n	Imag A_n
1	-0.26135071E-00	-0.36283383E-01
2	0.21063341E-00	0.14631177E-00
3	-0.23288849E-01	-0.27046179E-00
4	-0.27361685E-00	0.14737620E-00
5	0.25040132E-00	0.30112602E-00
6	0.36843674E-00	-0.38498398E-00
7	-0.58515138E 00	-0.48197028E-00
8	-0.65784762E 00	0.89123177E 00

$$q_o = 1.70$$

$$t_o = 3.2064956 + i0.00256434$$

n	Real A_n	Imag A_n
1	-0.24060726E-00	-0.25598344E-01
2	0.20283372E-00	0.10829071E-00
3	-0.75967736E-01	-0.23450982E-00
4	-0.17999380E-00	0.20653170E-00
5	0.32676045E-00	0.16711468E-00
6	0.13315720E-00	-0.46455469E-00
7	-0.71280074E 00	-0.76573469E-01
8	0.32455586E-01	0.10471603E 01

$$q_o = 1.75$$

$$t_o = 3.3674616 + i0.00152717$$

n	Real A_n	Imag A_n
1	-0.22098264E-00	-0.16420528E-01
2	0.18885200E-00	0.76375730E-01
3	-0.10761090E-00	-0.19023661E-00
4	-0.11636575E-00	0.23024698E-00
5	0.33064776E-00	0.24695143E-01
6	-0.99136052E-01	-0.44328199E-00
7	-0.57700423E 00	0.29053967E-00
8	0.59531185E 00	0.72523787E 00

Table C-2 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$q_o = 1.80$

$t_o = 3.5343704 + i0.000087475$

n	Real A_n	Imag A_n
1	-0.20295250E-00	-0.10094626E-01
2	0.17136483E-00	0.51285498E-01
3	-0.12297896E-00	-0.14460298E-00
4	-0.39660082E-01	0.22134382E-00
5	0.27531223E-00	-0.88861914E-01
6	-0.25280646E-00	-0.30294532E-00
7	-0.28909111E-00	0.47559384E-00
8	0.77518813E 00	0.19426705E-00

$q_o = 1,85$

$t_o = 3.7071357 + i0.00048151$

n	Real A_n	Imag A_n
1	-0.18674966E-00	-0.59432433E-02
2	0.15263959E-00	0.32779062E-01
3	-0.12470259E-00	-0.10320780E-00
4	0.18741711E-01	0.18989209E-00
5	0.18957840E-00	-0.15427920E-00
6	-0.30208256E-00	-0.13409275E-00
7	-0.40181538E-02	0.46048797E-00
8	0.60748155E 00	-0.23410610E-00

$q_o = 1.90$

$t_o = 3.8856643 + i0.00025453$

n	Real A_n	Imag A_n
1	-0.17240545E-00	-0.33494539E-02
2	0.13441952E-00	0.19943648E-01
3	-0.11694483E-00	-0.69325527E-01
4	0.54995800E-01	0.14618303E-00
5	0.10251427E-00	-0.17242768E-00
6	-0.26779618E-00	0.49741229E-02
7	0.17773409E-00	0.32196555E-00
8	0.29588679E-00	-0.41601094E-00

Table C-2 (Cont'd)

VALUES OF q_o, t_o, AND A_n

$q_o = 1.95$

$t_o = 4.0698658 + i0.00012912$

n	Real A_n	Imag A_n
1	-0.15981013E-00	-0.18062828E-02
2	0.11780544E-00	0.11554415E-01
3	-0.10400360E-00	-0.43924107E-01
4	0.71440397E-01	0.10648704E-00
5	0.33014268E-01	-0.15698161E-00
6	-0.17198830E-00	0.88272458E-01
7	0.23647793E-00	0.15606169E-00
8	0.24694245E-01	-0.38690419E-00

$q_o = 2.00$

$t_o = 4.2596573 + i0.00006282$

n	Real A_n	Imag A_n
1	-0.14877221E-00	-0.93175689E-03
2	0.10329467E-00	0.63760068E-02
3	-0.89270586E-01	-0.26302906E-01
4	0.73457155E-01	0.71675717E-01
5	-0.12091459E-01	-0.12431768E-00
6	-0.11120382E-00	0.11656965E-00
7	0.21370279E-00	0.25724519E-01
8	-0.12718074E-00	-0.25685050E-00

Appendix D

TABLES OF VALUES OF ROOTS OF $w_1'(t_s) - q\, w_1(t_s) = 0$

In this Appendix we present tables of values for the roots $t_s(q)$ of the equation

$$w_1'(t_s) - q\, w_1(t_s) = 0 \qquad \text{(D-1)}$$

which have been obtained by step-by-step integration of the Ricatti equation

$$\frac{dt}{dq} = \frac{1}{t - q^2} \qquad \text{(D-2)}$$

In each case we have held the phase of the impedance parameter q constant while varying the modulus. In the tables we write

$$t = Q\exp(iP)$$

where P is fixed and Q has been varied. The tables fall into two groups. In the first group, Tables D-1 through D-20, we give values for the roots for s = 1,2,3,4, and 5 for the phase of q equal to 45°, 0°, 90°, and 30°.

In Appendix C we have stressed the importance of the cases in which $P \approx 45^\circ$ and $P \approx 0^\circ$. We presented certain values of the roots and a set of coefficients which could be used in Eq. (C-5) to compute (on a manually-operated desk machine) the numerical value of the roots for values of q which are close to those values of q (for $P = 45^\circ$ and $P = 0^\circ$) which were tabulated. In this Appendix we present further values for the first root $t_1(q)$ for $P = 45^\circ$ and $P = 0^\circ$, as well as values for the next four roots.

For the case of some absorbing surfaces, the phase of q is approximately 90° and therefore we have given tables for the first five roots for this case. In following the literature, the author

has not found any discussion of the case in which the phase of q has been such that $P = 30^\circ$. The fact that no papers have appeared in which this case occurs is most likely due to the fact that only limiting cases have been considered in the theoretical work which has been reported in the literature. However, there are strong arguments for including this case in the present tabulation because it represents a case that lies between the cases $P = 0^\circ$ and $P = 45^\circ$. For example, the case of a perfectly conducting sphere which is coated with a thin (compared to wavelength) dielectric layer leads to a value of the phase of q which is close to 0°. The case of the dielectric interface is described by Eq. (C-3). A study of this case (which corresponds to vertical polarization) reveals that if we express q in terms of its modulus Q and phase P it is found that P lies in the range $45^\circ < P < 90^\circ$. Therefore, if one were to take a metallic sphere surrounded by a dielectric layer and permit the outer radius to remain fixed while permitting the inner core to shrink towards a radius of zero, it should be found that along the way one would observe a continuous transition from the case of $P \approx 0^\circ$ (for the very thin coating) to the range $45^\circ < P < 90^\circ$ for the "thick" coating. The tables for $P = 30^\circ$ which are contained herein will be helpful to anyone who sets out to study this transition.

The second set of tables, Tables D-21 through D-38, describe the loci* of the roots that start from $t_1^0 \approx 0.5094 + i0.8823$ for $Q = 0$ and then "fan out" as Q increases. If we observe the behavior depicted in Fig. 2-2 we see that for those loci for which $300^\circ < P < 360^\circ$ and $0^\circ < P < 19.2^\circ$ the process of "fanning out" continues to infinity. The reader will observe that Tables D-1, D-6, D-11, and D-16 provide additional tables for this second set of tables for the cases $P = 45^\circ$, 0°, 90°, and 30°.

*Tables in this group are for $P = 300^\circ$, 305°, 310°, 315°, 330°, 345°, 350°, 355°, 1°, 5°, 10°, 15°, 20°, 44°, 46°, 60°, 85°, and 120°

Table D-6 deserves some special discussion. This root locus is the one which is associated with the propagation of certain types of surface waves (on thin dielectric-covered surfaces, slightly rough surfaces, etc.). In Section 2 we denoted this root by the subscript "zero." In Eq. (2-25) we gave asymptotic expansions for $t_0(Q)$. Since the imaginary part of this root determines the attenuation of the surface wave, the determination of this imaginary part may be of interest in spite of the fact that it tends rapidly to 0 as Q increases. In the step-by-step integration procedure one finds that one loses accuracy when the root tends toward zero with increasing Q. The degradation is so extreme that in reproducing the computer output we have deleted some of the 8 figures in order to emphasize, in the vicinity of $Q = 2$, that the imaginary part is only known to about four significant figures. The data given in Table D-6b was obtained by means of Eq. (2-25). For $Q > 4$ (or $1/Q < 0.25$), the imaginary part is smaller than 10^{-38} which is the smallest number that the FORTRAN compiler will handle. Because of the extremely rapid variation of this root for large values of Q, we have included in Table D-6b an auxiliary set of numbers which vary sufficiently slowly that one can easily interpolate to find the root for intermediate value of $(1/Q)$. The auxiliary functions $R(Q)$ and $S(Q)$ are defined by means of the relation

$$t_1(Q) = t_0(Q) = Q^2R(Q) + 12Q^2\exp[-(\tfrac{4}{3}Q^3+1)]S(Q) \qquad \text{(D-3)}$$

The functions $R(Q)$ and $S(Q)$ provide good examples of slowly varying functions for which one should have no difficulty in using the theory of trigonometrical interpolation (in particular, the Chebyshev polynomial representation of Eq. (8-42)) to obtain an efficient means of computing $t_0(Q)$ on an electronic computer when Q is large.

The reader will observe that there are two types of computer out-out in the tables. Those numbers which appear in the form

0.50939649E+00

were generated with a single-precision FORTRAN II program which was run on the IBM 7090 electronic computer. Since these operations were carried out with a "word length" of 8 decimal digits, the entries in the tables are generally accurate to no more than 7 significant figures. On the other hand, the entries which appear in the form

.50939649+00

were generated with a double-precision FORTRAN IV program which was run on the UNIVAC 1107.* The results were converted to single precision before output and these entries should be, in most cases, accurate to within a digit in the 8th place.

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*The author would like to take this opportunity to point out that he feels that the IBM format is aesthetically superior to the UNIVAC format. The fact that a UNIVAC number requires a field width of 12 characters, whereas the IBM number requires an allowance for 14 characters, permits one to "pack" more output per line of computer output with the UNIVAC 1107.

Table D-1a

THE ROOTS $t_1[Q\exp(i45^\circ)]$

Q	Real t_1	Imag t_1
0.00	.50939648-00	.88230059-00
0.05	.55678568-00	.87077606-00
0.10	.60407776-00	.86158902-00
0.15	.65117583-00	.85471164-00
0.20	.69798310-00	.85011462-00
0.25	.74440256-00	.84776685-00
0.30	.79033688-00	.84763511-00
0.35	.83568820-00	.84968364-00
0.40	.88035798-00	.85387383-00
0.45	.92424691-00	.86016367-00
0.50	.96725480-00	.86850733-00
0.55	.10092806+01	.87885453-00
0.60	.10502224+01	.89114995-00
0.65	.10899776+01	.90533254-00
0.70	.11284431+01	.92133474-00
0.75	.11655156+01	.93908171-00
0.80	.12010925+01	.95849033-00
0.85	.12350718+01	.97946838-00
0.90	.12673541+01	.10019136+01
0.95	.12978431+01	.10257125+01
1.00	.13264475+01	.10507401+01
1.05	.13530829+01	.10768587+01
1.10	.13776745+01	.11039180+01
1.15	.14001590+01	.11317548+01
1.20	.14204884+01	.11601941+01
1.25	.14386330+01	.11890504+01
1.30	.14545842+01	.12181304+01
1.35	.14683577+01	.12472361+01
1.40	.14799952+01	.12761693+01
1.45	.14895654+01	.13047371+01
1.50	.14971640+01	.13327573+01
1.55	.15029108+01	.13600641+01
1.60	.15069468+01	.13865132+01
1.65	.15094288+01	.14119855+01
1.70	.15105235+01	.14363896+01
1.75	.15104014+01	.14596623+01
1.80	.15092312+01	.14817677+01
1.85	.15071744+01	.15026941+01
1.90	.15043815+01	.15224513+01
1.95	.15009896+01	.15410663+01
2.00	.14971207+01	.15585791+01

Table D-1a (Cont'd)

THE ROOTS $t_1[Q\exp(i45^\circ)]$

Q	Real t_1	Imag t_1
2.00	.14971207+01	.15585791+01
2.05	.14928815+01	.15750392+01
2.10	.14883638+01	.15905020+01
2.15	.14836453+01	.16050264+01
2.20	.14787908+01	.16186723+01
2.25	.14738542+01	.16314989+01
2.30	.14688792+01	.16435638+01
2.35	.14639013+01	.16549220+01
2.40	.14589487+01	.16656254+01
2.45	.14540439+01	.16757226+01
2.50	.14492043+01	.16852589+01
2.55	.14444432+01	.16942759+01
2.60	.14397705+01	.17028122+01
2.65	.14351936+01	.17109033+01
2.70	.14307174+01	.17185816+01
2.75	.14263454+01	.17258769+01
2.80	.14220794+01	.17328163+01
2.85	.14179201+01	.17394250+01
2.90	.14138674+01	.17457256+01
2.95	.14099204+01	.17517392+01
3.00	.14060778+01	.17574848+01
3.05	.14023377+01	[illegible]529800+01
3.10	.13986981+01	.17682411+01
3.15	.13951565+01	.17732827+01
3.20	.13917105+01	.17781185+01
3.25	.13883575+01	.17827612+01
3.30	.13850948+01	.17872221+01
3.35	.13819196+01	.17915121+01
3.40	.13788293+01	.17956409+01
3.45	.13758212+01	.17996178+01
3.50	.13728926+01	.18034510+01
3.55	.13700409+01	.18071486+01
3.60	.13672636+01	.18107177+01
3.65	.13645581+01	.18141652+01
3.70	.13619221+01	.18174973+01
3.75	.13593532+01	.18207199+01
3.80	.13568491+01	.18238385+01
3.85	.13544077+01	.18268581+01
3.90	.13520268+01	.18297837+01
3.95	.13497043+01	.18326195+01
4.00	.13474384+01	.18353699+01

Table D-1b

THE ROOTS $t_1[Q\exp(i45^{\circ})]$

1/Q	Real t_1	Imag t_1
0.00	.11690537+01	.20248604+01
0.01	.11761250+01	.20177886+01
0.02	.11831974+01	.20107123+01
0.03	.11902721+01	.20036268+01
0.04	.11973502+01	.19965279+01
0.05	.12044326+01	.19894107+01
0.06	.12115202+01	.19822706+01
0.07	.12186138+01	.19751030+01
0.08	.12257141+01	.19679031+01
0.09	.12328218+01	.19606660+01
0.10	.12399374+01	.19533867+01
0.11	.12470611+01	.19460603+01
0.12	.12541934+01	.19386815+01
0.13	.12613342+01	.19312450+01
0.14	.12684835+01	.19237455+01
0.15	.12756411+01	.19161772+01
0.16	.12828065+01	.19085346+01
0.17	.12899790+01	.19008116+01
0.18	.12971577+01	.18930022+01
0.19	.13043414+01	.18851002+01
0.20	.13115285+01	.18770991+01
0.21	.13187174+01	.18689923+01
0.22	.13259056+01	.18607731+01
0.23	.13330907+01	.18524344+01
0.24	.13402695+01	.18439690+01
0.25	.13474384+01	.18353698+01
0.26	.13545933+01	.18266292+01
0.27	.13617296+01	.18177397+01
0.28	.13688417+01	.18086935+01
0.29	.13759236+01	.17994830+01
0.30	.13829685+01	.17901004+01
0.31	.13899685+01	.17805380+01
0.32	.13969152+01	.17707884+01
0.33	.14037990+01	.17608440+01
0.34	.14106093+01	.17506978+01
0.35	.14173346+01	.17403433+01
0.36	.14239623+01	.17297743+01
0.37	.14304784+01	.17189854+01
0.38	.14368683+01	.17079720+01
0.39	.14431159+01	.16967306+01
0.40	.14492043+01	.16852587+01

Table D-2a

THE ROOTS $t_2[Q\exp(i45^{\circ})]$

Q	Real t_2	Imag t_2
0.00	.16240988+01	.28130215+01
0.05	.16389710+01	.28090750+01
0.10	.16538650+01	.28052077+01
0.15	.16688022+01	.28014266+01
0.20	.16838038+01	.27977397+01
0.25	.16988911+01	.27941562+01
0.30	.17140855+01	.27906864+01
0.35	.17294082+01	.27873422+01
0.40	.17448807+01	.27841372+01
0.45	.17605246+01	.27810871+01
0.50	.17763615+01	.27782098+01
0.55	.17924130+01	.27755260+01
0.60	.18087006+01	.27730592+01
0.65	.18252454+01	.27708366+01
0.70	.18420681+01	.27688890+01
0.75	.18591880+01	.27672519+01
0.80	.18766230+01	.27659655+01
0.85	.18943886+01	.27650756+01
0.90	.19124970+01	.27646335+01
0.95	.19309560+01	.27646975+01
1.00	.19497675+01	.27653318+01
1.05	.19689257+01	.27666074+01
1.10	.19884149+01	.27686017+01
1.15	.20082078+01	.27713972+01
1.20	.20282623+01	.27750804+01
1.25	.20485197+01	.27797394+01
1.30	.20689023+01	.27854607+01
1.35	.20893116+01	.27923246+01
1.40	.21096284+01	.28004010+01
1.45	.21297130+01	.28097428+01
1.50	.21494082+01	.28203803+01
1.55	.21685438+01	.28323163+01
1.60	.21869423+01	.28455209+01
1.65	.22044279+01	.28599295+01
1.70	.22208337+01	.28754424+01
1.75	.22360119+01	.28919274+01
1.80	.22498403+01	.29092241+01
1.85	.22622291+01	.29271516+01
1.90	.22731246+01	.29455166+01
1.95	.22825108+01	.29641222+01
2.00	.22904075+01	.29827774+01

Table D-2a (Cont'd)

THE ROOTS $t_2[Q\exp(i45^\circ)]$

Q	Real t_2	Imag t_2
2.00	.22904075+01	.29827774+01
2.05	.22968672+01	.30013044+01
2.10	.23019697+01	.30195449+01
2.15	.23058152+01	.30373643+01
2.20	.23085181+01	.30546537+01
2.25	.23102000+01	.30713305+01
2.30	.23109842+01	.30873366+01
2.35	.23109907+01	.31026358+01
2.40	.23103331+01	.31172112+01
2.45	.23091161+01	.31310615+01
2.50	.23074342+01	.31441972+01
2.55	.23053713+01	.31566383+01
2.60	.23030005+01	.31684112+01
2.65	.23003852+01	.31795465+01
2.70	.22975796+01	.31900775+01
2.75	.22946294+01	.32000382+01
2.80	.22915733+01	.32094632+01
2.85	.22884433+01	.32183861+01
2.90	.22852661+01	.32268394+01
2.95	.22820639+01	.32348541+01
3.00	.22788545+01	.32424597+01
3.05	.22756530+01	.32496837+01
3.10	.22724708+01	.32565517+01
3.15	.22693178+01	.32630878+01
3.20	.22662015+01	.32693141+01
3.25	.22631279+01	.32752513+01
3.30	.22601014+01	.32809184+01
3.35	.22571258+01	.32863329+01
3.40	.22542034+01	.32915112+01
3.45	.22513363+01	.32964681+01
3.50	.22485254+01	.33012175+01
3.55	.22457714+01	.33057722+01
3.60	.22430748+01	.33101439+01
3.65	.22404353+01	.33143436+01
3.70	.22378527+01	.33183815+01
3.75	.22353263+01	.33222666+01
3.80	.22328554+01	.33260079+01
3.85	.22304392+01	.33296134+01
3.90	.22280765+01	.33330903+01
3.95	.22257665+01	.33364457+01
4.00	.22235079+01	.33396860+01

Table D-2b

THE ROOTS $t_2[Q\exp(i45^\circ)]$

1/Q	Real t_2	Imag t_2
0.00	.20439747+01	.35402680+01
0.01	.20510460+01	.35331956+01
0.02	.20581196+01	.35261153+01
0.03	.20651972+01	.35190192+01
0.04	.20722808+01	.35118992+01
0.05	.20793723+01	.35047472+01
0.06	.20864733+01	.34975548+01
0.07	.20935854+01	.34903137+01
0.08	.21007102+01	.34830151+01
0.09	.21078488+01	.34756500+01
0.10	.21150023+01	.34682090+01
0.11	.21221714+01	.34606826+01
0.12	.21293567+01	.34530606+01
0.13	.21365584+01	.34453326+01
0.14	.21437761+01	.34374876+01
0.15	.21510093+01	.34295138+01
0.16	.21582568+01	.34213993+01
0.17	.21655165+01	.34131312+01
0.18	.21727861+01	.34046961+01
0.19	.21800619+01	.33960796+01
0.20	.21873397+01	.33872671+01
0.21	.21946138+01	.33782428+01
0.22	.22018771+01	.33689900+01
0.23	.22091211+01	.33594918+01
0.24	.22163355+01	.33497299+01
0.25	.22235078+01	.33396859+01
0.26	.22306231+01	.33293405+01
0.27	.22376636+01	.33186742+01
0.28	.22446087+01	.33076674+01
0.29	.22514341+01	.32963005+01
0.30	.22581118+01	.32845550+01
0.31	.22646093+01	.32724132+01
0.32	.22708900+01	.32598596+01
0.33	.22769123+01	.32468816+01
0.34	.22826298+01	.32334702+01
0.35	.22879915+01	.32196216+01
0.36	.22929423+01	.32053382+01
0.37	.22974233+01	.31906301+01
0.38	.23013733+01	.31755161+01
0.39	.23047306+01	.31600253+01
0.40	.23074339+01	.31441972+01

Table D-3a

THE ROOTS $t_3[Q\exp(i145^{\circ})]$

Q	Real t_3	Imag t_3
0	.24100496+01	.41743283+01
0.5	.24200711+01	.41716552+01
0.10	.24301028+01	.41690072+01
0.15	.24401549+01	.41663874+01
0.20	.24502376+01	.41637989+01
0.25	.24603611+01	.41612455+01
0.30	.24705358+01	.41587309+01
0.35	.24807721+01	.41562594+01
0.40	.24910806+01	.41538358+01
0.45	.25014721+01	.41514653+01
0.50	.25119575+01	.41491539+01
0.55	.25225481+01	.41469081+01
0.60	.25332553+01	.41447356+01
0.65	.25440909+01	.41426447+01
0.70	.25550667+01	.41406448+01
0.75	.25661951+01	.41387469+01
0.80	.25774883+01	.41369629+01
0.85	.25889591+01	.41353066+01
0.90	.26006201+01	.41337938+01
0.95	.26124838+01	.41324418+01
1.00	.26245628+01	.41312709+01
1.05	.26368689+01	.41303034+01
1.10	.26494135+01	.41295651+01
1.15	.26622069+01	.41290846+01
1.20	.26752578+01	.41288943+01
1.25	.26885727+01	.41290303+01
1.30	.27021552+01	.41295331+01
1.35	.27160052+01	.41304471+01
1.40	.27301175+01	.41318215+01
1.45	.27444809+01	.41337092+01
1.50	.27590764+01	.41361672+01
1.55	.27738757+01	.41392554+01
1.60	.27888399+01	.41430348+01
1.65	.28039176+01	.41475666+01
1.70	.28190434+01	.41529091+01
1.75	.28341380+01	.41591144+01
1.80	.28491071+01	.41662253+01
1.85	.28638435+01	.41742706+01
1.90	.28782289+01	.41832612+01
1.95	.28921374+01	.41931868+01
2.00	.29054411+01	.42040130+01

Table D-3a (Cont'd)

THE ROOTS $t_3[Q\exp(i45^\circ)]$

Q	Real t_3	Imag t_3
2.00	.29054411+01	.42040130+01
2.05	.29180156+01	.42156801+01
2.10	.29297467+01	.42281035+01
2.15	.29405362+01	.42411766+01
2.20	.29503075+01	.42547756+01
2.25	.29590092+01	.42687640+01
2.30	.29666167+01	.42830008+01
2.35	.29731322+01	.42973460+01
2.40	.29785825+01	.43116672+01
2.45	.29830153+01	.43258446+01
2.50	.29864949+01	.43397739+01
2.55	.29890973+01	.43533688+01
2.60	.29909055+01	.43665612+01
2.65	.29920051+01	.43793006+01
2.70	.29924808+01	.43915524+01
2.75	.29924145+01	.44032964+01
2.80	.29918824+01	.44145237+01
2.85	.29909545+01	.44252352+01
2.90	.29896940+01	.44354391+01
2.95	.29881567+01	.44451491+01
3.00	.29863920+01	.44543829+01
3.05	.29844428+01	.44631606+01
3.10	.29823460+01	.44715042+01
3.15	.29801332+01	.44794361+01
3.20	.29778317+01	.44869789+01
3.25	.29754640+01	.44941549+01
3.30	.29730497+01	.45009860+01
3.35	.29706048+01	.45074927+01
3.40	.29681430+01	.45136950+01
3.45	.29656755+01	.45196117+01
3.50	.29632117+01	.45252604+01
3.55	.29607593+01	.45306577+01
3.60	.29583245+01	.45358189+01
3.65	.29559125+01	.45407587+01
3.70	.29535276+01	.45454903+01
3.75	.29511730+01	.45500264+01
3.80	.29488512+01	.45543785+01
3.85	.29465645+01	.45585576+01
3.90	.29443141+01	.45625737+01
3.95	.29421014+01	.45664360+01
4.00	.29399270+01	.45701533+01

Table D-3b

THE ROOTS $t_3[Q\exp(i45^{\circ})]$

1/Q	Real t_3	Imag t_3
0.00	.27602798+01	.47809450+01
0.01	.27673514+01	.47738723+01
0.02	.27744258+01	.47667887+01
0.03	.27815057+01	.47596837+01
0.04	.27885938+01	.47525464+01
0.05	.27956927+01	.47453657+01
0.06	.28028046+01	.47381301+01
0.07	.28099317+01	.47308280+01
0.08	.28170760+01	.47234470+01
0.09	.28242391+01	.47159747+01
0.10	.28314223+01	.47083975+01
0.11	.28386266+01	.47007015+01
0.12	.28458522+01	.46928719+01
0.13	.28530989+01	.46848930+01
0.14	.28603658+01	.46767480+01
0.15	.28676511+01	.46684193+01
0.16	.28749515+01	.46598879+01
0.17	.28822631+01	.46511333+01
0.18	.28895798+01	.46421341+01
0.19	.28968938+01	.46328671+01
0.20	.29041951+01	.46233074+01
0.21	.29114708+01	.46134289+01
0.22	.29187044+01	.46032038+01
0.23	.29258759+01	.45926031+01
0.24	.29329602+01	.45815966+01
0.25	.29399269+01	.45701531+01
0.26	.29467390+01	.45582419+01
0.27	.29533520+01	.45458326+01
0.28	.29597130+01	.45328975+01
0.29	.29657604+01	.45194121+01
0.30	.29714220+01	.45053584+01
0.31	.29766163+01	.44907270+01
0.32	.29812519+01	.44755202+01
0.33	.29852298+01	.44597559+01
0.34	.29884454+01	.44434709+01
0.35	.29907931+01	.44267239+01
0.36	.29921714+01	.44095980+01
0.37	.29924902+01	.43922008+01
0.38	.29916768+01	.43746629+01
0.39	.29896840+01	.43571335+01
0.40	.29864950+01	.43397744+01

Table D-4a

THE ROOTS $t_4[Q\exp(i45^{\circ})]$

Q	Real t_4	Imag t_4
0.00	.30816536+01	.53375807+01
0.05	.30894908+01	.53354866+01
0.10	.30973342+01	.53334050+01
0.15	.31051903+01	.53313375+01
0.20	.31130653+01	.53292862+01
0.25	.31209655+01	.53272530+01
0.30	.31288974+01	.53252400+01
0.35	.31368674+01	.53232498+01
0.40	.31448821+01	.53212846+01
0.45	.31529482+01	.53193475+01
0.50	.31610724+01	.53174414+01
0.55	.31692617+01	.53155697+01
0.60	.31775233+01	.53137362+01
0.65	.31858644+01	.53119450+01
0.70	.31942924+01	.53102008+01
0.75	.32028151+01	.53085085+01
0.80	.32114404+01	.53068741+01
0.85	.32201762+01	.53053040+01
0.90	.32290308+01	.53038053+01
0.95	.32380128+01	.53023862+01
1.00	.32471307+01	.53010557+01
1.05	.32563933+01	.52998238+01
1.10	.32658093+01	.52987021+01
1.15	.32753877+01	.52977034+01
1.20	.32851370+01	.52968420+01
1.25	.32950658+01	.52961341+01
1.30	.33051821+01	.52955978+01
1.35	.33154934+01	.52952533+01
1.40	.33260062+01	.52951234+01
1.45	.33367257+01	.52952331+01
1.50	.33476555+01	.52956106+01
1.55	.33587968+01	.52962867+01
1.60	.33701480+01	.52972952+01
1.65	.33817039+01	.52986734+01
1.70	.33934546+01	.53004608+01
1.75	.34053849+01	.53026997+01
1.80	.34174726+01	.53054345+01
1.85	.34296882+01	.53087106+01
1.90	.34419933+01	.53125732+01
1.95	.34543399+01	.53170655+01
2.00	.34666695+01	.53222269+01

Table D-4a (Cont'd)

THE ROOTS $t_4[Q\exp(i45^{\circ})]$

Q	Real t_4	Imag t_4
2.00	.34666695+01	.53222269+01
2.05	.34789138+01	.53280903+01
2.10	.34909943+01	.53346793+01
2.15	.35028243+01	.53420057+01
2.20	.35143109+01	.53500666+01
2.25	.35253580+01	.53588418+01
2.30	.35358710+01	.53682936+01
2.35	.35457604+01	.53783657+01
2.40	.35549467+01	.53889853+01
2.45	.35633641+01	.54000645+01
2.50	.35709644+01	.54115051+01
2.55	.35777178+01	.54232025+01
2.60	.35836150+01	.54350505+01
2.65	.35886648+01	.54469458+01
2.70	.35928931+01	.54587923+01
2.75	.35963402+01	.54705039+01
2.80	.35990566+01	.54820062+01
2.85	.36011006+01	.54932383+01
2.90	.36025343+01	.55041519+01
2.95	.36034214+01	.55147113+01
3.00	.36038246+01	.55248919+01
3.05	.36038040+01	.55346790+01
3.10	.36034160+01	.55440660+01
3.15	.36027125+01	.55530529+01
3.20	.36017404+01	.55616453+01
3.25	.36005419+01	.55698524+01
3.30	.35991541+01	.55776863+01
3.35	.35976100+01	.55851613+01
3.40	.35959379+01	.55922930+01
3.45	.35941628+01	.55990975+01
3.50	.35923060+01	.56055910+01
3.55	.35903858+01	.56117901+01
3.60	.35884178+01	.56177107+01
3.65	.35864154+01	.56233682+01
3.70	.35843900+01	.56287776+01
3.75	.35823511+01	.56339531+01
3.80	.35803068+01	.56389079+01
3.85	.35782639+01	.56436550+01
3.90	.35762281+01	.56482062+01
3.95	.35742041+01	.56525727+01
4.00	.35721960+01	.56567650+01

Table D-4b

THE ROOTS $t_4[Q\exp(i45^\circ)]$

1/Q	Real t_4	Imag t_4
0.00	.33933540+01	.58774616+01
0.01	.34004257+01	.58703883+01
0.02	.34075008+01	.58633019+01
0.03	.34145828+01	.58561892+01
0.04	.34216749+01	.58490366+01
0.05	.34287801+01	.58418303+01
0.06	.34359016+01	.58345562+01
0.07	.34430417+01	.58271995+01
0.08	.34502030+01	.58197448+01
0.09	.34573872+01	.58121757+01
0.10	.34645956+01	.58044751+01
0.11	.34718291+01	.57966246+01
0.12	.34790876+01	.57886048+01
0.13	.34863701+01	.57603944+01
0.14	.34936742+01	.57719711+01
0.15	.35009962+01	.57633104+01
0.16	.35083304+01	.57543860+01
0.17	.35156687+01	.57451694+01
0.18	.35230001+01	.57356299+01
0.19	.35303100+01	.57257344+01
0.20	.35375794+01	.57154474+01
0.21	.35447838+01	.57047311+01
0.22	.35518919+01	.56935459+01
0.23	.35588645+01	.56818507+01
0.24	.35656525+01	.56696038+01
0.25	.35721957+01	.56567649+01
0.26	.35784206+01	.56432968+01
0.27	.35842390+01	.56291687+01
0.28	.35895470+01	.56143606+01
0.29	.35942251+01	.55988681+01
0.30	.35981397+01	.55827087+01
0.31	.36011472+01	.55659289+01
0.32	.36031002+01	.55486096+01
0.33	.36038586+01	.55308716+01
0.34	.36033016+01	.55128755+01
0.35	.36013412+01	.54948185+01
0.36	.35979360+01	.54769249+01
0.37	.35930993+01	.54594302+01
0.38	.35869020+01	.54425646+01
0.39	.35794684+01	.54265340+01
0.40	.35709650+01	.54115054+01

Table D-5a

THE ROOTS $t_5[Q\exp(i45^{\circ})]$

Q	Real t_5	Imag t_5
0.00	.36860886+01	.63844927+01
0.05	.36926405+01	.63827407+01
0.10	.36991968+01	.63809961+01
0.15	.37057620+01	.63792601+01
0.20	.37123404+01	.63775342+01
0.25	.37189365+01	.63758196+01
0.30	.37255549+01	.63741177+01
0.35	.37322000+01	.63724302+01
0.40	.37388765+01	.63707587+01
0.45	.37455890+01	.63691048+01
0.50	.37523423+01	.63674708+01
0.55	.37591412+01	.63658585+01
0.60	.37659907+01	.63642703+01
0.65	.37728959+01	.63627087+01
0.70	.37798618+01	.63611765+01
0.75	.37868939+01	.63596769+01
0.80	.37939977+01	.63582131+01
0.85	.38011787+01	.63567889+01
0.90	.38084427+01	.63554086+01
0.95	.38157958+01	.63540766+01
1.00	.38232439+01	.63527982+01
1.05	.38307933+01	.63515790+01
1.10	.38384503+01	.63504255+01
1.15	.38462215+01	.63493447+01
1.20	.38541135+01	.63483445+01
1.25	.38621330+01	.63474337+01
1.30	.38702864+01	.63466223+01
1.35	.38785806+01	.63459210+01
1.40	.38870218+01	.63453422+01
1.45	.38956164+01	.63448997+01
1.50	.39043701+01	.63446084+01
1.55	.39132879+01	.63444854+01
1.60	.39223745+01	.63445492+01
1.65	.39316329+01	.63448206+01
1.70	.39410650+01	.63453225+01
1.75	.39506707+01	.63460796+01
1.80	.39604478+01	.63471194+01
1.85	.39703908+01	.63484710+01
1.90	.39804910+01	.63501662+01
1.95	.39907353+01	.63522380+01
2.00	.40011056+01	.63547213+01

Table D-5a (Cont'd)

THE ROOTS $t_5[Q\exp(i45^o)]$

Q	Real t_5	Imag t_5
2.00	.40011056+01	.63547213+01
2.05	.40115779+01	.63576511+01
2.10	.40221217+01	.63610629+01
2.15	.40326992+01	.63649902+01
2.20	.40432654+01	.63694637+01
2.25	.40537669+01	.63745099+01
2.30	.40641435+01	.63801480+01
2.35	.40743282+01	.63863890+01
2.40	.40842489+01	.63932331+01
2.45	.40938309+01	.64006682+01
2.50	.41029990+01	.64086688+01
2.55	.41116813+01	.64171955+01
2.60	.41198119+01	.64261959+01
2.65	.41273343+01	.64356054+01
2.70	.41342041+01	.64453506+01
2.75	.41403906+01	.64553509+01
2.80	.41458779+01	.64655232+01
2.85	.41506647+01	.64757849+01
2.90	.41547638+01	.64860566+01
2.95	.41581993+01	.64962654+01
3.00	.41610056+01	.65063466+01
3.05	.41632242+01	.65162447+01
3.10	.41649015+01	.65259142+01
3.15	.41660864+01	.65353192+01
3.20	.41668285+01	.65444330+01
3.25	.41671766+01	.65532370+01
3.30	.41671775+01	.65617201+01
3.35	.41668751+01	.65698769+01
3.40	.41663099+01	.65777073+01
3.45	.41655190+01	.65852149+01
3.50	.41645354+01	.65924066+01
3.55	.41633891+01	.65992913+01
3.60	.41621061+01	.66058796+01
3.65	.41607098+01	.66121833+01
3.70	.41592205+01	.66182145+01
3.75	.41576556+01	.66239861+01
3.80	.41560305+01	.66295104+01
3.85	.41543584+01	.66348001+01
3.90	.41526508+01	.66398673+01
3.95	.41509172+01	.66447236+01
4.00	.41491662+01	.66493806+01

Table D-5b

THE ROOTS $t_5[Q\exp(i45^\circ)]$

1/Q	Real t_5	Imag t_5
0.00	.39720668+01	.68798215+01
0.01	.39791385+01	.68727479+01
0.02	.39862143+01	.68656589+01
0.03	.39932982+01	.68585389+01
0.04	.40003939+01	.68513723+01
0.05	.40075051+01	.68441426+01
0.06	.40146352+01	.68368331+01
0.07	.40217871+01	.68294260+01
0.08	.40289634+01	.68219028+01
0.09	.40361682+01	.68142437+01
0.10	.40433968+01	.68064277+01
0.11	.40506555+01	.67984322+01
0.12	.40579417+01	.67902327+01
0.13	.40652529+01	.67818027+01
0.14	.40725851+01	.67731135+01
0.15	.40799318+01	.67641335+01
0.16	.40872837+01	.67548288+01
0.17	.40946271+01	.67451619+01
0.18	.41019441+01	.67350923+01
0.19	.41092104+01	.67245764+01
0.20	.41163943+01	.67135672+01
0.21	.41234542+01	.67020154+01
0.22	.41303373+01	.66898700+01
0.23	.41369762+01	.66770799+01
0.24	.41432869+01	.66635972+01
0.25	.41491658+01	.66493805+01
0.26	.41544880+01	.66344012+01
0.27	.41591064+01	.66186509+01
0.28	.41628540+01	.66021506+01
0.29	.41655491+01	.65849617+01
0.30	.41670065+01	.65671948+01
0.31	.41670539+01	.65490173+01
0.32	.41655522+01	.65306525+01
0.33	.41624177+01	.65123713+01
0.34	.41576401+01	.64944722+01
0.35	.41512926+01	.64772541+01
0.36	.41435266+01	.64609867+01
0.37	.41345569+01	.64458855+01
0.38	.41246374+01	.64320956+01
0.39	.41140337+01	.64196884+01
0.40	.41029995+01	.64086684+01

Table D-6a

THE ROOTS $t_1[Q\exp(i0^{\circ})]$

Q	Real t_1	Imag t_1
0.00	.50939648+00	.88230058-00
0.05	.53512579-00	.83981264-00
0.10	.56326949-00	.79741255-00
0.15	.59387761-00	.75518902-00
0.20	.62700049-00	.71323170-00
0.25	.66268905-00	.67163127-00
0.30	.70099533-00	.63047944-00
0.35	.74197288-00	.58986899-00
0.40	.78567731-00	.54989382-00
0.45	.83216679-00	.51064891-00
0.50	.88150267-00	.47223038-00
0.55	.93375009-00	.43473542-00
0.60	.98897874-00	.39826214-00
0.65	.10472636+01	.36290947-00
0.70	.11086859+01	.32877682-00
0.75	.11733337+01	.29596369-00
0.80	.12413039+01	.26456921-00
0.85	.13127021+01	.23469114-00
0.90	.13876447+01	.20642491-00
0.95	.14662595+01	.17986219-00
1.00	.15486874+01	.15508894-00
1.05	.16350828+01	.13218309-00
1.10	.17256153+01	.11121146-00
1.15	.18204682+01	.9222614 -01
1.20	.19198395+01	.7526016 -01
1.25	.20239391+01	.6032268 -01
1.30	.21329851+01	.4739397 -01
1.35	.22471991+01	.3642072 -01
1.40	.23667974+01	.2731259 -01
1.45	.24919823+01	.1994098 -01
1.50	.26229306+01	.1414122 -01
1.55	.27597834+01	.971885 -02
1.60	.29026376+01	.646011 -02
1.65	.30515411+01	.414353 -02
1.70	.32064956+01	.256434 -02
1.75	.33674616+01	.152717 -02
1.80	.35343704+01	.87475 -03
1.85	.37071357+01	.48151 -03
1.90	.38856643+01	.25453 -03
1.95	.40698658+01	.12912 -03
2.00	.42596573+01	.6282 -04

Table D-6a (Cont'd)

THE ROOTS $t_1[Q\exp(i0^{\circ})]$

Q	Real t_1	Imag t_1
2.00	.42596573+01	.6282 -04
2.05	.44549658+01	.2929 -04
2.10	.46557305+01	.1308 -04
2.15	.48618994+01	.5590 -04
2.20	.50734290+01	.22844 -05
2.25	.52902835+01	.89183 -06
2.30	.55124316+01	.33235 -06
2.35	.57398493+01	.11811 -06
2.40	.59725127+01	.400085 -07
2.45	.62104024+01	.128998 -07
2.50	.64535013+01	.395597 -08
2.55	.67017940+01	.115280 -08
2.60	.69552670+01	.318918 -09
2.65	.72139081+01	.836787 -10
2.70	.74777062+01	.208041 -10
2.75	.77466512+01	.4896299 -11
2.80	.80207342+01	.10898139-11
2.85	.82999464+01	.22918461-12
2.90	.85842806+01	.45493291-13
2.95	.88737298+01	.85156374-14
3.00	.91682874+01	.15016693-14

Table D-6b

THE ROOTS $t_1[Q\exp(i0^\circ)]$

1/Q	Real t_1	Imag t_1	R(Q)	S(Q)
0.01	10000.005		1.0000005	0.99999942
0.02	2500.0100		1.0000040	0.99999534
0.03	1111.1261		1.0000134	0.99998426
0.04	625.02000	$< 10^{-38}$	1.0000320	0.99996267
0.05	400.02500		1.0000625	0.99992709
0.06	277.80778		1.0001080	0.99987398
0.07	204.11664		1.0001715	0.99979986
0.08	156.29000		1.0002560	0.99970122
0.09	123.50180		1.0003645	0.99957450
0.10	100.05001		1.0005001	0.99941620
0.11	82.699645		1.0006656	0.99922274
0.12	69.504471		1.0008644	0.99899058
0.13	59.236635		1.0010990	0.99871612
0.14	51.090457		1.0013729	0.99839574
0.15	44.519509		1.0016889	0.99802580
0.16	39.142581		1.0020500	0.99760263
0.17	34.687180		1.0024595	0.99712248
0.18	30.954330		1.0029203	0.99658161
0.19	27.795995		1.0034354	0.99597621
0.20	25.100201		1.0040081	0.99530235
0.21	22.780982		1.0046413	0.99455611
0.22	20.771454		1.0053383	0.99373345
0.23	19.018946		1.0061022	0.99283023
0.24	17.481533		1.0069363	0.99184224
0.25	16.125497	1.0163158×10^{-36}	1.0078435	0.99076514
0.26	14.923483	1.2196076×10^{-32}	1.0088275	0.98959443
0.27	13.853102	3.7987928×10^{-29}	1.0098911	0.98832554
0.28	12.895892	3.8751063×10^{-26}	1.0110379	0.98695366
0.29	12.036519	1.5593392×10^{-23}	1.0122712	0.98547384
0.30	11.262160	2.8760263×10^{-21}	1.0135943	0.98388091
0.31	10.562027	2.7466701×10^{-19}	1.0150108	0.98216945
0.32	9.9269933	1.5007935×10^{-17}	1.0165241	0.98033381
0.33	9.3492912	5.0936593×10^{-16}	1.0181378	0.97836800
0.34	8.8222793	1.1495057×10^{-14}	1.0198554	0.97626571
0.35	8.3402523	1.8256191×10^{-13}	1.0216808	0.97402023
0.36	7.8982858	2.1398222×10^{-12}	1.0236178	0.97162437
0.37	7.4921124	1.9266418×10^{-11}	1.0256701	0.96907046
0.38	7.1180189	1.3784520×10^{-10}	1.0278419	0.96635021
0.39	6.7727626	8.0658878×10^{-10}	1.0301372	0.96345460
0.40	6.4535012	3.9559756×10^{-9}	1.0325602	0.96037380

$$\{t_1[Q\exp(i0^\circ)] = Q^2R(Q) + i2Q^2\exp[-\tfrac{4}{3} - 1]S(Q)\}$$

Table D-7a

THE ROOTS $t_2[Q\exp(i10^{\circ})]$

Q	Real t_2	Imag t_2
0.00	.16240988+01	.28130215+01
0.05	.16318300+01	.27996875+01
0.10	.16396229+01	.27863333+01
0.15	.16474067+01	.27729385+01
0.20	.16553502+01	.27594817+01
0.25	.16632023+01	.27459411+01
0.30	.16711912+01	.27322935+01
0.35	.16791245+01	.27185158+01
0.40	.16870426+01	.27045833+01
0.45	.16949492+01	.26904707+01
0.50	.17028093+01	.26761520+01
0.55	.17106107+01	.26616002+01
0.60	.17183521+01	.26467875+01
0.65	.17259455+01	.26316852+01
0.70	.17334350+01	.26162645+01
0.75	.17407587+01	.26004960+01
0.80	.17478774+01	.25843503+01
0.85	.17547540+01	.25677992+01
0.90	.17613570+01	.25508158+01
0.95	.17675692+01	.25333756+01
1.00	.17733880+01	.25154589+01
1.05	.17787097+01	.24970514+01
1.10	.17834593+01	.24781477+01
1.15	.17875403+01	.24587540+01
1.20	.17908586+01	.24388913+01
1.25	.17932512+01	.24186000+01
1.30	.17947232+01	.23979435+01
1.35	.17950047+01	.23770119+01
1.40	.17942070+01	.23559242+01
1.45	.17920085+01	.23348290+01
1.50	.17885998+01	.23139003+01
1.55	.17837388+01	.22933309+01
1.60	.17776550+01	.22733205+01
1.65	.17702621+01	.22540620+01
1.70	.17617956+01	.22357265+01
1.75	.17523449+01	.22184510+01
1.80	.17421203+01	.22023304+01
1.85	.17312959+01	.21874151+01
1.90	.17200528+01	.21737145+01
1.95	.17085001+01	.21612025+01
2.00	.16969774+01	.21498270+01

Table D-7a (Cont'd)

THE ROOTS $t_2[Q\exp(i10^{\circ})]$

Q	Real t_2	Imag t_2
2.00	.16969774+01	.21498270+01
2.05	.16854026+01	.21395177+01
2.10	.16739414+01	.21301945+01
2.15	.16626686+01	.21217728+01
2.20	.16516397+01	.21141686+01
2.25	.16408940+01	.21073010+01
2.30	.16304577+01	.21010945+01
2.35	.16203465+01	.20954796+01
2.40	.16105683+01	.20903931+01
2.45	.16011248+01	.20857783+01
2.50	.15920132+01	.20815848+01
2.55	.15832275+01	.20777673+01
2.60	.15747594+01	.20742860+01
2.65	.15665990+01	.20711056+01
2.70	.15587355+01	.20681949+01
2.75	.15511573+01	.20655261+01
2.80	.15438530+01	.20630749+01
2.85	.15368107+01	.20608196+01
2.90	.15300190+01	.20587411+01
2.95	.15234666+01	.20568223+01
3.00	.15171427+01	.20550482+01
3.05	.15110366+01	.20534053+01
3.10	.15051383+01	.20518816+01
3.15	.14994383+01	.20504664+01
3.20	.14939274+01	.20491502+01
3.25	.14885968+01	.20479243+01
3.30	.14834382+01	.20467810+01
3.35	.14784439+01	.20457134+01
3.40	.14736064+01	.20447154+01
3.45	.14689187+01	.20437811+01
3.50	.14643740+01	.20429057+01
3.55	.14599661+01	.20420844+01
3.60	.14556891+01	.20413131+01
3.65	.14515372+01	.20405879+01
3.70	.14475052+01	.20399055+01
3.75	.14435880+01	.20392627+01
3.80	.14397807+01	.20386565+01
3.85	.14360789+01	.20380845+01
3.90	.14324783+01	.20375442+01
3.95	.14289748+01	.20370334+01
4.00	[illegible]	.20365500+01

Table D-7b

THE ROOTS $t_2[Q\exp(i0^\circ)]$

1/Q	Real t_2	Imag t_2
0.00	.20439747+01	.35402680+01
0.01	.205397[illegible]3+01	.35402692+01
0.02	.2063[illegible]1+01	.35402775+01
0.03	.20739932+01	.35402999+01
0.04	.208401[illegible]7+01	.35403439+01
0.05	.20940[illegible]9+01	.35404164+01
0.06	.210412[illegible]6+01	.35405252+01
0.07	.211421[illegible]5+01	.35406778+01
0.08	.2124[illegible]2+01	.35408821+01
0.09	.21344778+01	.35411461+01
0.10	.21446[illegible]1+01	.35414784+01
0.11	.2154[illegible]+01	.35418878+01
0.12	.216516[illegible]9+01	.35423835+01
0.13	.217547[illegible]2+01	.35429751+01
0.14	.218584[illegible]2+01	.35436727+01
0.15	.21962047+01	.35444871+01
0.16	.220073[illegible]5+01	.35454294+01
0.17	.22172[illegible]3+01	.35465116+01
0.18	.22278548+01	.35477462+01
0.19	.22384983+01	.35491463+01
0.20	.224919[illegible]0+01	.35507260+01
0.21	.225995[illegible]1+01	.35524998+01
0.22	.22707579+01	.35544830+01
0.23	.2281[illegible]+01	.35566917+01
0.24	.229250[illegible]1+01	.35591425+01
0.25	.23034330+01	.35616525+01
0.26	.231438[illegible]0+01	.35646393+01
0.27	.23253491+01	.35681206+01
0.28	.23363167+01	.35717142+01
0.29	.234725[illegible]7+01	.35756373+01
0.30	.235815[illegible]1+01	.35799067+01
0.31	.23689945+01	.35845379+01
0.32	.23797445+01	.35895443+01
0.33	.239037[illegible]5+01	.35949374+01
0.34	.24008549+01	.36007255+01
0.35	.24111472+01	.36069129+01
0.36	.24212149+01	.36134996+01
0.37	.24310184+01	.36204804+01
0.38	.24405177+01	.36278445+01
0.39	.24496730+01	.36355749+01
0.40	.24584462+01	.36436481+01

Table D-8a

THE ROOTS $t_3[Q\exp(i0^\circ)]$

Q	Real t_3	Imag t_3
0.00	.24100496+01	.41743283+01
0.05	.24152464+01	.41653433+01
0.10	.24204664+01	.41563491+01
0.15	.24256681+01	.41473360+01
0.20	.24309183+01	.41382948+01
0.25	.24361515+01	.41292158+01
0.30	.24413801+01	.41200889+01
0.35	.24465961+01	.41109043+01
0.40	.24517994+01	.41016517+01
0.45	.24569773+01	.40923210+01
0.50	.24621245+01	.40829017+01
0.55	.24672332+01	.40733832+01
0.60	.24722949+01	.40637550+01
0.65	.24773000+01	.40540062+01
0.70	.24822383+01	.40441262+01
0.75	.24870954+01	.40341043+01
0.80	.24918675+01	.40239301+01
0.85	.24965315+01	.40135931+01
0.90	.25010748+01	.40030836+01
0.95	.25054803+01	.39923923+01
1.00	.25097228+01	.39815106+01
1.05	.25137994+01	.39704312+01
1.10	.25176691+01	.39591479+01
1.15	.25213125+01	.39476566+01
1.20	.25247027+01	.39359550+01
1.25	.25278100+01	.39240438+01
1.30	.25306032+01	.39119271+01
1.35	.25330493+01	.38996127+01
1.40	.25351140+01	.38871125+01
1.45	.25367624+01	.38744444+01
1.50	.25379595+01	.38616313+01
1.55	.25386717+01	.38487026+01
1.60	.25388677+01	.38356940+01
1.65	.25385197+01	.38226476+01
1.70	.25376050+01	.38096116+01
1.75	.25361081+01	.37966395+01
1.80	.25340210+01	.37837891+01
1.85	.25313450+01	.37711204+01
1.90	.25280915+01	.37586945+01
1.95	.25242817+01	.37465703+01
2.00	.25199466+01	.37346039+01

Table D-8a (Cont'd)

THE ROOTS $t_3[Q\exp(i0^o)]$

Q	Real t_3	Imag t_3
2.00	.25199466+01	.37348039+01
2.05	.25151237+01	.37234449+01
2.10	.25098630+01	.37125356+01
2.15	.25042193+01	.37021101+01
2.20	.24982412+01	.36921928+01
2.25	.24919660+01	.36827991+01
2.30	.24855136+01	.36739352+01
2.35	.24786775+01	.36655996+01
2.40	.24721237+01	.36577835+01
2.45	.24652998+01	.36504726+01
2.50	.24584404+01	.36436482+01
2.55	.24515991+01	.36372881+01
2.60	.24447586+01	.36313681+01
2.65	.24380465+01	.36256629+01
2.70	.24313721+01	.36207464+01
2.75	.24248126+01	.36159929+01
2.80	.24186638+01	.36115771+01
2.85	.24120961+01	.36074746+01
2.90	.24058494+01	.36036624+01
2.95	.23997973+01	.36001185+01
3.00	.23938573+01	.35968225+01
3.05	.23881226+01	.35937551+01
3.10	.23825008+01	.35908986+01
3.15	.23770245+01	.35882365+01
3.20	.23716916+01	.35857537+01
3.25	.23665001+01	.35834362+01
3.30	.23614477+01	.35812712+01
3.35	.23565314+01	.35792468+01
3.40	.23517461+01	.35773525+01
3.45	.23470945+01	.35755780+01
3.50	.23425670+01	.35739145+01
3.55	.23381621+01	.35723536+01
3.60	.23338739+01	.35708877+01
3.65	.23297010+01	.35695098+01
3.70	.23256425+01	.35682135+01
3.75	.23216946+01	.35669930+01
3.80	.23178468+01	.35658427+01
3.85	.23141094+01	.35647580+01
3.90	.23104816+01	.35637341+01
3.95	.23069629+01	.35627668+01
4.00	.23035521+01	.35618524+01

Table D-8b

THE ROOTS $t_3[Q\exp(i0^\circ)]$

1/Q	Real t_3	Imag t_3
0.00	.27602798+01	.47809450+01
0.01	.27702808+01	.47809466+01
0.02	.27802873+01	.47809578+01
0.03	.27903048+01	.47809882+01
0.04	.28003391+01	.47810475+01
0.05	.28103955+01	.47811458+01
0.06	.28204795+01	.47812934+01
0.07	.28305963+01	.47815008+01
0.08	.28407510+01	.47817788+01
0.09	.28509484+01	.47821390+01
0.10	.28611930+01	.47825933+01
0.11	.28714890+01	.47831545+01
0.12	.28818399+01	.47838358+01
0.13	.28922486+01	.47846512+01
0.14	.29027174+01	.47856158+01
0.15	.29132476+01	.47867452+01
0.16	.29238396+01	.47880560+01
0.17	.29344922+01	.47895660+01
0.18	.29452030+01	.47912937+01
0.19	.29559677+01	.47932585+01
0.20	.29667800+01	.47954807+01
0.21	.29776310+01	.47979812+01
0.22	.29885095+01	.48007817+01
0.23	.29994009+01	.48039038+01
0.24	.30102872+01	.48073689+01
0.25	.30211471+01	.48111980+01
0.26	.30319548+01	.48154106+01
0.27	.30426807+01	.48200244+01
0.28	.30532909+01	.48250545+01
0.29	.30637477+01	.48305114+01
0.30	.30740093+01	.48364016+01
0.31	.30840313+01	.48427255+01
0.32	.30937674+01	.48494770+01
0.33	.31031700+01	.48566422+01
0.34	.31121927+01	.48641996+01
0.35	.31207911+01	.48721197+01
0.36	.31289253+01	.48803654+01
0.37	.31365609+01	.48888928+01
0.38	.31436708+01	.48976528+01
0.39	.31502359+01	.49065921+01
0.40	.31562462+01	.49156556+01

Table D-9a

THE ROOTS $t_4[Q\exp(i10^\circ)]$

Q	Real t_4	Imag t_4
0.00	.30816536+01	.53375807+01
0.05	.30857147+01	.53305541+01
0.10	.30897832+01	.53235219+01
0.15	.30938558+01	.53164781+01
0.20	.30979293+01	.53094171+01
0.25	.31020003+01	.53023331+01
0.30	.31060653+01	.52952201+01
0.35	.31101207+01	.52880721+01
0.40	.31141628+01	.52808832+01
0.45	.31181876+01	.52736473+01
0.50	.31221908+01	.52663583+01
0.55	.31261679+01	.52590100+01
0.60	.31301140+01	.52515963+01
0.65	.31340239+01	.52441109+01
0.70	.31378919+01	.52365478+01
0.75	.31417118+01	.52289008+01
0.80	.31454769+01	.52211639+01
0.85	.31491798+01	.52133310+01
0.90	.31528126+01	.52053963+01
0.95	.31563666+01	.51973543+01
1.00	.31598324+01	.51891996+01
1.05	.31631996+01	.51809274+01
1.10	.31664570+01	.51725331+01
1.15	.31695927+01	.51640130+01
1.20	.31725933+01	.51553639+01
1.25	.31754452+01	.51465837+01
1.30	.31781331+01	.51376711+01
1.35	.31806412+01	.51286265+01
1.40	.31829525+01	.51194513+01
1.45	.31850494+01	.51101491+01
1.50	.31869137+01	.51007251+01
1.55	.31885267+01	.50911868+01
1.60	.31898693+01	.50815443+01
1.65	.31909228+01	.50718102+01
1.70	.31916689+01	.50619999+01
1.75	.31920904+01	.50521316+01
1.80	.31921714+01	.50422268+01
1.85	.31918981+01	.50323094+01
1.90	.31912594+01	.50224061+01
1.95	.31902475+01	.50125460+01
2.00	.31888577+01	.50027600+01

Table D-9a (Cont'd)

THE ROOTS $t_4[Q\exp(i10^{\circ})]$

Q	Real t_4	Imag t_4
2.00	.31888577+01	.50027600+01
2.05	.31870901+01	.49930800+01
2.10	.31849487+01	.49835385+01
2.15	.31824422+01	.49741680+01
2.20	.31795837+01	.49649996+01
2.25	.31763905+01	.49560624+01
2.30	.31728840+01	.49473832+01
2.35	.31690884+01	.49389853+01
2.40	.31650309+01	.49308885+01
2.45	.31607402+01	.49231081+01
2.50	.31562463+01	.49156556+01
2.55	.31515795+01	.49085385+01
2.60	.31467695+01	.49017597+01
2.65	.31418452+01	.48953193+01
2.70	.31368340+01	.48892136+01
2.75	.31317614+01	.48834363+01
2.80	.31266510+01	.48779788+01
2.85	.31215240+01	.48728306+01
2.90	.31163994+01	.48679800+01
2.95	.31112940+01	.48634139+01
3.00	.31062222+01	.48591190+01
3.05	.31011967+01	.48550814+01
3.10	.30962279+01	.48512871+01
3.15	.30913246+01	.48477224+01
3.20	.30864942+01	.48443736+01
3.25	.30817425+01	.48412278+01
3.30	.30770738+01	.48382722+01
3.35	.30724920+01	.48354948+01
3.40	.30679993+01	.48328840+01
3.45	.30635975+01	.48304290+01
3.50	.30592877+01	.48281197+01
3.55	.30550703+01	.48259464+01
3.60	.30509451+01	.48238999+01
3.65	.30469117+01	.48219720+01
3.70	.30429693+01	.48201548+01
3.75	.30391166+01	.48184408+01
3.80	.30353524+01	.48168234+01
3.85	.30316751+01	.48152960+01
3.90	.30280830+01	.48138530+01
3.95	.30245743+01	.48124886+01
4.00	.30211472+01	.48111979+01

Table D-9b

THE ROOTS $t_4[Q\exp(i10^{\circ})]$

1/Q	Real t_4	Imag t_4
0.00	.33933540+01	.58774616+01
0.01	.34033551+01	.58774635+01
0.02	.34133631+01	.58774773+01
0.03	.34233847+01	.58775146+01
0.04	.34334265+01	.58775878+01
0.05	.34434955+01	.58777090+01
0.06	.34535979+01	.58778910+01
0.07	.34637401+01	.58781473+01
0.08	.34739277+01	.58784914+01
0.09	.34841663+01	.58789381+01
0.10	.34944607+01	.58795028+01
0.11	.35048151+01	.58802017+01
0.12	.35152327+01	.58810521+01
0.13	.35257157+01	.58820723+01
0.14	.35362649+01	.58832820+01
0.15	.35468797+01	.58847018+01
0.16	.35575573+01	.58863534+01
0.17	.35682929+01	.58882601+01
0.18	.35790791+01	.58904456+01
0.19	.35899048+01	.58929350+01
0.20	.36007562+01	.58957536+01
0.21	.36116149+01	.58989267+01
0.22	.36224581+01	.59024795+01
0.23	.36332583+01	.59064355+01
0.24	.36439829+01	.59108163+01
0.25	.36545938+01	.59156404+01
0.26	.36650480+01	.59209213+01
0.27	.36752980+01	.59266670+01
0.28	.36852925+01	.59328786+01
0.29	.36949777+01	.59395482+01
0.30	.37042991+01	.59466589+01
0.31	.37132034+01	.59541842+01
0.32	.37216411+01	.59620877+01
0.33	.37295680+01	.59703237+01
0.34	.37369483+01	.59788392+01
0.35	.37437554+01	.59875751+01
0.36	.37499733+01	.59964689+01
0.37	.37555969+01	.60054573+01
0.38	.37606318+01	.60144786+01
0.39	.37650929+01	.60234742+01
0.40	.37690035+01	.60323913+01

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Table D-10a

THE ROOTS $t_5[Q\exp(i0^\circ)]$

Q	Real t_5	Imag t_5
0.00	.36860886+01	.63844927+01
0.05	.36894825+01	.63786186+01
0.10	.36928803+01	.63727403+01
0.15	.36962797+01	.63668540+01
0.20	.36996786+01	.63609558+01
0.25	.37030744+01	.63550414+01
0.30	.37064648+01	.63491069+01
0.35	.37098473+01	.63431481+01
0.40	.37132192+01	.63371609+01
0.45	.37165778+01	.63311411+01
0.50	.37199203+01	.63250846+01
0.55	.37232437+01	.63189871+01
0.60	.37265446+01	.63128444+01
0.65	.37298196+01	.63066524+01
0.70	.37330651+01	.63004069+01
0.75	.37362769+01	.62941036+01
0.80	.37394509+01	.62877384+01
0.85	.37425824+01	.62813072+01
0.90	.37456664+01	.62748061+01
0.95	.37486976+01	.62682311+01
1.00	.37516701+01	.62615784+01
1.05	.37545777+01	.62548445+01
1.10	.37574136+01	.62480260+01
1.15	.37601706+01	.62411197+01
1.20	.37628408+01	.62341232+01
1.25	.37654158+01	.62270338+01
1.30	.37678869+01	.62198499+01
1.35	.37702444+01	.62125701+01
1.40	.37724783+01	.62051939+01
1.45	.37745780+01	.61977216+01
1.50	.37765322+01	.61901542+01
1.55	.37783295+01	.61824939+01
1.60	.37799578+01	.61747440+01
1.65	.37814049+01	.61669091+01
1.70	.37826582+01	.61589951+01
1.75	.37837055+01	.61510096+01
1.80	.37845346+01	.61429614+01
1.85	.37851334+01	.61348616+01
1.90	.37854911+01	.61267223+01
1.95	.37855974+01	.61185579+01
2.00	.37854432+01	.61103842+01

Table D-10a (Cont'd)

THE ROOTS $t_5[Q\exp(10^{\circ})]$

Q	Real t_5	Imag t_5
2.00	.37854432+01	.61103842+01
2.05	.37850212+01	.61022185+01
2.10	.37843257+01	.60940795+01
2.15	.37833533+01	.60859871+01
2.20	.37821025+01	.60779618+01
2.25	.37805746+01	.60700251+01
2.30	.37787735+01	.60621980+01
2.35	.37767056+01	.60545015+01
2.40	.37743800+01	.60469557+01
2.45	.37718081+01	.60395799+01
2.50	.37690035+01	.60323914+01
2.55	.37659821+01	.60254060+01
2.60	.37627609+01	.60186370+01
2.65	.37593585+01	.60120960+01
2.70	.37557939+01	.60057914+01
2.75	.37520871+01	.59997299+01
2.80	.37482575+01	.59939155+01
2.85	.37443247+01	.59883499+01
2.90	.37403075+01	.59830329+01
2.95	.37362238+01	.59779622+01
3.00	.37320907+01	.59731340+01
3.05	.37279237+01	.59685432+01
3.10	.37237375+01	.59641834+01
3.15	.37195452+01	.59600470+01
3.20	.37153587+01	.59561263+01
3.25	.37111886+01	.59524126+01
3.30	.37070441+01	.59488970+01
3.35	.37029333+01	.59455705+01
3.40	.36988633+01	.59424239+01
3.45	.36948400+01	.59394483+01
3.50	.36908685+01	.59366344+01
3.55	.36869528+01	.59339739+01
3.60	.36830965+01	.59314582+01
3.65	.36793022+01	.59290790+01
3.70	.36755719+01	.59268288+01
3.75	.36719073+01	.59246998+01
3.80	.36683095+01	.59226852+01
3.85	.36647790+01	.59207781+01
3.90	.36613163+01	.59189723+01
3.95	.36579213+01	.59172615+01
4.00	.36545939+01	.59156403+01

Table D-10b

THE ROOTS $t_5[Q\exp(i10^{\circ})]$

1/Q	Real t_5	Imag t_5
0.00	.39720668+01	.68798215+01
0.01	.39820681+01	.68798237+01
0.02	.39920773+01	.68798398+01
0.03	.40021025+01	.68798836+01
0.04	.40121514+01	.68799694+01
0.05	.40222318+01	.68801116+01
0.06	.40323509+01	.68803255+01
0.07	.40425158+01	.68806268+01
0.08	.40527331+01	.68810321+01
0.09	.40630084+01	.68815593+01
0.10	.40733467+01	.68822266+01
0.11	.40837520+01	.68830542+01
0.12	.40942267+01	.68840633+01
0.13	.41047717+01	.68852763+01
0.14	.41153859+01	.68867172+01
0.15	.41260657+01	.68884117+01
0.16	.41368047+01	.68903863+01
0.17	.41475926+01	.68926690+01
0.18	.41584157+01	.68952884+01
0.19	.41692553+01	.68982737+01
0.20	.41800872+01	.69016535+01
0.21	.41908821+01	.69054553+01
0.22	.42016040+01	.69097043+01
0.23	.42122108+01	.69144218+01
0.24	.42226545+01	.69196239+01
0.25	.42328812+01	.69253198+01
0.26	.42428328+01	.69315098+01
0.27	.42524485+01	.69381841+01
0.28	.42616669+01	.69453217+01
0.29	.42704288+01	.69528897+01
0.30	.42786801+01	.69608439+01
0.31	.42863745+01	.69691294+01
0.32	.42934760+01	.69776835+01
0.33	.42999607+01	.69864375+01
0.34	.43058176+01	.69953205+01
0.35	.43110484+01	.70042621+01
0.36	.43156666+01	.70131952+01
0.37	.43196961+01	.70220589+01
0.38	.43231685+01	.70307995+01
0.39	.43261217+01	.70393718+01
0.40	.43285970+01	.70477390+01

Table D-11a

THE ROOTS $t_1[Q\exp(i90^{\circ})]$

Q	Real t_1	Imag t_1
0.00	.50939648-00	.88230059-00
0.05	.55073144-00	.90683108-00
0.10	.58978919-00	.93131179-00
0.15	.62665622-00	.95569418-00
0.20	.66141845-00	.97993130-00
0.25	.69416117-00	.10039782+01
0.30	.72496902-00	.10277920+01
0.35	.75392590-00	.10513325+01
0.40	.78111483-00	.10745620+01
0.45	.80661786-00	.10974459+01
0.50	.83051591-00	.11199524+01
0.55	.85288865-00	.11420530+01
0.60	.87381428-00	.11637223+01
0.65	.89336942-00	.11849383+01
0.70	.91162886-00	.12056822+01
0.75	.92866546-00	.12259381+01
0.80	.94454993-00	.12456937+01
0.85	.95935072-00	.12649392+01
0.90	.97313388-00	.12836681+01
0.95	.98596291-00	.13018763+01
1.00	.99789871-00	.13195624+01
1.05	.10089994+01	.13367270+01
1.10	.10193206+01	.13533732+01
1.15	.10289147+01	.13695059+01
1.20	.10378318+01	.13851314+01
1.25	.10461190+01	.14002578+01
1.30	.10538206+01	.14148943+01
1.35	.10609785+01	.14290514+01
1.40	.10676316+01	.14427401+01
1.45	.10738168+01	.14559724+01
1.50	.10795682+01	.14687608+01
1.55	.10849177+01	.14811183+01
1.60	.10898950+01	.14930580+01
1.65	.10945276+01	.15045933+01
1.70	.10988413+01	.15157377+01
1.75	.11028597+01	.15265045+01
1.80	.11066048+01	.15369071+01
1.85	.11100970+01	.15469586+01
1.90	.11133550+01	.15566720+01
1.95	.11163964+01	.15660599+01
2.00	.11192370+01	.15751348+01

Table D-11a (Cont'd)

THE ROOTS $t_1[Q\exp(i90^o)]$

Q	Real t_1	Imag t_1
2.00	.11192370+01	.15751348+01
2.05	.11218918+01	.15839086+01
2.10	.11243743+01	.15923932+01
2.15	.11266972+01	.16005997+01
2.20	.11288721+01	.16085393+01
2.25	.11309098+01	.16162225+01
2.30	.11328202+01	.16236594+01
2.35	.11346122+01	.16308599+01
2.40	.11362945+01	.16378335+01
2.45	.11378746+01	.16445892+01
2.50	.11393599+01	.16511357+01
2.55	.11407569+01	.16574814+01
2.60	.11420716+01	.16636341+01
2.65	.11433099+01	.16696017+01
2.70	.11444767+01	.16753913+01
2.75	.11455770+01	.16810100+01
2.80	.11466153+01	.16864645+01
2.85	.11475956+01	.16917612+01
2.90	.11485217+01	.16969062+01
2.95	.11493973+01	.17019053+01
3.00	.11502254+01	.17067642+01
3.05	.11510093+01	.17114882+01
3.10	.11517517+01	.17160823+01
3.15	.11524551+01	.17205515+01
3.20	.11531221+01	.17249004+01
3.25	.11537549+01	.17291335+01
3.30	.11543556+01	.17332550+01
3.35	.11549261+01	.17372690+01
3.40	.11554683+01	.17411794+01
3.45	.11559837+01	.17449899+01
3.50	.11564741+01	.17487041+01
3.55	.11569409+01	.17523253+01
3.60	.11573854+01	.17558569+01
3.65	.11578089+01	.17593019+01
3.70	.11582126+01	.17626633+01
3.75	.11585976+01	.17659440+01
3.80	.11589651+01	.17691468+01
3.85	.11593158+01	.17722742+01
3.90	.11596508+01	.17753287+01
3.95	.11599709+01	.17783128+01
4.00	.11602769+01	.17812288+01

Table D-11b

THE ROOTS $t_1[Q\exp(i90^\circ)]$

1/Q	Real t_1	Imag t_1
0.00	.11690537+01	.20248604+01
0.01	.11690530+01	.20148608+01
0.02	.11690484+01	.20048635+01
0.03	.11690357+01	.19948709+01
0.04	.11690112+01	.19848854+01
0.05	.11689712+01	.19749093+01
0.06	.11689119+01	.19649450+01
0.07	.11688297+01	.19549949+01
0.08	.11687213+01	.19450615+01
0.09	.11685834+01	.19351472+01
0.10	.11684127+01	.19252546+01
0.11	.11682063+01	.19153862+01
0.12	.11679614+01	.19055446+01
0.13	.11676751+01	.18957322+01
0.14	.11673449+01	.18859518+01
0.15	.11669686+01	.18762060+01
0.16	.11665440+01	.18664973+01
0.17	.11660689+01	.18568285+01
0.18	.11655417+01	.18472022+01
0.19	.11649606+01	.18376211+01
0.20	.11643243+01	.18280877+01
0.21	.11636315+01	.18186048+01
0.22	.11628812+01	.18091749+01
0.23	.11620723+01	.17998006+01
0.24	.11612044+01	.17904844+01
0.25	.11602769+01	.17812288+01
0.26	.11592893+01	.17720362+01
0.27	.11582417+01	.17629091+01
0.28	.11571340+01	.17538497+01
0.29	.11559664+01	.17448602+01
0.30	.11547391+01	.17359427+01
0.31	.11534528+01	.17270994+01
0.32	.11521081+01	.17183322+01
0.33	.11507056+01	.17096430+01
0.34	.11492462+01	.17010335+01
0.35	.11477311+01	.16925054+01
0.36	.11461612+01	.16840602+01
0.37	.11445378+01	.16756993+01
0.38	.11428622+01	.16674242+01
0.39	.11411357+01	.16592360+01
0.40	.11393599+01	.16511358+01

Table D-12a

THE ROOTS $t_2[Q\exp(i90^\circ)]$

Q	Real t_2	Imag t_2
0.00	.16240988+01	.28130215+01
0.05	.16373899+01	.28207200+01
0.10	.16505887+01	.28284293+01
0.15	.16636761+01	.28361597+01
0.20	.16766335+01	.28439202+01
0.25	.16894426+01	.28517189+01
0.30	.17020858+01	.28595623+01
0.35	.17145458+01	.28674561+01
0.40	.17268061+01	.28754041+01
0.45	.17388508+01	.28834087+01
0.50	.17506648+01	.28914711+01
0.55	.17622338+01	.28995907+01
0.60	.17735447+01	.29077655+01
0.65	.17845851+01	.29159922+01
0.70	.17953442+01	.29242662+01
0.75	.18058122+01	.29325817+01
0.80	.18159807+01	.29409315+01
0.85	.18258424+01	.29493079+01
0.90	.18353920+01	.29577022+01
0.95	.18446250+01	.29661050+01
1.00	.18535387+01	.29745065+01
1.05	.18621318+01	.29828966+01
1.10	.18704041+01	.29912650+01
1.15	.18783571+01	.29996014+01
1.20	.18859931+01	.30078954+01
1.25	.18933159+01	.30161373+01
1.30	.19003303+01	.30243175+01
1.35	.19070418+01	.30324267+01
1.40	.19134571+01	.30404563+01
1.45	.19195833+01	.30483984+01
1.50	.19254284+01	.30562455+01
1.55	.19310008+01	.30639908+01
1.60	.19363093+01	.30716285+01
1.65	.19413629+01	.30791529+01
1.70	.19461711+01	.30865596+01
1.75	.19507433+01	.30938444+01
1.80	.19550891+01	.31010041+01
1.85	.19592180+01	.31080358+01
1.90	.19631394+01	.31149375+01
1.95	.19668627+01	.31217076+01
2.00	.19703971+01	.31283449+01

Table D-12a (Cont'd)

THE ROOTS $t_2[Q\exp(i90^{\circ})]$

Q	Real t_2	Imag t_2
2.00	.19703971+01	.31283449+01
2.05	.19737514+01	.31348488+01
2.10	.19769345+01	.31412192+01
2.15	.19799548+01	.31474563+01
2.20	.19828204+01	.31535608+01
2.25	.19855393+01	.31595334+01
2.30	.19881190+01	.31653754+01
2.35	.19905668+01	.31710882+01
2.40	.19928897+01	.31766734+01
2.45	.19950942+01	.31821331+01
2.50	.19971868+01	.31874689+01
2.55	.19991735+01	.31926833+01
2.60	.20010599+01	.31977783+01
2.65	.20028516+01	.32027563+01
2.70	.20045537+01	.32076198+01
2.75	.20061710+01	.32123711+01
2.80	.20077083+01	.32170129+01
2.85	.20091698+01	.32215475+01
2.90	.20105598+01	.32259775+01
2.95	.20118821+01	.32303054+01
3.00	.20131405+01	.32345339+01
3.05	.20143383+01	.32386653+01
3.10	.20154789+01	.32427022+01
3.15	.20165654+01	.32466470+01
3.20	.20176007+01	.32505021+01
3.25	.20185876+01	.32542700+01
3.30	.20195286+01	.32579529+01
3.35	.20204262+01	.32615531+01
3.40	.20212828+01	.32650730+01
3.45	.20221003+01	.32685146+01
3.50	.20228811+01	.32718801+01
3.55	.20236269+01	.32751718+01
3.60	.20243396+01	.32783915+01
3.65	.20250209+01	.32815412+01
3.70	.20256723+01	.32846230+01
3.75	.20262956+01	.32876386+01
3.80	.20268920+01	.32905900+01
3.85	.20274629+01	.32934788+01
3.90	.20280097+01	.32963068+01
3.95	.20285335+01	.32990758+01
4.00	.20290355+01	.33017873+01

Table D-12b

THE ROOTS $t_2[Q\exp(i90^{\circ})]$

1/Q	Real t_2	Imag t_2
0.00	.20439747+01	.35402680+01
0.01	.20439735+01	.35302687+01
0.02	.20439653+01	.35202735+01
0.03	.20439431+01	.35102864+01
0.04	.20439000+01	.35003117+01
0.05	.20438296+01	.34903537+01
0.06	.20437253+01	.34804164+01
0.07	.20435807+01	.34705043+01
0.08	.20433900+01	.34606218+01
0.09	.20431474+01	.34507734+01
0.10	.20428476+01	.34409638+01
0.11	.20424856+01	.34311976+01
0.12	.20420567+01	.34214798+01
0.13	.20415565+01	.34118150+01
0.14	.20409812+01	.34022083+01
0.15	.20403274+01	.33926647+01
0.16	.20395921+01	.33831892+01
0.17	.20387726+01	.33737867+01
0.18	.20378670+01	.33644625+01
0.19	.20368735+01	.33552213+01
0.20	.20357911+01	.33460682+01
0.21	.20346190+01	.33370079+01
0.22	.20333570+01	.33280452+01
0.23	.20320053+01	.33191847+01
0.24	.20305644+01	.33104306+01
0.25	.20290355+01	.33017874+01
0.26	.20274199+01	.32932588+01
0.27	.20257194+01	.32848486+01
0.28	.20239362+01	.32765604+01
0.29	.20220727+01	.32683972+01
0.30	.20201316+01	.32603621+01
0.31	.20181160+01	.32524576+01
0.32	.20160288+01	.32446860+01
0.33	.20138735+01	.32370492+01
0.34	.20116536+01	.32295490+01
0.35	.20093727+01	.32221867+01
0.36	.20070346+01	.32149633+01
0.37	.20046432+01	.32078795+01
0.38	.20022022+01	.32009358+01
0.39	.19997155+01	.31941324+01
0.40	.19971868+01	.31874690+01

Table D-13a

THE ROOTS $t_3[Q\exp(i90^\circ)]$

Q	Real t_3	Imag t_3
0.00	.24100496+01	.41743283+01
0.05	.24190204+01	.41795158+01
0.10	.24279597+01	.41847085+01
0.15	.24368585+01	.41899115+01
0.20	.24457077+01	.41951293+01
0.25	.24544986+01	.42003668+01
0.30	.24632221+01	.42056277+01
0.35	.24718698+01	.42109157+01
0.40	.24804330+01	.42162342+01
0.45	.24889033+01	.42215855+01
0.50	.24972726+01	.42269722+01
0.55	.25055327+01	.42323957+01
0.60	.25136758+01	.42378571+01
0.65	.25216945+01	.42433567+01
0.70	.25295814+01	.42488945+01
0.75	.25373298+01	.42544696+01
0.80	.25449331+01	.42600808+01
0.85	.25523853+01	.42657263+01
0.90	.25596808+01	.42714034+01
0.95	.25668143+01	.42771094+01
1.00	.25737813+01	.42828409+01
1.05	.25805777+01	.42885939+01
1.10	.25872001+01	.42943645+01
1.15	.25936455+01	.43001480+01
1.20	.25999114+01	.43059397+01
1.25	.26059964+01	.43117345+01
1.30	.26118991+01	.43175276+01
1.35	.26176190+01	.43233135+01
1.40	.26231561+01	.43290869+01
1.45	.26285110+01	.43348427+01
1.50	.26336848+01	.43405757+01
1.55	.26386791+01	.43462807+01
1.60	.26434958+01	.43519527+01
1.65	.26481374+01	.43575872+01
1.70	.26526069+01	.43631793+01
1.75	.26569074+01	.43687249+01
1.80	.26610424+01	.43742198+01
1.85	.26650157+01	.43796605+01
1.90	.26688313+01	.43850431+01
1.95	.26724935+01	.43903647+01
2.00	.26760065+01	.43956221+01

Table D-13a (Cont'd)

THE ROOTS $t_3[Q\exp(i90^{\circ})]$

Q	Real t_3	Imag t_3
2.00	.26760065+01	.43956221+01
2.05	.26793749+01	.44008129+01
2.10	.26826032+01	.44059348+01
2.15	.26856962+01	.44109857+01
2.20	.26886583+01	.44159638+01
2.25	.26914944+01	.44208678+01
2.30	.26942090+01	.44256963+01
2.35	.26968066+01	.44304484+01
2.40	.26992921+01	.44351234+01
2.45	.27016696+01	.44397208+01
2.50	.27039438+01	.44442401+01
2.55	.27061187+01	.44486815+01
2.60	.27081986+01	.44530447+01
2.65	.27101876+01	.44573302+01
2.70	.27120896+01	.44615381+01
2.75	.27139084+01	.44656691+01
2.80	.27156477+01	.44697236+01
2.85	.27173112+01	.44737026+01
2.90	.27189021+01	.44776066+01
2.95	.27204239+01	.44814365+01
3.00	.27218796+01	.44851935+01
3.05	.27232722+01	.44888784+01
3.10	.27246048+01	.44924925+01
3.15	.27258801+01	.44960366+01
3.20	.27271008+01	.44995121+01
3.25	.27282693+01	.45029200+01
3.30	.27293883+01	.45062618+01
3.35	.27304598+01	.45095385+01
3.40	.27314862+01	.45127513+01
3.45	.27324696+01	.45159016+01
3.50	.27334120+01	.45189906+01
3.55	.27343153+01	.45220196+01
3.60	.27351813+01	.45249898+01
3.65	.27360117+01	.45279025+01
3.70	.27368084+01	.45307588+01
3.75	.27375726+01	.45335601+01
3.80	.27383061+01	.45363075+01
3.85	.27390102+01	.45390023+01
3.90	.27396861+01	.45416456+01
3.95	.27403353+01	.45442387+01
4.00	.27409590+01	.45467824+01

Table D-13b

THE ROOTS $t_3[Q\exp(i90^\circ)]$

1/Q	Real t_3	Imag t_3
0.00	.27602798+01	.47809450+01
0.01	.27602783+01	.47709460+01
0.02	.27602672+01	.47609524+01
0.03	.27602372+01	.47509699+01
0.04	.27601791+01	.47410042+01
0.05	.27600839+01	.47310609+01
0.06	.27599429+01	.47211460+01
0.07	.27597480+01	.47112653+01
0.08	.27594912+01	.47014252+01
0.09	.27591651+01	.46916317+01
0.10	.27587628+01	.46818916+01
0.11	.27582781+01	.46722112+01
0.12	.27577054+01	.46625976+01
0.13	.27570396+01	.46530573+01
0.14	.27562764+01	.46435975+01
0.15	.27554122+01	.46342251+01
0.16	.27544441+01	.46249472+01
0.17	.27533703+01	.46157707+01
0.18	.27521891+01	.46067023+01
0.19	.27509002+01	.45977487+01
0.20	.27495036+01	.45889166+01
0.21	.27480002+01	.45802120+01
0.22	.27463916+01	.45716410+01
0.23	.27446799+01	.45632090+01
0.24	.27428679+01	.45549213+01
0.25	.27409590+01	.45467825+01
0.26	.27389570+01	.45387969+01
0.27	.27368661+01	.45309682+01
0.28	.27346909+01	.45232997+01
0.29	.27324364+01	.45157941+01
0.30	.27301078+01	.45084534+01
0.31	.27277102+01	.45012794+01
0.32	.27252495+01	.44942732+01
0.33	.27227309+01	.44874354+01
0.34	.27201603+01	.44807661+01
0.35	.27175429+01	.44742649+01
0.36	.27148844+01	.44679311+01
0.37	.27121900+01	.44617635+01
0.38	.27094651+01	.44557604+01
0.39	.27067148+01	.44499201+01
0.40	.27039438+01	.44442402+01

Table D-14a

THE ROOTS $t_4[Q\exp(i90^\circ)]$

Q	Real t_4	Imag t_4
0.00	.30816536+01	.53375807+01
0.05	.30886730+01	.53416375+01
0.10	.30956761+01	.53456976+01
0.15	.31026572+01	.53497639+01
0.20	.31096108+01	.53538398+01
0.25	.31165313+01	.53579280+01
0.30	.31234132+01	.53620312+01
0.35	.31302510+01	.53661520+01
0.40	.31370393+01	.53702927+01
0.45	.31437729+01	.53744553+01
0.50	.31504462+01	.53786417+01
0.55	.31570540+01	.53828534+01
0.60	.31635914+01	.53870916+01
0.65	.31700533+01	.53913572+01
0.70	.31764346+01	.53956509+01
0.75	.31827307+01	.53999729+01
0.80	.31889369+01	.54043232+01
0.85	.31950489+01	.54087012+01
0.90	.32010623+01	.54131063+01
0.95	.32069733+01	.54175375+01
1.00	.32127779+01	.54219936+01
1.05	.32184727+01	.54264727+01
1.10	.32240544+01	.54309730+01
1.15	.32295202+01	.54354922+01
1.20	.32348672+01	.54400281+01
1.25	.32400932+01	.54445779+01
1.30	.32451962+01	.54491387+01
1.35	.32501744+01	.54537077+01
1.40	.32550267+01	.54582818+01
1.45	.32597519+01	.54628575+01
1.50	.32643494+01	.54674318+01
1.55	.32688188+01	.54720011+01
1.60	.32731601+01	.54765622+01
1.65	.32773736+01	.54811117+01
1.70	.32814599+01	.54856462+01
1.75	.32854199+01	.54901624+01
1.80	.32892546+01	.54946572+01
1.85	.32929657+01	.54991274+01
1.90	.32965545+01	.55035700+01
1.95	.33000231+01	.55079822+01
2.00	.33033734+01	.55123612+01

Table D-14a (Cont'd)

THE ROOTS $t_4[Q\exp(i90^o)]$

Q	Real t_4	Imag t_4
2.00	.33033734+01	.55123612+01
2.05	.33066077+01	.55167043+01
2.10	.33097284+01	.55210094+01
2.15	.33127379+01	.55252739+01
2.20	.33156388+01	.55294959+01
2.25	.33184340+01	.55336734+01
2.30	.33211262+01	.55378047+01
2.35	.33237182+01	.55418881+01
2.40	.33262130+01	.55459224+01
2.45	.33286133+01	.55499062+01
2.50	.33309224+01	.55538385+01
2.55	.33331429+01	.55577182+01
2.60	.33352780+01	.55615447+01
2.65	.33373304+01	.55653173+01
2.70	.33393031+01	.55690355+01
2.75	.33411988+01	.55726989+01
2.80	.33430205+01	.55763072+01
2.85	.33447707+01	.55798603+01
2.90	.33464523+01	.55833582+01
2.95	.33480678+01	.55868008+01
3.00	.33496197+01	.55901884+01
3.05	.33511106+01	.55935213+01
3.10	.33525429+01	.55967996+01
3.15	.33539189+01	.56000238+01
3.20	.33552409+01	.56031942+01
3.25	.33565110+01	.56063114+01
3.30	.33577313+01	.56093760+01
3.35	.33589041+01	.56123884+01
3.40	.33600311+01	.56153493+01
3.45	.33611143+01	.56182595+01
3.50	.33621554+01	.56211194+01
3.55	.33631564+01	.56239299+01
3.60	.33641188+01	.56266917+01
3.65	.33650442+01	.56294055+01
3.70	.33659343+01	.56320722+01
3.75	.33667904+01	.56346925+01
3.80	.33676141+01	.56372671+01
3.85	.33684066+01	.56397968+01
3.90	.33691694+01	.56422825+01
3.95	.33699036+01	.56447249+01
4.00	.33706104+01	.56471249+01

Table D-14b

THE ROOTS $t_4[Q\exp(i90^\circ)]$

Q	Real t_4	Imag t_4
0.00	.33933540+01	.58774616+01
0.01	.33933521+01	.58674627+01
0.02	.33933384+01	.58574706+01
0.03	.33933015+01	.58474922+01
0.04	.33932301+01	.58375344+01
0.05	.33931132+01	.58276044+01
0.06	.33929402+01	.58177093+01
0.07	.33927012+01	.58076567+01
0.08	.33923870+01	.57980543+01
0.09	.33919885+01	.57883102+01
0.10	.33914980+01	.57786323+01
0.11	.33909085+01	.57690293+01
0.12	.33902137+01	.57595095+01
0.13	.33894084+01	.57500817+01
0.14	.33884884+01	.57407545+01
0.15	.33874505+01	.57315366+01
0.16	.33862925+01	.57224368+01
0.17	.33850135+01	.57134634+01
0.18	.33836132+01	.57046246+01
0.19	.33820928+01	.56959285+01
0.20	.33804540+01	.56873823+01
0.21	.33786996+01	.56789931+01
0.22	.33768333+01	.56707674+01
0.23	.33748596+01	.56627108+01
0.24	.33727834+01	.56548285+01
0.25	.33706104+01	.56471250+01
0.26	.33683468+01	.56396038+01
0.27	.33659989+01	.56322680+01
0.28	.33635735+01	.56251196+01
0.29	.33610777+01	.56181600+01
0.30	.33585184+01	.56113901+01
0.31	.33559028+01	.56048098+01
0.32	.33532379+01	.55984186+01
0.33	.33505305+01	.55922150+01
0.34	.33477874+01	.55861974+01
0.35	.33450151+01	.55803634+01
0.36	.33422198+01	.55747103+01
0.37	.33394075+01	.55692349+01
0.38	.33365837+01	.55639337+01
0.39	.33337537+01	.55588029+01
0.40	.33309224+01	.55538385+01

Table D-15a

THE ROOTS $t_5[Q\exp(i90^\circ)]$

Q	Real t_5	Imag t_5
0.00	.36860886+01	.63844927+01
0.05	.36919584+01	.63878842+01
0.10	.36978181+01	.63912781+01
0.15	.37036636+01	.63946763+01
0.20	.37094910+01	.63980813+01
0.25	.37152964+01	.64014950+01
0.30	.37210759+01	.64049195+01
0.35	.37268256+01	.64083566+01
0.40	.37325418+01	.64118081+01
0.45	.37382205+01	.64152755+01
0.50	.37438580+01	.64187605+01
0.55	.37494504+01	.64222641+01
0.60	.37549941+01	.64257877+01
0.65	.37604855+01	.64293320+01
0.70	.37659209+01	.64328977+01
0.75	.37712968+01	.64364855+01
0.80	.37766098+01	.64400955+01
0.85	.37818565+01	.64437281+01
0.90	.37870337+01	.64473829+01
0.95	.37921382+01	.64510596+01
1.00	.37971670+01	.64547577+01
1.05	.38021173+01	.64584764+01
1.10	.38069863+01	.64622149+01
1.15	.38117716+01	.64659718+01
1.20	.38164707+01	.64697460+01
1.25	.38210814+01	.64735358+01
1.30	.38256019+01	.64773397+01
1.35	.38300303+01	.64811558+01
1.40	.38343649+01	.64849821+01
1.45	.38386045+01	.64888166+01
1.50	.38427479+01	.64926571+01
1.55	.38467941+01	.64965013+01
1.60	.38507426+01	.65003470+01
1.65	.38545927+01	.65041917+01
1.70	.38583442+01	.65080331+01
1.75	.38619971+01	.65118687+01
1.80	.38655514+01	.65156960+01
1.85	.38690076+01	.65195129+01
1.90	.38723662+01	.65233168+01
1.95	.38756279+01	.65271054+01
2.00	.38787936+01	.65308765+01

Table D-15a (Cont'd)

THE ROOTS $t_5[Q\exp(i90^\circ)]$

Q	Real t_5	Imag t_5
2.00	.38787936+01	.65308765+01
2.05	.38818643+01	.65346278+01
2.10	.38848413+01	.65383573+01
2.15	.38877258+01	.65420628+01
2.20	.38905195+01	.65457425+01
2.25	.38932239+01	.65493946+01
2.30	.38958406+01	.65530171+01
2.35	.38983713+01	.65566087+01
2.40	.39008180+01	.65601676+01
2.45	.39031826+01	.65636926+01
2.50	.39054671+01	.65671824+01
2.55	.39076734+01	.65706357+01
2.60	.39098036+01	.65740515+01
2.65	.39118596+01	.65774287+01
2.70	.39138439+01	.65807666+01
2.75	.39157581+01	.65840645+01
2.80	.39176047+01	.65873217+01
2.85	.39193855+01	.65905376+01
2.90	.39211027+01	.65937118+01
2.95	.39227584+01	.65968439+01
3.00	.39243545+01	.65999337+01
3.05	.39258929+01	.66029808+01
3.10	.39273758+01	.66059853+01
3.15	.39288050+01	.66089471+01
3.20	.39301823+01	.66118661+01
3.25	.39315097+01	.66147424+01
3.30	.39327889+01	.66175762+01
3.35	.39340217+01	.66203676+01
3.40	.39352096+01	.66231169+01
3.45	.39363545+01	.66258244+01
3.50	.39374579+01	.66284903+01
3.55	.39385214+01	.66311151+01
3.60	.39395465+01	.66336990+01
3.65	.39405345+01	.66362425+01
3.70	.39414871+01	.66387460+01
3.75	.39424054+01	.66412100+01
3.80	.39432908+01	.66436350+01
3.85	.39441445+01	.66460215+01
3.90	.39449679+01	.66483700+01
3.95	.39457621+01	.66506810+01
4.00	.39465282+01	.66529550+01

Table D-15b

THE ROOTS $t_5[Q\exp(i90^{\circ})]$

1/Q	Real t_5	Imag t_5
0.00	.39720668+01	.68798215+01
0.01	.39720645+01	.68698228+01
0.02	.39720485+01	.68598321+01
0.03	.39720053+01	.68498574+01
0.04	.39719217+01	.68399069+01
0.05	.39717851+01	.68299889+01
0.06	.39715831+01	.68201120+01
0.07	.39713044+01	.68102854+01
0.08	.39709383+01	.68005180+01
0.09	.39704749+01	.67908193+01
0.10	.39699058+01	.67811994+01
0.11	.39692233+01	.67716679+01
0.12	.39684210+01	.67622352+01
0.13	.39674938+01	.67529117+01
0.14	.39664380+01	.67437074+01
0.15	.39652510+01	.67346327+01
0.16	.39639319+01	.67256975+01
0.17	.39624807+01	.67169115+01
0.18	.39608990+01	.67082839+01
0.19	.39591896+01	.66998232+01
0.20	.39573561+01	.66915375+01
0.21	.39554036+01	.66834339+01
0.22	.39533377+01	.66755190+01
0.23	.39511649+01	.66677980+01
0.24	.39488925+01	.66602755+01
0.25	.39465282+01	.66529551+01
0.26	.39440800+01	.66458394+01
0.27	.39415563+01	.66389300+01
0.28	.39389654+01	.66322275+01
0.29	.39363158+01	.66257318+01
0.30	.39336158+01	.66194420+01
0.31	.39308736+01	.66133560+01
0.32	.39280970+01	.66074716+01
0.33	.39252937+01	.66017856+01
0.34	.39224707+01	.65962943+01
0.35	.39196347+01	.65909936+01
0.36	.39167923+01	.65858792+01
0.37	.39139491+01	.65809460+01
0.38	.39111106+01	.65761890+01
0.39	.39082818+01	.65716030+01
0.40	.39054671+01	.65671824+01

Table D-16a

THE ROOTS $t_1[Q\exp(i30^\circ)]$

Q	Real t_1	Imag t_1
0.00	.50939648-00	.88230058-00
0.05	.55247547-00	.85879382-00
0.10	.59664887-00	.83738442-00
0.15	.64182816-00	.81812159-00
0.20	.68792360-00	.80105397-00
0.25	.73484389-00	.78622971-00
0.30	.78249577-00	.77369668-00
0.35	.83078354-00	.76350248-00
0.40	.87960867-00	.75569461-00
0.45	.92886920-00	.75032056-00
0.50	.97845920-00	.74742780-00
0.55	.10282681+01	.74706384-00
0.60	.10781797+01	.74927623-00
0.65	.11280719+01	.75411252-00
0.70	.11778149+01	.76162010-00
0.75	.12272703+01	.77184601-00
0.80	.12762901+01	.78483663-00
0.85	.13247144+01	.80063715-00
0.90	.13723696+01	.81929082-00
0.95	.14190663+01	.84083785-00
1.00	.14645960+01	.86531381-00
1.05	.15087282+01	.89274730-00
1.10	.15512062+01	.92315654-00
1.15	.15917425+01	.95654450-00
1.20	.16300139+01	.99289173-00
1.25	.16656562+01	.10321461+01
1.30	.16982596+01	.10742079+01
1.35	.17273668+01	.11189089+01
1.40	.17524767+01	.11659839+01
1.45	.17730599+01	.12150335+01
1.50	.17885961+01	.12654821+01
1.55	.17986419+01	.13165425+01
1.60	.18029370+01	.13672092+01
1.65	.18015315+01	.14163190+01
1.70	.17948830+01	.14626985+01
1.75	.17838520+01	.15053746+01
1.80	.17695618+01	.15437548+01
1.85	.17531776+01	.15776854+01
1.90	.17357157+01	.16073740+01
1.95	.17179481+01	.16332454+01
2.00	.17003997+01	.16558069+01

Table D-16a (Cont'd)

THE ROOTS $t_1[Q\exp(i30^o)]$

Q	Real t_1	Imag t_1
2.00	.17003997+01	.16558069+01
2.05	.16833934+01	.16755590+01
2.10	.16671073+01	.16929507+01
2.15	.16516230+01	.17083649+01
2.20	.16369626+01	.17221196+01
2.25	.16231125+01	.17344751+01
2.30	.16100395+01	.17456437+01
2.35	.15977006+01	.17557986+01
2.40	.15860488+01	.17650818+01
2.45	.15750364+01	.17736100+01
2.50	.15646173+01	.17814799+01
2.55	.15547478+01	.17887720+01
2.60	.15453874+01	.17955540+01
2.65	.15364984+01	.18018830+01
2.70	.15280464+01	.18078078+01
2.75	.15199997+01	.18133700+01
2.80	.15123295+01	.18186053+01
2.85	.15050096+01	.18235448+01
2.90	.14980159+01	.18282156+01
2.95	.14913265+01	.18326413+01
3.00	.14849214+01	.18368427+01
3.05	.14787821+01	.18408382+01
3.10	.14728920+01	.18446441+01
3.15	.14672356+01	.18482750+01
3.20	.14617988+01	.18517437+01
3.25	.14565684+01	.18550621+01
3.30	.14515327+01	.18582406+01
3.35	.14466804+01	.18612886+01
3.40	.14420013+01	.18642148+01
3.45	.14374860+01	.18670270+01
3.50	.14331257+01	.18697322+01
3.55	.14289121+01	.18723371+01
3.60	.14248379+01	.18748475+01
3.65	.14208958+01	.18772689+01
3.70	.14170794+01	18796063+01
3.75	.14133824+01	.18818644+01
3.80	.14097993+01	.18840474+01
3.85	.14063245+01	.18861593+01
3.90	.14029530+01	.18882037+01
3.95	.13996803+01	.18901842+01
4.00	.13965018+01	.18921037+01

Table D-16b

THE ROOTS $t_1[Q\exp(i30^o)]$

1/Q	Real t_1	Imag t_1
0.00	.11690537+01	.20248604+01
0.01	.11777146+01	.20198600+01
0.02	.11863796+01	.20148572+01
0.03	.11950526+01	.20098497+01
0.04	.12037377+01	.20048349+01
0.05	.12124389+01	.19998102+01
0.06	.12211601+01	.19947730+01
0.07	.12299055+01	.19897205+01
0.08	.12386792+01	.19846500+01
0.09	.12474853+01	.19795584+01
0.10	.12563279+01	.19744427+01
0.11	.12652113+01	.19692996+01
0.12	.12741398+01	.19641258+01
0.13	.12831180+01	.19589176+01
0.14	.12921502+01	.19536714+01
0.15	.13012411+01	.19483831+01
0.16	.13103955+01	.19430485+01
0.17	.13196183+01	.19376632+01
0.18	.13289144+01	.19322224+01
0.19	.13382891+01	.19267210+01
0.20	.13477478+01	.19211536+01
0.21	.13572960+01	.19155142+01
0.22	.13669394+01	.19097967+01
0.23	.13766841+01	.19039941+01
0.24	.13865361+01	.18980991+01
0.25	.13965019+01	.18921037+01
0.26	.14065881+01	.18859993+01
0.27	.14168016+01	.18797762+01
0.28	.14271495+01	.18734242+01
0.29	.14376391+01	.18669318+01
0.30	.14482782+01	.18602865+01
0.31	.14590743+01	.18534745+01
0.32	.14700356+01	.18464806+01
0.33	.14811701+01	.18392877+01
0.34	.14924859+01	.18318771+01
0.35	.15039912+01	.18242279+01
0.36	.15156939+01	.18163167+01
0.37	.15276014+01	.18081174+01
0.38	.15397208+01	.17996007+01
0.39	.15520580+01	.17907338+01
0.40	.15646174+01	.17814798+01

Table D-17a

THE ROOTS $t_2[Q\exp(i30^{\circ})]$

Q	Real t_2	Imag t_2
0.00	.16240988+01	.28130215+01
0.05	.16374512+01	.28053547+01
0.10	.16508599+01	.27977401+01
0.15	.16643454+01	.27901672+01
0.20	.16779288+01	.27826262+01
0.25	.16916322+01	.27751074+01
0.30	.17054787+01	.27676020+01
0.35	.17194929+01	.27601010+01
0.40	.17337010+01	.27525963+01
0.45	.17481313+01	.27450798+01
0.50	.17628143+01	.27375440+01
0.55	.17777832+01	.27299816+01
0.60	.17930745+01	.27223863+01
0.65	.18087286+01	.27147522+01
0.70	.18247902+01	.27070741+01
0.75	.18413094+01	.26993487+01
0.80	.18583428+01	.26915736+01
0.85	.18759541+01	.26837492+01
0.90	.18942161+01	.26758788+01
0.95	.19132123+01	.26679699+01
1.00	.19330387+01	.26600364+01
1.05	.19538065+01	.26521003+01
1.10	.19756448+01	.26441956+01
1.15	.19987042+01	.26363732+01
1.20	.20231599+01	.26287080+01
1.25	.20492156+01	.26213090+01
1.30	.20771051+01	.26143342+01
1.35	.21070924+01	.26080109+01
1.40	.21394638+01	.26026639+01
1.45	.21745084+01	.25987510+01
1.50	.22124757+01	.25969035+01
1.55	.22535001+01	.25979593+01
1.60	.22974873+01	.26029634+01
1.65	.23439753+01	.26131012+01
1.70	.23920241+01	.26295376+01
1.75	.24402045+01	.26531870+01
1.80	.24867208+01	.26845004+01
1.85	.25296171+01	.27233545+01
1.90	.25669681+01	.27690477+01
1.95	.25970158+01	.28203254+01
2.00	.26183032+01	.28753721+01

Table D-17a (Cont'd)

THE ROOTS $t_2[Q\exp(i30^\circ)]$

Q	Real t_2	Imag t_2
2.00	.26183032+01	.28753721+01
2.05	.26299094+01	.29318044+01
2.10	.26318134+01	.29868337+01
2.15	.26251809+01	.30377754+01
2.20	.26121923+01	.30827685+01
2.25	.25953516+01	.31211828+01
2.30	.25767729+01	.31534132+01
2.35	.25578839+01	.31803638+01
2.40	.25394989+01	.32030272+01
2.45	.25220152+01	.32222806+01
2.50	.25055866+01	.32388306+01
2.55	.24902362+01	.32532249+01
2.60	.24759234+01	.32658826+01
2.65	.24625793+01	.32771245+01
2.70	.24501255+01	.32871978+01
2.75	.24384837+01	.32962952+01
2.80	.24275799+01	.33045684+01
2.85	.24173462+01	.33121382+01
2.90	.24077214+01	.33191023+01
2.95	.23986506+01	.33255401+01
3.00	.23900852+01	.33315171+01
3.05	.23819812+01	.33370879+01
3.10	.23743003+01	.33422980+01
3.15	.23670077+01	.33471862+01
3.20	.23600727+01	.33517855+01
3.25	.23534675+01	.33561242+01
3.30	.23471674+01	.33602268+01
3.35	.23411503+01	.33641146+01
3.40	.23353959+01	.33678062+01
3.45	.23298860+01	.33713181+01
3.50	.23246043+01	.33746646+01
3.55	.23195358+01	.33778585+01
3.60	.23146667+01	.33809114+01
3.65	.23099848+01	.33838336+01
3.70	.23054786+01	.33866342+01
3.75	.23011377+01	.33893215+01
3.80	.22969525+01	.33919030+01
3.85	.22929141+01	.33943855+01
3.90	.22890145+01	.33967753+01
3.95	.22852461+01	.33990779+01
4.00	.22816020+01	.34012985+01

Table D-17b

THE ROOTS $t_2[Q\exp(i30^\circ)]$

1/Q	Real t_2	Imag t_2
0.00	.20439747+01	.35402680+01
0.01	.20526361+01	.35352673+01
0.02	.20613046+01	.35302626+01
0.03	.20699873+01	.35252494+01
0.04	.20786912+01	.35202237+01
0.05	.20874236+01	.35151809+01
0.06	.20961917+01	.35101167+01
0.07	.21050031+01	.35050261+01
0.08	.21138652+01	.34999043+01
0.09	.21227861+01	.34947462+01
0.10	.21317738+01	.34895464+01
0.11	.21408367+01	.34842990+01
0.12	.21499836+01	.34789980+01
0.13	.21592237+01	.34736368+01
0.14	.21685667+01	.34682085+01
0.15	.21780229+01	.34627055+01
0.16	.21876028+01	.34571197+01
0.17	.21973182+01	.34514421+01
0.18	.22071813+01	.34456631+01
0.19	.22172052+01	.34397718+01
0.20	.22274040+01	.34337564+01
0.21	.22377930+01	.34276038+01
0.22	.22483885+01	.34212994+01
0.23	.22592087+01	.34148265+01
0.24	.22702727+01	.34081665+01
0.25	.22816020+01	.34012984+01
0.26	.22932199+01	.33941978+01
0.27	.23051517+01	.33868369+01
0.28	.23174255+01	.33791834+01
0.29	.23300723+01	.33711996+01
0.30	.23431260+01	.33628412+01
0.31	.23566242+01	.33540556+01
0.32	.23706077+01	.33447800+01
0.33	.23851214+01	.33349385+01
0.34	.24002136+01	.33244389+01
0.35	.24159356+01	.33131678+01
0.36	.24323398+01	.33009843+01
0.37	.24494766+01	.32877123+01
0.38	.24673882+01	.32731295+01
0.39	.24860974+01	.32569542+01
0.40	.25055870+01	.32388290+01

Table D-18a

THE ROOTS $t_3[Q\exp(i30^\circ)]$

Q	Real t_3	Imag t_3
0.00	.24100496+01	.41743283+01
0.05	.24190401+01	.41691505+01
0.10	.24280512+01	.41639868+01
0.15	.24370920+01	.41588320+01
0.20	.24461722+01	.41536812+01
0.25	.24553016+01	.41485292+01
0.30	.24644904+01	.41433711+01
0.35	.24737491+01	.41382020+01
0.40	.24830888+01	.41330168+01
0.45	.24925210+01	.41278105+01
0.50	.25020580+01	.41225780+01
0.55	.25117127+01	.41173140+01
0.60	.25214990+01	.41120132+01
0.65	.25314319+01	.41066701+01
0.70	.25415273+01	.41012792+01
0.75	.25518027+01	.40958346+01
0.80	.25622770+01	.40903302+01
0.85	.25729711+01	.40847600+01
0.90	.25839080+01	.40791173+01
0.95	.25951130+01	.40733957+01
1.00	.26066144+01	.40675880+01
1.05	.26184440+01	.40616874+01
1.10	.26306373+01	.40556867+01
1.15	.26432350+01	.40495785+01
1.20	.26562833+01	.40433559+01
1.25	.26698352+01	.40370119+01
1.30	.26839520+01	.40305404+01
1.35	.26987051+01	.40239369+01
1.40	.27141783+01	.40171984+01
1.45	.27304707+01	.40103255+01
1.50	.27477003+01	.40033245+01
1.55	.27660090+01	.39962097+01
1.60	.27855688+01	.39890088+01
1.65	.28065889+01	.39817704+01
1.70	.28293265+01	.39745761+01
1.75	.28540967+01	.39675600+01
1.80	.28812850+01	.39609402+01
1.85	.29113527+01	.39550694+01
1.90	.29448268+01	.39505135+01
1.95	.29822454+01	.39481645+01
2.00	.30240083+01	.39493725+01

Table D-18a (Cont'd)

THE ROOTS $t_3[Q\exp(i30^\circ)]$

Q	Real t_3	Imag t_3
2.00	.30240083+01	.39493725+01
2.05	.30700730+01	.39560249+01
2.10	.31195027+01	.39703957+01
2.15	.31700949+01	.39945961+01
2.20	.32184828+01	.40297535+01
2.25	.32607992+01	.40754271+01
2.30	.32934771+01	.41295420+01
2.35	.33139005+01	.41885970+01
2.40	.33210863+01	.42480398+01
2.45	.33163414+01	.43032195+01
2.50	.33030265+01	.43509350+01
2.55	.32850573+01	.43903140+01
2.60	.32654504+01	.44222204+01
2.65	.32459583+01	.44481330+01
2.70	.32274046+01	.44694521+01
2.75	.32100924+01	.44872850+01
2.80	.31940723+01	.45024518+01
2.85	.31792845+01	.45155488+01
2.90	.31656281+01	.45270112+01
2.95	.31529924+01	.45371602+01
3.00	.31412712+01	.45462369+01
3.05	.31303676+01	.45544253+01
3.10	.31201953+01	.45618679+01
3.15	.31106785+01	.45686768+01
3.20	.31017511+01	.45749419+01
3.25	.30933554+01	.45807356+01
3.30	.30854409+01	.45861172+01
3.35	.30779635+01	.45911361+01
3.40	.30708844+01	.45958332+01
3.45	.30641694+01	.46002432+01
3.50	.30577881+01	.46043957+01
3.55	.30517139+01	.46083159+01
3.60	.30459229+01	.46120257+01
3.65	.30403937+01	.46155441+01
3.70	.30351070+01	.46188875+01
3.75	.30300458+01	.46220706+01
3.80	.30251945+01	.46251060+01
3.85	.30205390+01	.46280056+01
3.90	.30160666+01	.46307792+01
3.95	.30117656+01	.46334360+01
4.00	.30076254+01	.46359841+01

Table D-18b

THE ROOTS $t_3[Q\exp(i30^{\circ})]$

1/Q	Real t_3	Imag t_3
0.00	.27602798+01	.47809450+01
0.01	.27689417+01	.47759441+01
0.02	.27776131+01	.47709376+01
0.03	.27863037+01	.47659199+01
0.04	.27950231+01	.47608853+01
0.05	.28037813+01	.47558276+01
0.06	.28125881+01	.47507410+01
0.07	.28214542+01	.47456188+01
0.08	.28303902+01	.47404543+01
0.09	.28394073+01	.47352404+01
0.10	.28485174+01	.47299696+01
0.11	.28577327+01	.47246335+01
0.12	.28670666+01	.47192237+01
0.13	.28765329+01	.47137304+01
0.14	.28861468+01	.47081433+01
0.15	.28959244+01	.47024512+01
0.16	.29058836+01	.46966412+01
0.17	.29160433+01	.46906995+01
0.18	.29264248+01	.46846104+01
0.19	.29370511+01	.46783560+01
0.20	.29479480+01	.46719162+01
0.21	.29591440+01	.46652680+01
0.22	.29706713+01	.46583846+01
0.23	.29825658+01	.46512350+01
0.24	.29948684+01	.46437825+01
0.25	.30076256+01	.46359839+01
0.26	.30208907+01	.46277869+01
0.27	.30347248+01	.46191283+01
0.28	.30491990+01	.46099301+01
0.29	.30643955+01	.46000952+01
0.30	.30804104+01	.45895003+01
0.31	.30973553+01	.45779864+01
0.32	.31153598+01	.45653447+01
0.33	.31345718+01	.45512947+01
0.34	.31551537+01	.45354516+01
0.35	.31772678+01	.45172755+01
0.36	.32010361+01	.44959942+01
0.37	.32264362+01	.44704911+01
0.38	.32530577+01	.44391806+01
0.39	.32795877+01	.43999964+01
0.40	.33030050+01	.43509232+01

Table D-19a

THE ROOTS $t_4[Q\exp(i30^{o})]$

Q	Real t_4	Imag t_4
0.00	.30816536+01	.53375807+01
0.05	.30886829+01	.53335285+01
0.10	.30957232+01	.53294823+01
0.15	.31027803+01	.53254390+01
0.20	.31098599+01	.53213952+01
0.25	.31169681+01	.53173479+01
0.30	.31241109+01	.53132937+01
0.35	.31312947+01	.53092296+01
0.40	.31385258+01	.53051523+01
0.45	.31458110+01	.53010586+01
0.50	.31531574+01	.52969449+01
0.55	.31605723+01	.52928079+01
0.60	.31680634+01	.52886441+01
0.65	.31756388+01	.52844498+01
0.70	.31833076+01	.52802213+01
0.75	.31910786+01	.52759547+01
0.80	.31989620+01	.52716456+01
0.85	.32069683+01	.52672900+01
0.90	.32151092+01	.52628834+01
0.95	.32233971+01	.52584210+01
1.00	.32318456+01	.52538977+01
1.05	.32404697+01	.52493083+01
1.10	.32492856+01	.52446472+01
1.15	.32583114+01	.52399084+01
1.20	.32675672+01	.52350855+01
1.25	.32770752+01	.52301718+01
1.30	.32868605+01	.52251599+01
1.35	.32969513+01	.52200425+01
1.40	.33073794+01	.52148110+01
1.45	.33181813+01	.52094571+01
1.50	.33293987+01	.52039717+01
1.55	.33410798+01	.51983453+01
1.60	.33532806+01	.51925684+01
1.65	.33660670+01	.51866313+01
1.70	.33795167+01	.51805252+01
1.75	.33937223+01	.51742423+01
1.80	.34087958+01	.51677776+01
1.85	.34248731+01	.51611308+01
1.90	.34421217+01	.51543099+01
1.95	.34607498+01	.51473370+01
2.00	.34810195+01	.51402584+01

+133

Table D-19a (Cont'd)

THE ROOTS $t_4[Q\exp(i30^\circ)]$

Q	Real t_4	Imag t_4
2.00	.34810195+01	.51402584+01
2.05	.35032633+01	.51331623+01
2.10	.35279060+01	.51262084+01
2.15	.35554887+01	.51196823+01
2.20	.35866860+01	.51140899+01
2.25	.36222872+01	.51103179+01
2.30	.36630611+01	.51098797+01
2.35	.37093508+01	.51151808+01
2.40	.37602566+01	.51294973+01
2.45	.38127143+01	.51561013+01
2.50	.38615120+01	.51965401+01
2.55	.39007554+01	.52493513+01
2.60	.39257294+01	.53100950+01
2.65	.39344409+01	.53722598+01
2.70	.39287383+01	.54291839+01
2.75	.39135679+01	.54768170+01
2.80	.38941148+01	.55147400+01
2.85	.38737795+01	.55446123+01
2.90	.38541975+01	.55684316+01
2.95	.38359728+01	.55878211+01
3.00	.38192324+01	.56039493+01
3.05	.38039112+01	.56176317+01
3.10	.37898793+01	.56294383+01
3.15	.37769945+01	.56397747+01
3.20	.37651225+01	.56489351+01
3.25	.37541439+01	.56571379+01
3.30	.37439546+01	.56645481+01
3.35	.37344652+01	.56712933+01
3.40	.37255989+01	.56774732+01
3.45	.37172897+01	.56831678+01
3.50	.37094810+01	.56884414+01
3.55	.37021235+01	.56933466+01
3.60	.36951748+01	.56979271+01
3.65	.36885975+01	.57022194+01
3.70	.36823591+01	.57062541+01
3.75	.36764310+01	.57100576+01
3.80	.36707878+01	.57136523+01
3.85	.36654072+01	.57170577+01
3.90	.36602689+01	.57202904+01
3.95	.36553551+01	.57233655+01
4.00	.36506499+01	.57262959+01

Table D-19b

THE ROOTS $t_4[Q\exp(i30^\circ)]$

1/Q	Real t_4	Imag t_4
0.00	.33933540+01	.58774616+01
0.01	.34020162+01	.58724605+01
0.02	.34106902+01	.58674525+01
0.03	.34193878+01	.58624308+01
0.04	.34281209+01	.58573882+01
0.05	.34369019+01	.58523174+01
0.06	.34457434+01	.58472107+01
0.07	.34546584+01	.58420602+01
0.08	.34636606+01	.58368574+01
0.09	.34727646+01	.58315931+01
0.10	.34819856+01	.58262579+01
0.11	.34913401+01	.58208410+01
0.12	.35008456+01	.58153311+01
0.13	.35105213+01	.58097155+01
0.14	.35203879+01	.58039802+01
0.15	.35304687+01	.57981095+01
0.16	.35407888+01	.57920860+01
0.17	.35513766+01	.57858895+01
0.18	.35622638+01	.57794970+01
0.19	.35734862+01	.57728823+01
0.20	.35850848+01	.57660140+01
0.21	.35971060+01	.57588558+01
0.22	.36096044+01	.57513639+01
0.23	.36226428+01	.57434857+01
0.24	.36362953+01	.57351564+01
0.25	.36506502+01	.57262956+01
0.26	.36658125+01	.57168016+01
0.27	.36819101+01	.57065429+01
0.28	.36990982+01	.56953464+01
0.29	.37175683+01	.56829779+01
0.30	.37375564+01	.56691108+01
0.31	.37593526+01	.56532750+01
0.32	.37833036+01	.56347667+01
0.33	.38097867+01	.56124876+01
0.34	.38390822+01	.55846553+01
0.35	.38709112+01	.55483254+01
0.36	.39030024+01	.54989717+01
0.37	.39280373+01	.54320033+01
0.38	.39330282+01	.53497346+01
0.39	.39094511+01	.52660362+01
0.40	.38616068+01	.51965476+01

Table D-20a

THE ROOTS $t_5[Q\exp(i30^\circ)]$

Q	Real t_5	Imag t_5
0.00	.36860886+01	.63844927+01
0.05	.36919644+01	.63811040+01
0.10	.36978474+01	.63777183+01
0.15	.37037414+01	.63743335+01
0.20	.37096506+01	.63709473+01
0.25	.37155791+01	.63675574+01
0.30	.37215311+01	.63641616+01
0.35	.37275109+01	.63607576+01
0.40	.37335230+01	.63573430+01
0.45	.37395718+01	.63539157+01
0.50	.37456621+01	.63504730+01
0.55	.37517989+01	.63470126+01
0.60	.37579873+01	.63435320+01
0.65	.37642326+01	.63400284+01
0.70	.37705406+01	.63364994+01
0.75	.37769171+01	.63329420+01
0.80	.37833685+01	.63293534+01
0.85	.37899014+01	.63257305+01
0.90	.37965231+01	.63220702+01
0.95	.38032412+01	.63183691+01
1.00	.38100639+01	.63146237+01
1.05	.38170002+01	.63108301+01
1.10	.38240594+01	.63069846+01
1.15	.38312522+01	.63030829+01
1.20	.38385899+01	.62991205+01
1.25	.38460849+01	.62950926+01
1.30	.38537509+01	.62909941+01
1.35	.38616029+01	.62868194+01
1.40	.38696577+01	.62825628+01
1.45	.38779338+01	.62782176+01
1.50	.38864522+01	.62737771+01
1.55	.38952363+01	.62692339+01
1.60	.39043125+01	.62645797+01
1.65	.39137109+01	.62598060+01
1.70	.39234661+01	.62549033+01
1.75	.39336176+01	.62498615+01
1.80	.39442117+01	.62446697+01
1.85	.39553019+01	.62393163+01
1.90	.39669520+01	.62337892+01
1.95	.39792371+01	.62280761+01
2.00	.39922481+01	.62221648+01

Table D-20a (Cont'd)

THE ROOTS $t_5[Q\exp(i30^\circ)]$

Q	Real t_5	Imag t_5
2.00	.39922481+01	.62221648+01
2.05	.40060949+01	.62160441+01
2.10	.40209129+01	.62097059+01
2.15	.40368702+01	.62031475+01
2.20	.40541793+01	.61963770+01
2.25	.40731118+01	.61894222+01
2.30	.40940199+01	.61823474+01
2.35	.41173665+01	.61752838+01
2.40	.41437627+01	.61684883+01
2.45	.41740085+01	.61624543+01
2.50	.42091029+01	.61581213+01
2.55	.42501146+01	.61572386+01
2.60	.42976388+01	.61628367+01
2.65	.43504927+01	.61792988+01
2.70	.44041432+01	.62108760+01
2.75	.44510215+01	.62586676+01
2.80	.44835041+01	.63188283+01
2.85	.44972403+01	.63834389+01
2.90	.44934171+01	.64434163+01
2.95	.44781222+01	.64929494+01
3.00	.44580112+01	.65314188+01
3.05	.44371939+01	.65610217+01
3.10	.44174349+01	.65842213+01
3.15	.43992761+01	.66028871+01
3.20	.43827611+01	.66182957+01
3.25	.43677612+01	.66313031+01
3.30	.43541038+01	.66424917+01
3.35	.43416197+01	.66522669+01
3.40	.43301578+01	.66609187+01
3.45	.43195882+01	.66686599+01
3.50	.43098007+01	.66756499+01
3.55	.43007020+01	.66820111+01
3.60	.42922134+01	.66878391+01
3.65	.42842676+01	.66932096+01
3.70	.42768078+01	.66981839+01
3.75	.42697846+01	.67028116+01
3.80	.42631559+01	.67071340+01
3.85	.42568850+01	.67111856+01
3.90	.42509397+01	.67149952+01
3.95	.42452920+01	.67185877+01
4.00	.42399172+01	.67219841+01

+135

Table D-20b

THE ROOTS $t_5[Q\exp(i30^\circ)]$

1/Q	Real t_5	Imag t_5
0.00	.39720668+01	.68798215+01
0.01	.39807294+01	.68748201+01
0.02	.39894056+01	.68698109+01
0.03	.39981096+01	.68647854+01
0.04	.40068553+01	.68597355+01
0.05	.40156574+01	.68546526+01
0.06	.40245306+01	.68495275+01
0.07	.40334909+01	.68443509+01
0.08	.40425546+01	.68391125+01
0.09	.40517394+01	.68338014+01
0.10	.40610643+01	.68284059+01
0.11	.40705496+01	.68229130+01
0.12	.40802177+01	.68173083+01
0.13	.40900930+01	.68115760+01
0.14	.41002028+01	.68056981+01
0.15	.41105775+01	.67996541+01
0.16	.41212513+01	.67934211+01
0.17	.41322633+01	.67869715+01
0.18	.41436579+01	.67802741+01
0.19	.41554870+01	.67732913+01
0.20	.41678108+01	.67659780+01
0.21	.41807006+01	.67582797+01
0.22	.41942415+01	.67501286+01
0.23	.42085360+01	.67414402+01
0.24	.42237096+01	.67321059+01
0.25	.42399178+01	.67219835+01
0.26	.42573560+01	.67108820+01
0.27	.42762739+01	.66985368+01
0.28	.42969945+01	.66845676+01
0.29	.43199405+01	.66684037+01
0.30	.43456624+01	.66491415+01
0.31	.43748431+01	.66252597+01
0.32	.44081480+01	.65940139+01
0.33	.44453201+01	.65502166+01
0.34	.44812629+01	.64849944+01
0.35	.44974878+01	.63925967+01
0.36	.44713742+01	.62912076+01
0.37	.44071042+01	.62130000+01
0.38	.43306223+01	.61715592+01
0.39	.42628334+01	.61579540+01
0.40	.42090850+01	.61581145+01

Table D-21

THE ROOTS $t_1[Q\exp(i300^\circ)]$

Q	Real t_1	Imag t_1
0	0.50939649E 00	0.88230059E 00
0.5	0.48425817E-00	0.83875976E 00
0.10	0.45788710E-00	0.79308371E 00
0.15	0.43023229E-00	0.74518418E 00
0.20	0.40124264E-00	0.69497262E 00
0.25	0.37086708E-00	0.64236061E 00
0.30	0.33905483E-00	0.58726015E 00
0.35	0.30575553E-00	0.52958407E 00
0.40	0.27091953E-00	0.46924635E-00
0.45	0.23449802E-00	0.40616243E-00
0.50	0.19644319E-00	0.34024952E-00
0.55	0.15670846E-00	0.27142695E-00
0.60	0.11524856E-00	0.19961629E-00
0.65	0.72019693E-01	0.12474169E-00
0.70	0.26979624E-01	0.46730001E-01
0.75	-0.19912204E-01	-0.34489032E-01
0.80	-0.68694678E-01	-0.11898274E-00
0.85	-0.11940488E-00	-0.20681539E-00
0.90	-0.17207811E-00	-0.29804809E-00
0.95	-0.22674790E-00	-0.39273890E-00
1.00	-0.28344598E-00	-0.49094280E-00
1.05	-0.34220229E-00	-0.59271164E 00
1.10	-0.40304510E-00	-0.69809435E 00
1.15	-0.46600099E-00	-0.80713695E 00
1.20	-0.53109486E 00	-0.91988255E 00
1.25	-0.59835024E 00	-0.10363717E 01
1.30	-0.66778903E 00	-0.11566425E 01
1.35	-0.73943189E 00	-0.12807303E 01
1.40	-0.81329826E 00	-0.14086686E 01
1.45	-0.88940658E 00	-0.15404887E 01
1.50	-0.96777467E 00	-0.16762204E 01
1.55	-0.10484203E 01	-0.18158925E 01
1.60	-0.11313625E 01	-0.19595339E 01
1.65	-0.12166236E 01	-0.21071758E 01
1.70	-0.13042349E 01	-0.22588560E 01
1.75	-0.13942472E 01	-0.24146291E 01
1.80	-0.14867578E 01	-0.25745891E 01
1.85	-0.15819736E 01	-0.27389237E 01
1.90	-0.16803730E 01	-0.29080495E 01
1.95	-0.17831670E 01	-0.30829714E 01
2.00	-0.18938587E 01	-0.32663264E 01

Table D-22

THE ROOTS $t_1[Q\exp(i305^{\circ})]$

Q	Real t_1	Imag t_1
0	0.50939649E 00	0.88230059E 00
0.5	0.48824671E-00	0.83669402E 00
0.10	0.46626214E-00	0.78876795E 00
0.15	0.44341645E-00	0.73842390E 00
0.20	0.41968318E-00	0.68556309E 00
0.25	0.39503582E-00	0.63008691E 00
0.30	0.36944775E-00	0.57189726E 00
0.35	0.34289240E-00	0.51089697E 00
0.40	0.31534317E-00	0.44699021E-00
0.45	0.28677358E-00	0.38008288E-00
0.50	0.25715727E-00	0.31008293E-00
0.55	0.22646802E-00	0.23690071E-00
0.60	0.19467987E-00	0.16044921E-00
0.65	0.16176715E-00	0.80644389E-01
0.70	0.12770452E-00	-0.25946805E-02
0.75	0.92467077E-01	-0.89345606E-01
0.80	0.56030356E-01	-0.17968255E-00
0.85	0.18370441E-01	-0.27367620E-00
0.90	-0.20536020E-01	-0.37139367E-00
0.95	-0.60711714E-01	-0.47289844E-00
1.00	-0.10217869E-00	-0.57825056E 00
1.05	-0.14495825E-00	-0.68750644E 00
1.10	-0.18907097E-00	-0.80071922E 00
1.15	-0.23453663E-00	-0.91793869E 00
1.20	-0.28137419E-00	-0.10392114E 01
1.25	-0.32960181E-00	-0.11645809E 01
1.30	-0.37923677E-00	-0.12940878E 01
1.35	-0.43029550E-00	-0.14277703E 01
1.40	-0.48279350E-00	-0.15656637E 01
1.45	-0.53674508E 00	-0.17078011E 01
1.50	-0.59216332E 00	-0.18542141E 01
1.55	-0.64905923E 00	-0.20049318E 01
1.60	-0.70744068E 00	-0.21599827E 01
1.65	-0.76730975E 00	-0.23193938E 01
1.70	-0.82865708E 00	-0.24831901E 01
1.75	-0.89145035E 00	-0.26513918E 01
1.80	-0.95560941E 00	-0.28240015E 01
1.85	-0.10209555E 01	-0.30009662E 01
1.90	-0.10871132E 01	-0.31820671E 01
1.95	-0.11533467E 01	-0.33666115E 01
2.00	-0.12184072E 01	-0.35526696E 01

Table D-23

THE ROOTS $t_1[Q\exp(i310^\circ)]$

Q	Real t_1	Imag t_1
0	0.50939649E 00	0.88230059E 00
0.5	0.49240567E-00	0.83500167E 00
0.10	0.47500439E-00	0.78527314E 00
0.15	0.45719276E-00	0.73301307E 00
0.20	0.43897082E-00	0.67811918E 00
0.25	0.42033835E-00	0.62048927E 00
0.30	0.40129482E-00	0.56002155E 00
0.35	0.38183913E-00	0.49661510E-00
0.40	0.36196961E-00	0.43017018E-00
0.45	0.34168380E-00	0.36058863E-00
0.50	0.32097848E-00	0.28777424E-00
0.55	0.29984948E-00	0.21163306E-00
0.60	0.27829172E-00	0.13207366E-00
0.65	0.25629914E-00	0.49007501E-01
0.70	0.23386472E-00	-0.37650917E-01
0.75	0.21098050E-00	-0.12798378E-00
0.80	0.18763766E-00	-0.22206990E-00
0.85	0.16382650E-00	-0.31998455E-00
0.90	0.13953663E-00	-0.42179927E-00
0.95	0.11475698E-00	-0.52758197E 00
1.00	0.89475925E-01	-0.63739695E 00
1.05	0.63681416E-01	-0.75130469E 00
1.10	0.37361048E-01	-0.86936221E 00
1.15	0.10502201E-01	-0.99162298E 00
1.20	-0.16907834E-01	-0.11181369E 01
1.25	-0.44881810E-01	-0.12489508E 01
1.30	-0.73432365E-01	-0.13841081E 01
1.35	-0.10257192E-00	-0.15236495E 01
1.40	-0.13231253E-00	-0.16676124E 01
1.45	-0.16266576E-00	-0.18160316E 01
1.50	-0.19364257E-00	-0.19689389E 01
1.55	-0.22525304E-00	-0.21263628E 01
1.60	-0.25750633E-00	-0.22883283E 01
1.65	-0.29041118E-00	-0.24548545E 01
1.70	-0.32397731E-00	-0.26259528E 01
1.75	-0.35822076E-00	-0.28016209E 01
1.80	-0.39317731E-00	-0.29818370E 01
1.85	-0.42893478E-00	-0.31665549E 01
1.90	-0.46570464E-00	-0.33557170E 01
1.95	-0.50396582E 00	-0.35493349E 01
2.00	-0.54473260E 00	-0.37477726E 01

Table D-24

THE ROOTS $t_1[Q\exp(i315^o)]$

Q	Real t_1	Imag t_1
0	0.50939649E 00	0.88230059E 00
0.5	0.49669862E-00	0.83369682E 00
0.10	0.48402708E-00	0.78263102E 00
0.15	0.47140844E-00	0.72900476E 00
0.20	0.45886923E-00	0.67271921E 00
0.25	0.44643567E-00	0.61367548E 00
0.30	0.43413338E-00	0.55177484E 00
0.35	0.42198709E-00	0.48691906E-00
0.40	0.41002037E-00	0.41901073E-00
0.45	0.39825537E-00	0.34795349E-00
0.50	0.38671260E-00	0.27365240E-00
0.55	0.37541074E-00	0.19601418E-00
0.60	0.36436646E-00	0.11494756E-00
0.65	0.35359431E-00	0.30363480E-01
0.70	0.34310667E-00	-0.57824623E-01
0.75	0.33291370E-00	-0.14970052E-00
0.80	0.32302330E-00	-0.24534516E-00
0.85	0.31344124E-00	-0.34483628E-00
0.90	0.30417116E-00	-0.44824830E-00
0.95	0.29521473E-00	-0.55565240E 00
1.00	0.28657177E-00	-0.66711618E 00
1.05	0.27824040E-00	-0.78270371E 00
1.10	0.27021722E-00	-0.90247561E 00
1.15	0.26249748E-00	-0.10264889E 01
1.20	0.25507531E-00	-0.11547971E 01
1.25	0.24794378E-00	-0.12874503E 01
1.30	0.24109524E-00	-0.14244954E 01
1.35	0.23452132E-00	-0.15659760E 01
1.40	0.22821315E-00	-0.17119323E 01
1.45	0.22216141E-00	-0.18624017E 01
1.50	0.21635631E-00	-0.20174186E 01
1.55	0.21078748E-00	-0.21770147E 01
1.60	0.20544371E-00	-0.23412193E 01
1.65	0.20031247E-00	-0.25100598E 01
1.70	0.19537951E-00	-0.26835637E 01
1.75	0.19062909E-00	-0.28617609E 01
1.80	0.18604647E-00	-0.30446893E 01
1.85	0.18162673E-00	-0.32324005E 01
1.90	0.17739627E-00	-0.34249634E 01
1.95	0.17345548E-00	-0.36224477E 01
2.00	0.17003874E-00	-0.38248476E 01

Table D-25

THE ROOTS $t_1[Q\exp(i330^\circ)]$

Q	Real t_1	Imag t_1
0	0.50939649E 00	0.88230059E 00
0.5	0.51000212E 00	0.83219077E 00
0.10	0.51187769E 00	0.77998306E 00
0.15	0.51511170E 00	0.72562680E 00
0.20	0.51979306E 00	0.66907095E 00
0.25	0.52601074E 00	0.61026385E 00
0.30	0.53385338E 00	0.54915298E 00
0.35	0.54340898E 00	0.48568481E-00
0.40	0.55476443E 00	0.41980458E-00
0.45	0.56800520E 00	0.35145611E-00
0.50	0.58321481E 00	0.28058167E-00
0.55	0.60047454E 00	0.20712193E-00
0.60	0.61986291E 00	0.13101584E-00
0.65	0.64145532E 00	0.52200682E-01
0.70	0.66532360E 00	-0.29387904E-01
0.75	0.69153564E 00	-0.11381571E-00
0.80	0.72015504E 00	-0.20114993E-00
0.85	0.75124077E 00	-0.29145877E-00
0.90	0.78484690E 00	-0.38481111E-00
0.95	0.82102241E 00	-0.48127632E-00
1.00	0.85981107E 00	-0.58092370E 00
1.05	0.90125130E 00	-0.68382210E 00
1.10	0.94537632E 00	-0.79003951E 00
1.15	0.99221415E 00	-0.89964259E 00
1.20	0.10417877E 01	-0.10126961E 01
1.25	0.10941155E 01	-0.11292630E 01
1.30	0.11492112E 01	-0.12494035E 01
1.35	0.12070847E 01	-0.13731752E 01
1.40	0.12677421E 01	-0.15006327E 01
1.45	0.13311864E 01	-0.16318273E 01
1.50	0.13974176E 01	-0.17668073E 01
1.55	0.14664335E 01	-0.19056176E 01
1.60	0.15382298E 01	-0.20482999E 01
1.65	0.16128006E 01	-0.21948925E 01
1.70	0.16901387E 01	-0.23454308E 01
1.75	0.17702358E 01	-0.24999468E 01
1.80	0.18530831E 01	-0.26584703E 01
1.85	0.19386708E 01	-0.28210282E 01
1.90	0.20269890E 01	-0.29876448E 01
1.95	0.21180280E 01	-0.31583431E 01
2.00	0.22117778E 01	-0.33331431E 01

Table D-26

THE ROOTS $t_1[Q\exp(i345^\circ)]$

Q	Real t_1	Imag t_1
0	0.50939649E 00	0.88230059E 00
0.5	0.52313869E 00	0.83430852E 00
0.10	0.53902606E 00	0.78516090E 00
0.15	0.55715689E 00	0.73488475E 00
0.20	0.57763024E 00	0.68350731E 00
0.25	0.60054610E 00	0.63105556E 00
0.30	0.62600543E 00	0.57755590E 00
0.35	0.65411032E 00	0.52303369E 00
0.40	0.68496415E 00	0.46751276E-00
0.45	0.71867163E 00	0.41101496E-00
0.50	0.75533883E 00	0.35355965E-00
0.55	0.79507322E 00	0.29516312E-00
0.60	0.83798364E 00	0.23583809E-00
0.65	0.88418011E 00	0.17559306E-00
0.70	0.93377370E 00	0.11443170E-00
0.75	0.98687612E 00	0.52352326E-01
0.80	0.10435994E 01	-0.10652851E-01
0.85	0.11040553E 01	-0.74598142E-01
0.90	0.11683545E 01	-0.13950493E-00
0.95	0.12366059E 01	-0.20540212E-00
1.00	0.13089158E 01	-0.27232634E-00
1.05	0.13853861E 01	-0.34032205E-00
1.10	0.14661144E 01	-0.40944146E-00
1.15	0.15511917E 01	-0.47974406E-00
1.20	0.16407014E 01	-0.55129609E 00
1.25	0.17347192E 01	-0.62416948E 00
1.30	0.18333108E 01	-0.69844089E 00
1.35	0.19365325E 01	-0.77419027E 00
1.40	0.20444301E 01	-0.85149921E 00
1.45	0.21570391E 01	-0.93044962E 00
1.50	0.22743854E 01	-0.10111220E 01
1.55	0.23964852E 01	-0.10935944E 01
1.60	0.25233468E 01	-0.11779407E 01
1.65	0.26549710E 01	-0.12642302E 01
1.70	0.27913523E 01	-0.13525265E 01
1.75	0.29324803E 01	-0.14428876E 01
1.80	0.30783417E 01	-0.15353658E 01
1.85	0.32289194E 01	-0.16300070E 01
1.90	0.33841958E 01	-0.17268525E 01
1.95	0.35441513E 01	-0.18259376E 01
2.00	0.37087664E 01	-0.19272934E 01

Table D-27

THE ROOTS $t_1[Q\exp(i350^\circ)]$

Q	Real t_1	Imag t_1
0	0.50939649E 00	0.88230059E 00
0.5	0.52730741E 00	0.83578679E 00
0.10	0.54752740E 00	0.78851534E 00
0.15	0.57014437E 00	0.74053787E 00
0.20	0.59524691E 00	0.69190650E 00
0.25	0.62292461E 00	0.64267351E 00
0.30	0.65326834E 00	0.59289110E 00
0.35	0.68637057E 00	0.54261092E 00
0.40	0.72232564E 00	0.49188385E-00
0.45	0.76123014E 00	0.44075949E-00
0.50	0.80318311E 00	0.38928573E-00
0.55	0.84828637E 00	0.33750825E-00
0.60	0.89664481E 00	0.28546997E-00
0.65	0.94836644E 00	0.23321036E-00
0.70	0.10035627E 01	0.18076472E-00
0.75	0.10623483E 01	0.12816345E-00
0.80	0.11248414E 01	0.75431076E-01
0.85	0.11911632E 01	0.22585367E-01
0.90	0.12614375E 01	-0.30363645E-01
0.95	0.13357901E 01	-0.83414733E-01
1.00	0.14143480E 01	-0.13657641E-00
1.05	0.14972385E 01	-0.18986784E-00
1.10	0.15845870E 01	-0.24331959E-00
1.15	0.16765160E 01	-0.29697409E-00
1.20	0.17731429E 01	-0.35088566E-00
1.25	0.18745777E 01	-0.40512038E-00
1.30	0.19809213E 01	-0.45975504E-00
1.35	0.20922633E 01	-0.51487599E 00
1.40	0.22086799E 01	-0.57057687E 00
1.45	0.23302338E 01	-0.62695663E 00
1.50	0.24569722E 01	-0.68411651E 00
1.55	0.25889280E 01	-0.74215757E 00
1.60	0.27261198E 01	-0.80117789E 00
1.65	0.28685538E 01	-0.86127046E 00
1.70	0.30162253E 01	-0.92252108E 00
1.75	0.31691208E 01	-0.98500760E 00
1.80	0.33272206E 01	-0.10487990E 01
1.85	0.34905001E 01	-0.11139555E 01
1.90	0.36589321E 01	-0.11805287E 01
1.95	0.38324884E 01	-0.12485625E 01
2.00	0.40111401E 01	-0.13180937E 01

Table D-27 (Cont'd)

THE ROOTS $t_1[Q\exp(i350^{\circ})]$

Q	Real t_1	Imag t_1
2.05	0.41948597E 01	-0.13891532E 01
2.10	0.43836205E 01	-0.14617665E 01
2.15	0.45773977E 01	-0.15359551E 01
2.20	0.47761680E 01	-0.16117364E 01
2.25	0.49799103E 01	-0.16891254E 01
2.30	0.51886044E 01	-0.17681344E 01
2.35	0.54022331E 01	-0.18487737E 01
2.40	0.56207791E 01	-0.19310521E 01
2.45	0.58442281E 01	-0.20149770E 01
2.50	0.60725667E 01	-0.21005550E 01
2.55	0.63057818E 01	-0.21877914E 01
2.60	0.65438624E 01	-0.22766910E 01
2.65	0.67867981E 01	-0.23672579E 01
2.70	0.70345793E 01	-0.24594960E 01
2.75	0.72871969E 01	-0.25534081E 01
2.80	0.75446432E 01	-0.26489973E 01
2.85	0.78069106E 01	-0.27462660E 01
2.90	0.80739928E 01	-0.28452166E 01
2.95	0.83458821E 01	-0.29458509E 01
3.00	0.86225738E 01	-0.30481708E 01
3.05	0.89040617E 01	-0.31521779E 01
3.10	0.91903421E 01	-0.32578740E 01
3.15	0.94814083E 01	-0.33652599E 01
3.20	0.97772570E 01	-0.34743372E 01
3.25	0.10077885E 02	-0.35851072E 01
3.30	0.10383286E 02	-0.36975702E 01
3.35	0.10693458E 02	-0.38117277E 01
3.40	0.11008398E 02	-0.39275806E 01
3.45	0.11328103E 02	-0.40451295E 01
3.50	0.11652568E 02	-0.41643751E 01
3.55	0.11981792E 02	-0.42853184E 01
3.60	0.12315772E 02	-0.44079597E 01
3.65	0.12654506E 02	-0.45322997E 01
3.70	0.12997991E 02	-0.46583391E 01
3.75	0.13346225E 02	-0.47860783E 01
3.80	0.13699206E 02	-0.49155176E 01
3.85	0.14056931E 02	-0.50466578E 01
3.90	0.14419401E 02	-0.51794992E 01
3.95	0.14786611E 02	-0.53140417E 01
4.00	0.15158561E 02	-0.54502864E 01

Table D-28

THE ROOTS $t_1[Q\exp(i355^{\circ})]$

Q	Real t_1	Imag t_1
0	0.50939649E 00	0.88230059E 00
0.5	0.53131361E 00	0.83762775E 00
0.10	0.55563027E 00	0.79261623E 00
0.15	0.58241788E 00	0.74733871E 00
0.20	0.61174843E 00	0.70186862E 00
0.25	0.64369476E 00	0.65628002E 00
0.30	0.67833108E 00	0.61064740E 00
0.35	0.71573333E 00	0.56504557E 00
0.40	0.75597963E 00	0.51954938E 00
0.45	0.79915078E 00	0.47423360E-00
0.50	0.84533074E 00	0.42917255E-00
0.55	0.89460714E 00	0.38443977E-00
0.60	0.94707184E 00	0.34010768E-00
0.65	0.10028214E 01	0.29624700E-00
0.70	0.10619577E 01	0.25292611E-00
0.75	0.11245885E 01	0.21021041E-00
0.80	0.11908277E 01	0.16816122E-00
0.85	0.12607960E 01	0.12683485E-00
0.90	0.13346209E 01	0.86281236E-01
0.95	0.14124369E 01	0.46542361E-01
1.00	0.14943855E 01	0.76506638E-02
1.05	0.15806143E 01	-0.30372903E-01
1.10	0.16712762E 01	-0.67520928E-01
1.15	0.17665277E 01	-0.10380143E-00
1.20	0.18665268E 01	-0.13923988E-00
1.25	0.19714298E 01	-0.17388061E-00
1.30	0.20813880E 01	-0.20778771E-00
1.35	0.21965434E 01	-0.24104501E-00
1.40	0.23170245E 01	-0.27375481E-00
1.45	0.24429415E 01	-0.30603542E-00
1.50	0.25743841E 01	-0.33801726E-00
1.55	0.27114173E 01	-0.36983784E-00
1.60	0.28540824E 01	-0.40163629E-00
1.65	0.30023966E 01	-0.43354771E-00
1.70	0.31563564E 01	-0.46569832E-00
1.75	0.33159410E 01	-0.49820174E-00
1.80	0.34811168E 01	-0.53115678E 00
1.85	0.36518419E 01	-0.56464661E 00
1.90	0.38280699E 01	-0.59873942E 00
1.95	0.40097535E 01	-0.63348979E 00
2.00	0.41968466E 01	-0.66894055E 00

Table D-28 (Cont'd)

THE ROOTS $t_1[Q\exp(i355^\circ)]$

Q	Real t_1	Imag t_1
2.05	0.43893055E 01	-0.70512490E 00
2.10	0.45870909E 01	-0.74206850E 00
2.15	0.47901667E 01	-0.77979091E 00
2.20	0.49985008E 01	-0.81830736E 00
2.25	0.52120642E 01	-0.85762940E 00
2.30	0.54308319E 01	-0.89776643E 00
2.35	0.56547813E 01	-0.93872569E 00
2.40	0.58838923E 01	-0.98051303E 00
2.45	0.61181471E 01	-0.10231332E 01
2.50	0.63575301E 01	-0.10665905E 01
2.55	0.66020263E 01	-0.11108878E 01
2.60	0.68516232E 01	-0.11560282E 01
2.65	0.71063393E 01	-0.12020140E 01
2.70	0.73660733E 01	-0.12488473E 01
2.75	0.76309060E 01	-0.12965298E 01
2.80	0.79007985E 01	-0.13450634E 01
2.85	0.81757426E 01	-0.13944490E 01
2.90	0.84557310E 01	-0.14446882E 01
2.95	0.87407574E 01	-0.14957820E 01
3.00	0.90308147E 01	-0.15477312E 01
3.05	0.93258969E 01	-0.16005369E 01
3.10	0.96260001E 01	-0.16541998E 01
3.15	0.99311178E 01	-0.17087206E 01
3.20	0.10241245E 02	-0.17641000E 01
3.25	0.10556380E 02	-0.18203385E 01
3.30	0.10876516E 02	-0.18774368E 01
3.35	0.11201651E 02	-0.19353952E 01
3.40	0.11531781E 02	-0.19942144E 01
3.45	0.11866902E 02	-0.20538945E 01
3.50	0.12207011E 02	-0.21144361E 01
3.55	0.12552106E 02	-0.21758395E 01
3.60	0.12902185E 02	-0.22381050E 01
3.65	0.13257243E 02	-0.23012330E 01
3.70	0.13617279E 02	-0.23652238E 01
3.75	0.13982291E 02	-0.24300775E 01
3.80	0.14352276E 02	-0.24957945E 01
3.85	0.14727233E 02	-0.25623750E 01
3.90	0.15107160E 02	-0.26298191E 01
3.95	0.15492054E 02	-0.26981271E 01
4.00	0.15881915E 02	-0.27672993E 01

Table D-29

THE ROOTS $t_1[Q\exp(i1^{6})]$

Q	Real t_1	Imag t_1
0	0.50939649E 00	0.88230059E 00
0.5	0.53586233E 00	0.84028906E 00
0.10	0.56473643E 00	0.79845038E 00
0.15	0.59606411E 00	0.75687576E 00
0.20	0.62989088E 00	0.71565741E 00
0.25	0.66626280E 00	0.67488857E 00
0.30	0.70522692E 00	0.63466366E 00
0.35	0.74683178E 00	0.59507831E 00
0.40	0.79112784E 00	0.55622942E 00
0.45	0.83816805E 00	0.51821530E 00
0.50	0.88800842E 00	0.48113564E-00
0.55	0.94070870E 00	0.44509157E-00
0.60	0.99633317E 00	0.41018563E-00
0.65	0.10549512E 01	0.37652168E-00
0.70	0.11166386E 01	0.34420469E-00
0.75	0.11814782E 01	0.31334051E-00
0.80	0.12495613E 01	0.28403536E-00
0.85	0.13209886E 01	0.25639518E-00
0.90	0.13958719E 01	0.23052464E-00
0.95	0.14743352E 01	0.20652590E-00
1.00	0.15565167E 01	0.18449675E-00
1.05	0.16425694E 01	0.16452830E-00
1.10	0.17326630E 01	0.14670191E-00
1.15	0.18269844E 01	0.13108536E-00
1.20	0.19257376E 01	0.11772807E-00
1.25	0.20291422E 01	0.10665561E-00
1.30	0.21374312E 01	0.97863609E-01
1.35	0.22508450E 01	0.91311550E-01
1.40	0.23696236E 01	0.86917552E-01
1.45	0.24939957E 01	0.84555227E-01
1.50	0.26241653E 01	0.84054349E-01
1.55	0.27602994E 01	0.85206583E-01
1.60	0.29025146E 01	0.87776831E-01
1.65	0.30508714E 01	0.91519276E-01
1.70	0.32053736E 01	0.96195456E-01
1.75	0.33659758E 01	0.10159086E-00
1.80	0.35325966E 01	0.10752660E-00
1.85	0.37051331E 01	0.11386447E-00
1.90	0.38834761E 01	0.12050562E-00
1.95	0.40675204E 01	0.12738470E-00
2.00	0.42571715E 01	0.13446207E-00

Table D-29 (Cont'd)

THE ROOTS t_1 Qexp(i1°)

Q	Real t_1	Imag t_1
2.05	0.44523486E 01	0.14171591E-00
2.10	0.46529850E 01	0.14913570E-00
2.15	0.48590256E 01	0.15671743E-00
2.20	0.50704252E 01	0.16446042E-00
2.25	0.52871472E 01	0.17236555E-00
2.30	0.55091601E 01	0.18043413E-00
2.35	0.57364381E 01	0.18866756E-00
2.40	0.59689590E 01	0.19706707E-00
2.45	0.62067029E 01	0.20563369E-00
2.50	0.64496528E 01	0.21436831E-00
2.55	0.66977932E 01	0.22327162E-00
2.60	0.69511110E 01	0.23234427E-00
2.65	0.72095934E 01	0.24158671E-00
2.70	0.74732296E 01	0.25099942E-00
2.75	0.77420100E 01	0.26058277E-00
2.80	0.80159245E 01	0.27033707E-00
2.85	0.82949657E 01	0.28026263E-00
2.90	0.85791261E 01	0.29035969E-00
2.95	0.88683976E 01	0.30062846E-00
3.00	0.91627748E 01	0.31106917E-00
3.05	0.94622514E 01	0.32168198E-00
3.10	0.97668222E 01	0.33246706E-00
3.15	0.10076481E 02	0.34342454E-00
3.20	0.10391225E 02	0.35455458E-00
3.25	0.10711049E 02	0.36585729E-00
3.30	0.11035948E 02	0.37733276E-00
3.35	0.11365918E 02	0.38898110E-00
3.40	0.11700958E 02	0.40080243E-00
3.45	0.12041062E 02	0.41279680E-00
3.50	0.12386228E 02	0.42496430E-00
3.55	0.12736453E 02	0.43730500E-00
3.60	0.13091735E 02	0.44981898E-00
3.65	0.13452070E 02	0.46250629E-00
3.70	0.13817456E 02	0.47536698E-00
3.75	0.14187891E 02	0.48840111E-00
3.80	0.14563373E 02	0.50160875E 00
3.85	0.14943900E 02	0.51498991E 00
3.90	0.15329470E 02	0.52854465E 00
3.95	0.15720080E 02	0.54227304E 00
4.00	0.16115730E 02	0.55617508E 00

Table D-29 (Cont'd)

THE ROOTS $t_1[Q\exp(i1^{\circ})]$

Q	Real t_1	Imag t_1
2.0	0.42570173E 01	0.13450941E-00
2.1	0.46529218E 01	0.14915717E-00
2.2	0.50703993E 01	0.16446946E-00
2.3	0.55091490E 01	0.18043794E-00
2.4	0.59689538E 01	0.19706874E-00
2.5	0.64496504E 01	0.21436910E-00
2.6	0.69511098E 01	0.23234465E-00
2.7	0.74732292E 01	0.25099963E-00
2.8	0.80159244E 01	0.27033719E-00
2.9	0.85791258E 01	0.29035974E-00
3.0	0.91627748E 01	0.31106920E-00
3.1	0.97668220E 01	0.33246708E-00
3.2	0.10391225E 02	0.35455460E-00
3.3	0.11035948E 02	0.37733278E-00
3.4	0.11700958E 02	0.40080244E-00
3.5	0.12386228E 02	0.42496431E-00
3.6	0.13091735E 02	0.44981899E-00
3.7	0.13817456E 02	0.47536699E-00
3.8	0.14563373E 02	0.50160876E 00
3.9	0.15329470E 02	0.52854468E 00
4.0	0.16115730E 02	0.55617509E 00
4.1	0.16922140E 02	0.58450028E 00
4.2	0.17748690E 02	0.61352053E 00
4.3	0.18595367E 02	0.64323607E 00
4.4	0.19462162E 02	0.67364711E 00
4.5	0.20349065E 02	0.70475381E 00
4.6	0.21256070E 02	0.73655640E 00
4.7	0.22183166E 02	0.76905494E 00
4.8	0.23130351E 02	0.80224965E 00
4.9	0.24097616E 02	0.83614061E 00
5.0	0.25084955E 02	0.87072793E 00

Table D-30

THE ROOTS $t_1[Q\exp(i5^{\circ})]$

Q	Real t_1	Imag t_1
0	0.50939649E 00	0.88230059E 00
0.5	0.53871473E 00	0.84232002E 00
0.10	0.57038693E 00	0.80284781E 00
0.15	0.60443838E 00	0.76398256E 00
0.20	0.64089420E 00	0.72582394E 00
0.25	0.67977976E 00	0.68847287E 00
0.30	0.72112097E 00	0.65203176E 00
0.35	0.76494474E 00	0.61660475E 00
0.40	0.81127940E 00	0.58229791E 00
0.45	0.86015518E 00	0.54921965E 00
0.50	0.91160472E 00	0.51748081E 00
0.55	0.96566369E 00	0.48719514E-00
0.60	0.10223715E 01	0.45847946E-00
0.65	0.10817723E 01	0.43145397E-00
0.70	0.11439154E 01	0.40624256E-00
0.75	0.12088571E 01	0.38297296E-00
0.80	0.12766615E 01	0.36177697E-00
0.85	0.13474023E 01	0.34279038E-00
0.90	0.14211646E 01	0.32615291E-00
0.95	0.14980470E 01	0.31200776E-00
1.00	0.15781645E 01	0.30050079E-00
1.05	0.16616514E 01	0.29177903E-00
1.10	0.17486645E 01	0.28598859E-00
1.15	0.18393870E 01	0.28327103E-00
1.20	0.19340325E 01	0.28375841E-00
1.25	0.20328485E 01	0.28756609E-00
1.30	0.21361182E 01	0.29478282E-00
1.35	0.22441611E 01	0.30545802E-00
1.40	0.23573282E 01	0.31958605E-00
1.45	0.24759898E 01	0.33708924E-00
1.50	0.26005156E 01	0.35780281E-00
1.55	0.27312435E 01	0.38146701E-00
1.60	0.28684420E 01	0.40773373E-00
1.65	0.30122749E 01	0.43619268E-00
1.70	0.31627791E 01	0.46641569E-00
1.75	0.33198686E 01	0.49800924E-00
1.80	0.34833638E 01	0.53065798E 00
1.85	0.36530364E 01	0.56414604E 00
1.90	0.38286541E 01	0.59835336E 00
1.95	0.40100119E 01	0.63323421E 00
2.00	0.41969465E 01	0.66878983E 00

Table D-30 (Cont'd)

THE ROOTS $t_1[Q\exp(i5^{\circ})]$

Q	Real t_1	Imag t_1
2.05	0.43893366E 01	0.70504472E 00
2.10	0.45870964E 01	0.74203002E 00
2.15	0.47901648E 01	0.77977456E 00
2.20	0.49984981E 01	0.81830158E 00
2.25	0.52120622E 01	0.85762823E 00
2.30	0.54308308E 01	0.89776705E 00
2.35	0.56547807E 01	0.93872695E 00
2.40	0.58838919E 01	0.98051453E 00
2.45	0.61181469E 01	0.10231348E 01
2.50	0.63575300E 01	0.10665921E 01
2.55	0.66020259E 01	0.11108895E 01
2.60	0.68516230E 01	0.11560300E 01
2.65	0.71063089E 01	0.12020159E 01
2.70	0.73660729E 01	0.12488492E 01
2.75	0.76309057E 01	0.12965319E 01
2.80	0.79007982E 01	0.13450654E 01
2.85	0.81757422E 01	0.13944511E 01
2.90	0.84557308E 01	0.14446905E 01
2.95	0.87407570E 01	0.14957842E 01
3.00	0.90308142E 01	0.15477335E 01
3.05	0.93258966E 01	0.16005393E 01
3.10	0.96259999E 01	0.16542023E 01
3.15	0.99311174E 01	0.17087232E 01
3.20	0.10241245E 02	0.17641027E 01
3.25	0.10556380E 02	0.18203414E 01
3.30	0.10876516E 02	0.18774396E 01
3.35	0.11201651E 02	0.19353982E 01
3.40	0.11531780E 02	0.19942174E 01
3.45	0.11866901E 02	0.20538977E 01
3.50	0.12207010E 02	0.21144393E 01
3.55	0.12552106E 02	0.21758428E 01
3.60	0.12902184E 02	0.22381085E 01
3.65	0.13257242E 02	0.23012366E 01
3.70	0.13617278E 02	0.23652274E 01
3.75	0.13982290E 02	0.24300813E 01
3.80	0.14352275E 02	0.24957983E 01
3.85	0.14727233E 02	0.25623789E 01
3.90	0.15107159E 02	0.26298232E 01
3.95	0.15492053E 02	0.26981312E 01
4.00	0.15881914E 02	0.27673035E 01

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Table D-30 (Cont'd)

THE ROOTS $t_1[Q\exp(i5^{\circ})]$

Q	Real t_1	Imag t_1
2.0	0.41969408E 01	0.66898195E 00
2.1	0.45871055E 01	0.74210170E 00
2.2	0.49985020E 01	0.81832830E 00
2.3	0.54308318E 01	0.89777808E 00
2.4	0.58838920E 01	0.98051957E 00
2.5	0.63575298E 01	0.10665945E 01
2.6	0.68516230E 01	0.11560312E 01
2.7	0.73660731E 01	0.12488499E 01
2.8	0.79007983E 01	0.13450658E 01
2.9	0.84557310E 01	0.14446906E 01
3.0	0.90308142E 01	0.15477336E 01
3.1	0.96259997E 01	0.16542023E 01
3.2	0.10241245E 02	0.17641027E 01
3.3	0.10876516E 02	0.18774397E 01
3.4	0.11531781E 02	0.19942174E 01
3.5	0.12207011E 02	0.21144393E 01
3.6	0.12902185E 02	0.22381084E 01
3.7	0.13617279E 02	0.23652274E 01
3.8	0.14352276E 02	0.24957984E 01
3.9	0.15107160E 02	0.26298232E 01
4.0	0.15881914E 02	0.27673034E 01
4.1	0.16676527E 02	0.29082408E 01
4.2	0.17490985E 02	0.30526365E 01
4.3	0.18325279E 02	0.32004917E 01
4.4	0.19179398E 02	0.33518074E 01
4.5	0.20053334E 02	0.35065846E 01
4.6	0.20947076E 02	0.36648241E 01
4.7	0.21860623E 02	0.38265266E 01
4.8	0.22793964E 02	0.39916929E 01
4.9	0.23747091E 02	0.41603234E 01
5.0	0.24720001E 02	0.43324189E 01

Table D-31

THE ROOTS $t_1[Q\exp(i110^{o})]$

Q	Real t_1	Imag t_1
0	0.50939649E 00	0.88230059E 00
0.5	0.54205377E 00	0.84512601E 00
0.10	0.57693213E 00	0.80886113E 00
0.15	0.61403035E 00	0.77360766E 00
0.20	0.65334666E 00	0.73946831E 00
0.25	0.69487895E 00	0.70654710E 00
0.30	0.73862506E 00	0.67494974E 00
0.35	0.78458298E 00	0.64478402E 00
0.40	0.83275110E 00	0.61616022E 00
0.45	0.88312855E 00	0.58919162E 00
0.50	0.93571553E 00	0.56399497E 00
0.55	0.99051370E 00	0.54069111E 00
0.60	0.10475266E 01	0.51940558E 00
0.65	0.11067604E 01	0.50026932E 00
0.70	0.11682244E 01	0.48341949E-00
0.75	0.12319318E 01	0.46900029E-00
0.80	0.12979010E 01	0.45716406E-00
0.85	0.13661571E 01	0.44807230E-00
0.90	0.14367330E 01	0.44189689E-00
0.95	0.15096724E 01	0.43882143E-00
1.00	0.15850321E 01	0.43904270E-00
1.05	0.16628863E 01	0.44277193E-00
1.10	0.17433311E 01	0.45023623E-00
1.15	0.18264917E 01	0.46167929E-00
1.20	0.19125301E 01	0.47736143E-00
1.25	0.20016582E 01	0.49755768E-00
1.30	0.20941512E 01	0.52255260E 00
1.35	0.21903668E 01	0.55262919E 00
1.40	0.22907673E 01	0.58804732E 00
1.45	0.23959403E 01	0.62900607E 00
1.50	0.25066108E 01	0.67558241E 00
1.55	0.26236220E 01	0.72764245E 00
1.60	0.27478562E 01	0.78473941E 00
1.65	0.28800628E 01	0.84604859E 00
1.70	0.30206237E 01	0.91042999E 00
1.75	0.31693781E 01	0.97667549E 00
1.80	0.33256678E 01	0.10438470E 01
1.85	0.34886098E 01	0.11114877E 01
1.90	0.36574047E 01	0.11795870E 01
1.95	0.38315036E 01	0.12483915E 01
2.00	0.40106160E 01	0.13182141E 01

Table D-31 (Cont'd)

THE ROOTS $t_1[Q\exp(i10^{\circ})]$

Q	Real t_1	Imag t_1
2.05	0.41946356E 01	0.13893206E 01
2.10	0.43835534E 01	0.14618903E 01
2.15	0.45773943E 01	0.15360238E 01
2.20	0.47761803E 01	0.16117664E 01
2.25	0.49799208E 01	0.16891357E 01
2.30	0.51886096E 01	0.17681371E 01
2.35	0.54022343E 01	0.18487749E 01
2.40	0.56207787E 01	0.19310535E 01
2.45	0.58442272E 01	0.20149790E 01
2.50	0.60725658E 01	0.21005574E 01
2.55	0.63057306E 01	0.21877938E 01
2.60	0.65438615E 01	0.22766937E 01
2.65	0.67867971E 01	0.23672608E 01
2.70	0.70345782E 01	0.24594988E 01
2.75	0.72871958E 01	0.25534111E 01
2.80	0.75446422E 01	0.26490004E 01
2.85	0.78069095E 01	0.27462692E 01
2.90	0.80739911E 01	0.28452199E 01
2.95	0.83458810E 01	0.29458544E 01
3.00	0.86225725E 01	0.30481744E 01
3.05	0.89040606E 01	0.31521817E 01
3.10	0.91903404E 01	0.32578778E 01
3.15	0.94814069E 01	0.33652639E 01
3.20	0.97772560E 01	0.34743415E 01
3.25	0.10077883E 02	0.35851111E 01
3.30	0.10383284E 02	0.36975745E 01
3.35	0.10693456E 02	0.38117322E 01
3.40	0.11008597E 02	0.39275852E 01
3.45	0.11328101E 02	0.40451343E 01
3.50	0.11652566E 02	0.41643800E 01
3.55	0.11981791E 02	0.42853235E 01
3.60	0.12315771E 02	0.44079649E 01
3.65	0.12654504E 02	0.45323050E 01
3.70	0.12997989E 02	0.46583445E 01
3.75	0.13346224E 02	0.47860838E 01
3.80	0.13699204E 02	0.49155234E 01
3.85	0.14056930E 02	0.50466637E 01
3.90	0.14419398E 02	0.51795051E 01
3.95	0.14786609E 02	0.53140481E 01
4.00	0.15158559E 02	0.54502928E 01

Table D-31 (Cont'd)

THE ROOTS $t_1[Q\exp(i10^o)]$

Q	Real t_1	Imag t_1
2.0	0.40109725E 01	0.13181297E 01
2.1	0.43836296E 01	0.14618231E 01
2.2	0.47761938E 01	0.16117514E 01
2.3	0.51886161E 01	0.17681370E 01
2.4	0.56207831E 01	0.19310543E 01
2.5	0.60725679E 01	0.21005576E 01
2.6	0.65438627E 01	0.22766938E 01
2.7	0.70345789E 01	0.24594989E 01
2.8	0.75446427E 01	0.26490004E 01
2.9	0.80739915E 01	0.28452198E 01
3.0	0.86225725E 01	0.30481744E 01
3.1	0.91903404E 01	0.32578778E 01
3.2	0.97772562E 01	0.34743414E 01
3.3	0.10383284E 02	0.36975745E 01
3.4	0.11008397E 02	0.39275853E 01
3.5	0.11652566E 02	0.41643800E 01
3.6	0.12315771E 02	0.44079649E 01
3.7	0.12997989E 02	0.46583445E 01
3.8	0.13699204E 02	0.49155234E 01
3.9	0.14419399E 02	0.51795051E 01
4.0	0.15158559E 02	0.54502928E 01
4.1	0.15916672E 02	0.57278895E 01
4.2	0.16693725E 02	0.60122977E 01
4.3	0.17489709E 02	0.63035199E 01
4.4	0.18304615E 02	0.66015578E 01
4.5	0.19138432E 02	0.69064133E 01
4.6	0.19991153E 02	0.72180882E 01
4.7	0.20862772E 02	0.75365836E 01
4.8	0.21753281E 02	0.78619015E 01
4.9	0.22662675E 02	0.81940422E 01
5.0	0.23590948E 02	0.85330077E 01

Table D-32

THE ROOTS $t_1[Q\exp(i15^{o})]$

Q	Real t_1	Imag t_1
0	0.50939649E 00	0.88230059E 00
0.5	0.54511897E 00	0.84820469E 00
0.10	0.58286220E 00	0.81538835E 00
0.15	0.62259859E 00	0.78394940E 00
0.20	0.66429963E 00	0.75398639E 00
0.25	0.70793594E 00	0.72559905E 00
0.30	0.75347733E 00	0.69888870E 00
0.35	0.80089285E 00	0.67395864E 00
0.40	0.85015082E 00	0.65091478E 00
0.45	0.90121883E 00	0.62986609E 00
0.50	0.95406388E 00	0.61092525E 00
0.55	0.10086522E 01	0.59420940E 00
0.60	0.10649499E 01	0.57984087E 00
0.65	0.11229222E 01	0.56794818E 00
0.70	0.11825342E 01	0.55866706E 00
0.75	0.12437510E 01	0.55214176E 00
0.80	0.13065375E 01	0.54852656E 00
0.85	0.13708597E 01	0.54798758E 00
0.90	0.14366841E 01	0.55070504E 00
0.95	0.15039796E 01	0.55687594E 00
1.00	0.15727177E 01	0.56671755E 00
1.05	0.16428749E 01	0.58047155E 00
1.10	0.17144350E 01	0.59840963E 00
1.15	0.17873929E 01	0.62084026E 00
1.20	0.18617604E 01	0.64811797E 00
1.25	0.19375756E 01	0.68065530E 00
1.30	0.20149175E 01	0.71893819E 00
1.35	0.20939311E 01	0.76354700E 00
1.40	0.21748681E 01	0.81518222E 00
1.45	0.22581613E 01	0.87469430E 00
1.50	0.23445567E 01	0.94310667E 00
1.55	0.24353559E 01	0.10215885E 01
1.60	0.25328308E 01	0.11112340E 01
1.65	0.26407389E 01	0.12122399E 01
1.70	0.27639673E 01	0.13218964E 01
1.75	0.29048711E 01	0.14330178E 01
1.80	0.30589309E 01	0.15385686E 01
1.85	0.32189775E 01	0.16378651E 01
1.90	0.33811571E 01	0.17342103E 01
1.95	0.35448062E 01	0.18307966E 01
2.00	0.37106340E 01	0.19295880E 01

Table D-32 (Cont'd)

THE ROOTS $t_1[Q\exp(i115^{\circ})]$

Q	Real t_1	Imag t_1
2.05	0.38796823E 01	0.20314784E 01
2.10	0.40528535E 01	0.21366268E 01
2.15	0.42307068E 01	0.22447891E 01
2.20	0.44134344E 01	0.23556141E 01
2.25	0.46009796E 01	0.24688467E 01
2.30	0.47932009E 01	0.25843885E 01
2.35	0.49899842E 01	0.27022515E 01
2.40	0.51912750E 01	0.28224852E 01
2.45	0.53970608E 01	0.29451295E 01
2.50	0.56073425E 01	0.30702033E 01
2.55	0.58221170E 01	0.31977122E 01
2.60	0.60413779E 01	0.33276594E 01
2.65	0.62651153E 01	0.34600480E 01
2.70	0.64933200E 01	0.35948820E 01
2.75	0.67259841E 01	0.37321661E 01
2.80	0.69631007E 01	0.38719041E 01
2.85	0.72046623E 01	0.40140989E 01
2.90	0.74506642E 01	0.41587542E 01
2.95	0.77010994E 01	0.43058719E 01
3.00	0.79559629E 01	0.44554551E 01
3.05	0.82152497E 01	0.46075056E 01
3.10	0.84789559E 01	0.47620259E 01
3.15	0.87470762E 01	0.49190171E 01
3.20	0.90196065E 01	0.50784815E 01
3.25	0.92965437E 01	0.52404206E 01
3.30	0.95778836E 01	0.54048353E 01
3.35	0.98636232E 01	0.55717278E 01
3.40	0.10153759E 02	0.57410986E 01
3.45	0.10448288E 02	0.59129489E 01
3.50	0.10747208E 02	0.60872801E 01
3.55	0.11050516E 02	0.62640929E 01
3.60	0.11358210E 02	0.64433884E 01
3.65	0.11670288E 02	0.66251673E 01
3.70	0.11986746E 02	0.68094303E 01
3.75	0.12307583E 02	0.69961782E 01
3.80	0.12632797E 02	0.71854119E 01
3.85	0.12962386E 02	0.73771320E 01
3.90	0.13296349E 02	0.75713386E 01
3.95	0.13634682E 02	0.77680326E 01
4.00	0.13977388E 02	0.79672150E 01

Table D-32 (Cont'd)

THE ROOTS $t_1[Q\exp(i15^{\circ})]$

Q	Real t_1	Imag t_1
2.0	0.37096836E 01	0.19283783E 01
2.1	0.40523782E 01	0.21367379E 01
2.2	0.44134422E 01	0.23557464E 01
2.3	0.47932413E 01	0.25843836E 01
2.4	0.51912767E 01	0.28224722E 01
2.5	0.56073410E 01	0.30702011E 01
2.6	0.60413782E 01	0.33276586E 01
2.7	0.64933203E 01	0.35948814E 01
2.8	0.69631007E 01	0.38719036E 01
2.9	0.74506642E 01	0.41587537E 01
3.0	0.79559632E 01	0.44554549E 01
3.1	0.84789559E 01	0.47620255E 01
3.2	0.90196066E 01	0.50784814E 01
3.3	0.95778836E 01	0.54048353E 01
3.4	0.10153759E 02	0.57410984E 01
3.5	0.10747208E 02	0.60872801E 01
3.6	0.11358210E 02	0.64433882E 01
3.7	0.11986746E 02	0.68094302E 01
3.8	0.12632797E 02	0.71854119E 01
3.9	0.13296349E 02	0.75713386E 01
4.0	0.13977388E 02	0.79672150E 01
4.1	0.14675900E 02	0.83730450E 01
4.2	0.15391877E 02	0.87888329E 01
4.3	0.16125306E 02	0.92145818E 01
4.4	0.16876180E 02	0.96502946E 01
4.5	0.17644489E 02	0.10095973E 02
4.6	0.18430226E 02	0.10551621E 02
4.7	0.19233384E 02	0.11017240E 02
4.8	0.20053958E 02	0.11492832E 02
4.9	0.20891940E 02	0.11978399E 02
5.0	0.21747325E 02	0.12473942E 02

Table D-33

THE ROOTS $t_1[Q\exp(i120^{\circ})]$

Q	Real t_1	Imag t_1
0	0.50939649E 00	0.88230059E 00
0.5	0.54788925E 00	0.85152842E 00
0.10	0.58814208E 00	0.82236311E 00
0.15	0.63010296E 00	0.79489185E 00
0.20	0.67371870E 00	0.76920222E 00
0.25	0.71893478E 00	0.74538255E 00
0.30	0.76569525E 00	0.72352232E 00
0.35	0.81394254E 00	0.70371262E 00
0.40	0.86361726E 00	0.68604656E 00
0.45	0.91465793E 00	0.67061978E 00
0.50	0.96700083E 00	0.65753107E 00
0.55	0.10205795E 01	0.64688294E 00
0.60	0.10753246E 01	0.63878234E 00
0.65	0.11311634E 01	0.63334156E 00
0.70	0.11880191E 01	0.63067911E 00
0.75	0.12458103E 01	0.63092095E 00
0.80	0.13044506E 01	0.63420185E 00
0.85	0.13638472E 01	0.64066709E 00
0.90	0.14239002E 01	0.65047454E 00
0.95	0.14845009E 01	0.66379725E 00
1.00	0.15455304E 01	0.68082676E 00
1.05	0.16068573E 01	0.70177733E 00
1.10	0.16683343E 01	0.72689169E 00
1.15	0.17297951E 01	0.75644836E 00
1.20	0.17910484E 01	0.79077213E 00
1.25	0.18518706E 01	0.83024863E 00
1.30	0.19119938E 01	0.87534557E 00
1.35	0.19710898E 01	0.92664495E 00
1.40	0.20287409E 01	0.98489403E 00
1.45	0.20843913E 01	0.10510906E 01
1.50	0.21372553E 01	0.11266363E 01
1.55	0.21861168E 01	0.12136395E 01
1.60	0.22288193E 01	0.13155930E 01
1.65	0.22605315E 01	0.14371842E 01
1.70	0.22638530E 01	0.15999239E 01
1.75	0.21735803E 01	0.17132550E 01
1.80	0.20310266E 01	0.17349278E 01
1.85	0.19598942E 01	0.17546788E 01
1.90	0.19061477E 01	0.17673491E 01
1.95	0.18629371E 01	0.17769817E 01
2.00	0.18268178E 01	0.17849336E 01

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Table D-33 (Cont'd)

THE ROOTS $t_1[Q\exp(i120^\circ)]$

Q	Real t_1	Imag t_1
2.05	0.17958205E 01	0.17918021E 01
2.10	0.17687099E 01	0.17978998E 01
2.15	0.17446566E 01	0.18034106E 01
2.20	0.17230751E 01	0.18084536E 01
2.25	0.17035351E 01	0.18131100E 01
2.30	0.16857107E 01	0.18174394E 01
2.35	0.16693490E 01	0.18214867E 01
2.40	0.16542488E 01	0.18252864E 01
2.45	0.16402481E 01	0.18288672E 01
2.50	0.16272143E 01	0.18322517E 01
2.55	0.16150365E 01	0.18354592E 01
2.60	0.16036223E 01	0.18385062E 01
2.65	0.15928932E 01	0.18414064E 01
2.70	0.15827818E 01	0.18441720E 01
2.75	0.15732303E 01	0.18468135E 01
2.80	0.15641880E 01	0.18493403E 01
2.85	0.15556112E 01	0.18517604E 01
2.90	0.15474609E 01	0.18540817E 01
2.95	0.15397031E 01	0.18563104E 01
3.00	0.15323074E 01	0.18584526E 01
3.05	0.15252466E 01	0.18605138E 01
3.10	0.15184965E 01	0.18624987E 01
3.15	0.15120351E 01	0.18644119E 01
3.20	0.15058434E 01	0.18662579E 01
3.25	0.14999027E 01	0.18680399E 01
3.30	0.14941972E 01	0.18697616E 01
3.35	0.14887121E 01	0.18714263E 01
3.40	0.14834340E 01	0.18730369E 01
3.45	0.14783506E 01	0.18745963E 01
3.50	0.14734502E 01	0.18761065E 01
3.55	0.14687229E 01	0.18775706E 01
3.60	0.14641589E 01	0.18789904E 01
3.65	0.14597493E 01	0.18803681E 01
3.70	0.14554863E 01	0.18817056E 01
3.75	0.14513617E 01	0.18830047E 01
3.80	0.14473687E 01	0.18842670E 01
3.85	0.14435009E 01	0.18854943E 01
3.90	0.14397521E 01	0.18866882E 01
3.95	0.14361165E 01	0.18878496E 01
4.00	0.14325889E 01	0.18889805E 01

Table D-33 (Cont'd)

THE ROOTS $t_1[Q\exp(i20^{\circ})]$

Q	Real t_1	Imag t_1
2.0	0.33104712E 01	0.24694440E 01
2.1	0.35990214E 01	0.27407619E 01
2.2	0.39190886E 01	0.30292844E 01
2.3	0.42577693E 01	0.33221941E 01
2.4	0.46087171E 01	0.36269369E 01
2.5	0.49759050E 01	0.39457256E 01
2.6	0.53594987E 01	0.42766944E 01
2.7	0.57587586E 01	0.46201694E 01
2.8	0.61738424E 01	0.49763100E 01
2.9	0.66046794E 01	0.53450790E 01
3.0	0.70512225E 01	0.57265215E 01
3.1	0.75134376E 01	0.61206563E 01
3.2	0.79912909E 01	0.65275021E 01
3.3	0.84847539E 01	0.69470746E 01
3.4	0.89938012E 01	0.73793877E 01
3.5	0.95184101E 01	0.78244522E 01
3.6	0.10058561E 02	0.82822781E 01
3.7	0.10614236E 02	0.87528755E 01
3.8	0.11185418E 02	0.92362508E 01
3.9	0.11772093E 02	0.97324110E 01
4.0	0.12374249E 02	0.10241361E 02
4.1	0.12991873E 02	0.10763108E 02
4.2	0.13624956E 02	0.11297656E 02
4.3	0.14273486E 02	0.11845008E 02
4.4	0.14937457E 02	0.12405170E 02
4.5	0.15616859E 02	0.12978143E 02
4.6	0.16311585E 02	0.13563931E 02
4.7	0.17021929E 02	0.14162537E 02
4.8	0.17747585E 02	0.14773965E 02
4.9	0.18488649E 02	0.15398215E 02
5.0	0.19245112E 02	0.16035289E 02

Table D-34

THE ROOTS $t_1[Q\exp(i44^{\circ})]$

Q	Real t_1	Imag t_1
0	0.50939649E 00	0.88230059E 00
0.5	0.55659781E 00	0.86995077E 00
0.10	0.60378324E 00	0.85994212E 00
0.15	0.65085455E 00	0.85225192E 00
0.20	0.69771345E 00	0.84685599E 00
0.25	0.74426132E 00	0.84372834E 00
0.30	0.79039906E 00	0.84284094E 00
0.35	0.83602680E 00	0.84416331E 00
0.40	0.88104378E 00	0.84766213E 00
0.45	0.92534816E 00	0.85330080E 00
0.50	0.96883687E 00	0.86103895E 00
0.55	0.10114055E 01	0.87083185E 00
0.60	0.10529487E 01	0.88262985E 00
0.65	0.10933592E 01	0.89637755E 00
0.70	0.11325293E 01	0.91201306E 00
0.75	0.11703501E 01	0.92946712E 00
0.80	0.12067122E 01	0.94866216E 00
0.85	0.12415066E 01	0.96951113E 00
0.90	0.12746254E 01	0.99191654E 00
0.95	0.13059630E 01	0.10157692E 01
1.00	0.13354173E 01	0.10409474E 01
1.05	0.13628925E 01	0.10673155E 01
1.10	0.13883007E 01	0.10947239E 01
1.15	0.14115651E 01	0.11230082E 01
1.20	0.14326237E 01	0.11519897E 01
1.25	0.14514323E 01	0.11814768E 01
1.30	0.14679686E 01	0.12112668E 01
1.35	0.14822359E 01	0.12411496E 01
1.40	0.14942658E 01	0.12709122E 01
1.45	0.15041198E 01	0.13003443E 01
1.50	0.15118899E 01	0.13292450E 01
1.55	0.15176971E 01	0.13574295E 01
1.60	0.15216870E 01	0.13847347E 01
1.65	0.15240251E 01	0.14110252E 01
1.70	0.15248899E 01	0.14361960E 01
1.75	0.15244661E 01	0.14601739E 01
1.80	0.15229364E 01	0.14829162E 01
1.85	0.15204774E 01	0.15044086E 01
1.90	0.15172531E 01	0.15246608E 01
1.95	0.15134124E 01	0.15437026E 01
2.00	0.15090879E 01	0.15615778E 01

Table D-34 (Cont'd)

THE ROOTS $t_1[Q\exp(i44^{\circ})]$

Q	Real t_1	Imag t_1
2.05	0.15043942E 01	0.15783416E 01
2.10	0.14994296E 01	0.15940554E 01
2.15	0.14942764E 01	0.16087842E 01
2.20	0.14890031E 01	0.16225935E 01
2.25	0.14836653E 01	0.16355483E 01
2.30	0.14783081E 01	0.16477112E 01
2.35	0.14729675E 01	0.16591417E 01
2.40	0.14676718E 01	0.16698954E 01
2.45	0.14624430E 01	0.16800247E 01
2.50	0.14572979E 01	0.16895778E 01
2.55	0.14522489E 01	0.16985987E 01
2.60	0.14473052E 01	0.17071286E 01
2.65	0.14424732E 01	0.17152043E 01
2.70	0.14377566E 01	0.17228604E 01
2.75	0.14331582E 01	0.17301275E 01
2.80	0.14286789E 01	0.17370340E 01
2.85	0.14243182E 01	0.17436059E 01
2.90	0.14200755E 01	0.17498668E 01
2.95	0.14159489E 01	0.17558384E 01
3.00	0.14119365E 01	0.17615400E 01
3.05	0.14080359E 01	0.17669903E 01
3.10	0.14042440E 01	0.17722052E 01
3.15	0.14005581E 01	0.17772002E 01
3.20	0.13969750E 01	0.17819890E 01
3.25	0.13934916E 01	0.17865846E 01
2.30	0.13901050E 01	0.17909983E 01
3.35	0.13868119E 01	0.17952412E 01
3.40	0.13836090E 01	0.17993234E 01
3.45	0.13804937E 01	0.18032541E 01
3.50	0.13774627E 01	0.18070416E 01
3.55	0.13745131E 01	0.18106941E 01
3.60	0.13716422E 01	0.18142187E 01
3.65	0.13688470E 01	0.18176223E 01
3.70	0.13661250E 01	0.18209114E 01
3.75	0.13634738E 01	0.18240912E 01
3.80	0.13608906E 01	0.18271681E 01
3.85	0.13583730E 01	0.18301468E 01
3.90	0.13559192E 01	0.18330323E 01
3.95	0.13535266E 01	0.18358283E 01
4.00	0.13511929E 01	0.18385402E 01

Table D-35

THE ROOTS $t_1[Q\exp(i46^\circ)]$

Q	Real t_1	Imag t_1
0	0.50939649E 00	0.88230059E 00
0.5	0.55695913E 00	0.87160375E 00
0.10	0.60434368E 00	0.86323792E 00
0.15	0.65145490E 00	0.85717026E 00
0.20	0.69819770E 00	0.85336641E 00
0.25	0.74447697E 00	0.85179024E 00
0.30	0.79019745E 00	0.85240344E 00
0.35	0.83526354E 00	0.85516517E 00
0.40	0.87957921E 00	0.86003161E 00
0.45	0.92304796E 00	0.86695559E 00
0.50	0.96557277E 00	0.87588596E 00
0.55	0.10070561E 01	0.88676716E 00
0.60	0.10474002E 01	0.89953852E 00
0.65	0.10865069E 01	0.91413372E 00
0.70	0.11242783E 01	0.93047989E 00
0.75	0.11606171E 01	0.94849707E 00
0.80	0.11954270E 01	0.96809721E 00
0.85	0.12286140E 01	0.98918354E 00
0.90	0.12600867E 01	0.10116496E 01
0.95	0.12897585E 01	0.10353787E 01
1.00	0.13175484E 01	0.10602434E 01
1.05	0.13433833E 01	0.10861047E 01
1.10	0.13672003E 01	0.11128125E 01
1.15	0.13889492E 01	0.11402062E 01
1.20	0.14085951E 01	0.11681148E 01
1.25	0.14261208E 01	0.11963599E 01
1.30	0.14415300E 01	0.12247574E 01
1.35	0.14548487E 01	0.12531213E 01
1.40	0.14661273E 01	0.12812677E 01
1.45	0.14754399E 01	0.13090196E 01
1.50	0.14828841E 01	0.13362120E 01
1.55	0.14885785E 01	0.13626959E 01
1.60	0.14926589E 01	0.13883434E 01
1.65	0.14952738E 01	0.14130496E 01
1.70	0.14965792E 01	0.14367350E 01
1.75	0.14967334E 01	0.14593449E 01
1.80	0.14958920E 01	0.14808489E 01
1.85	0.14942037E 01	0.15012382E 01
1.90	0.14918070E 01	0.15205223E 01
1.95	0.14888285E 01	0.15387264E 01
2.00	0.14853809E 01	0.15558864E 01

Table D-35 (Cont'd)

THE ROOTS $t_1[Q\exp(i46^\circ)]$

Q	Real t_1	Imag t_1
2.05	0.14815637E 01	0.15720475E 01
2.10	0.14774625E 01	0.15872605E 01
2.15	0.14731507E 01	0.16015783E 01
2.20	0.14686901E 01	0.16150559E 01
2.25	0.14641321E 01	0.16277479E 01
2.30	0.14595193E 01	0.16397069E 01
2.35	0.14548863E 01	0.16509841E 01
2.40	0.14502615E 01	0.16616277E 01
2.45	0.14456673E 01	0.16716832E 01
2.50	0.14411214E 01	0.16811930E 01
2.55	0.14366380E 01	0.16901962E 01
2.60	0.14322276E 01	0.16987297E 01
3.65	0.14278982E 01	0.17068268E 01
2.70	0.14236559E 01	0.17145185E 01
2.75	0.14195047E 01	0.17218333E 01
2.80	0.14154472E 01	0.17287977E 01
2.85	0.14114852E 01	0.17354348E 01
2.90	0.14076189E 01	0.17417680E 01
2.95	0.14038485E 01	0.17478166E 01
3.00	0.14001732E 01	0.17535996E 01
3.05	0.13965916E 01	0.17591333E 01
3.10	0.13931024E 01	0.17644347E 01
3.15	0.13897038E 01	0.17695175E 01
3.20	0.13863936E 01	0.17743950E 01
3.25	0.13831699E 01	0.17790797E 01
3.30	0.13800301E 01	0.17835831E 01
3.35	0.13769723E 01	0.17879153E 01
3.40	0.13739940E 01	0.17920867E 01
3.45	0.13710928E 01	0.17961055E 01
3.50	0.13682663E 01	0.17999807E 01
3.55	0.13655124E 01	0.18037197E 01
3.60	0.13628287E 01	0.18073298E 01
3.65	0.13602129E 01	0.18108178E 01
3.70	0.13576629E 01	0.18141899E 01
3.75	0.13551766E 01	0.18174518E 01
3.80	0.13527516E 01	0.18206090E 01
3.85	0.13503865E 01	0.18236671E 01
3.90	0.13480789E 01	0.18266304E 01
3.95	0.13458268E 01	0.18295027E 01
4.00	0.13436288E 01	0.18322894E 01

Table D-36

THE ROOTS $t_1[Q\exp(i60^{\circ})]$

Q	Real t_1	Imag t_1
0	0.50939649E 00	0.88230059E 00
0.5	0.55787475E 00	0.88330974E 00
0.10	0.60512120E 00	0.88627898E 00
0.15	0.65108723E 00	0.89112059E 00
0.20	0.69572591E 00	0.89774631E 00
0.25	0.73899173E 00	0.90606698E 00
0.30	0.78084134E 00	0.91599235E 00
0.35	0.82123359E 00	0.92743080E 00
0.40	0.86012989E 00	0.94028924E 00
0.45	0.89749453E 00	0.95447290E 00
0.50	0.93329515E 00	0.96988524E 00
0.55	0.96750307E 00	0.98642788E 00
0.60	0.10000937E 01	0.10040007E 01
0.65	0.10310474E 01	0.10225017E 01
0.70	0.10603493E 01	0.10418278E 01
0.75	0.10879903E 01	0.10618740E 01
0.80	0.11139675E 01	0.10825350E 01
0.85	0.11382845E 01	0.11037046E 01
0.90	0.11609519E 01	0.11252772E 01
0.95	0.11819877E 01	0.11471478E 01
1.00	0.12014173E 01	0.11692132E 01
1.05	0.12192739E 01	0.11913729E 01
1.10	0.12355982E 01	0.12135297E 01
1.15	0.12504381E 01	0.12355911E 01
1.20	0.12638487E 01	0.12574701E 01
1.25	0.12758910E 01	0.12790859E 01
1.30	0.12866317E 01	0.13003649E 01
1.35	0.12961422E 01	0.13212414E 01
1.40	0.13044969E 01	0.13416579E 01
1.45	0.13117730E 01	0.13615653E 01
1.50	0.13180488E 01	0.13809235E 01
1.55	0.13234024E 01	0.13997006E 01
1.60	0.13279115E 01	0.14178729E 01
1.65	0.13316514E 01	0.14354243E 01
1.70	0.13346948E 01	0.14523458E 01
1.75	0.13371111E 01	0.14686345E 01
1.80	0.13389657E 01	0.14842934E 01
1.85	0.13403198E 01	0.14993300E 01
1.90	0.13412301E 01	0.15137556E 01
1.95	0.13417490E 01	0.15275852E 01
2.00	0.13419241E 01	0.15408358E 01

Table D-36 (Cont'd)

THE ROOTS $t_1[Q\exp(i60^{\circ})]$

Q	Real t_1	Imag t_1
2.05	0.13417988E 01	0.15535264E 01
2.10	0.13414124E 01	0.15656778E 01
2.15	0.13407996E 01	0.15773114E 01
2.20	0.13399923E 01	0.15884490E 01
2.25	0.13390183E 01	0.15991122E 01
2.30	0.13379026E 01	0.16093231E 01
2.35	0.13366670E 01	0.16191033E 01
2.40	0.13353314E 01	0.16284736E 01
2.45	0.13339123E 01	0.16374546E 01
2.50	0.13324250E 01	0.16460656E 01
2.55	0.13308825E 01	0.16543256E 01
2.60	0.13292961E 01	0.16622528E 01
2.65	0.13276760E 01	0.16698641E 01
2.70	0.13260309E 01	0.16771757E 01
2.75	0.13243681E 01	0.16842032E 01
2.80	0.13226943E 01	0.16909612E 01
2.85	0.13210151E 01	0.16974635E 01
2.90	0.13193352E 01	0.17037234E 01
2.95	0.13176589E 01	0.17097531E 01
3.00	0.13159896E 01	0.17155641E 01
3.05	0.13143306E 01	0.17211673E 01
3.10	0.13126843E 01	0.17265733E 01
3.15	0.13110528E 01	0.17317916E 01
3.20	0.13094380E 01	0.17368315E 01
3.25	0.13078413E 01	0.17417016E 01
3.30	0.13062641E 01	0.17464100E 01
3.35	0.13047074E 01	0.17509641E 01
3.40	0.13031720E 01	0.17553715E 01
3.45	0.13016584E 01	0.17596387E 01
3.50	0.13001673E 01	0.17637721E 01
3.55	0.12986987E 01	0.17677780E 01
3.60	0.12972533E 01	0.17716620E 01
3.65	0.12958307E 01	0.17754293E 01
3.70	0.12944315E 01	0.17790854E 01
3.75	0.12930553E 01	0.17826346E 01
3.80	0.12917022E 01	0.17860818E 01
3.85	0.12903719E 01	0.17894312E 01
3.90	0.12890643E 01	0.17926870E 01
3.95	0.12877794E 01	0.17958525E 01
4.00	0.12865165E 01	0.17989323E 01

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Table D-37

THE ROOTS $t_1[Q\exp(i85^o)]$

Q	Real t_1	Imag t_1
0	0.50939649E 00	0.88230059E 00
0.5	0.55272371E 00	0.90323513E 00
0.10	0.59379385E 00	0.92450963E 00
0.15	0.63267811E 00	0.94605481E 00
0.20	0.66944774E 00	0.96780308E 00
0.25	0.70417416E 00	0.98968890E 00
0.30	0.73692904E 00	0.10116489E 01
0.35	0.76778425E 00	0.10336226E 01
0.40	0.79681196E 00	0.10555521E 01
0.45	0.82408451E 00	0.10773826E 01
0.50	0.84967438E 00	0.10990626E 01
0.55	0.87365402E 00	0.11205443E 01
0.60	0.89609571E 00	0.11417833E 01
0.65	0.91707145E 00	0.11627391E 01
0.70	0.93665271E 00	0.11833750E 01
0.75	0.95491019E 00	0.12036581E 01
0.80	0.97191364E 00	0.12235598E 01
0.85	0.98773161E 00	0.12430547E 01
0.90	0.10024311E 01	0.12621218E 01
0.95	0.10160779E 01	0.12807435E 01
1.00	0.10287353E 01	0.12989057E 01
1.05	0.10404652E 01	0.13165977E 01
1.10	0.10513267E 01	0.13338120E 01
1.15	0.10613769E 01	0.13505439E 01
1.20	0.10706703E 01	0.13667914E 01
1.25	0.10792591E 01	0.13825552E 01
1.30	0.10871927E 01	0.13978378E 01
1.35	0.10945178E 01	0.14126440E 01
1.40	0.11012786E 01	0.14269800E 01
1.45	0.11075169E 01	0.14408536E 01
1.50	0.11132713E 01	0.14542738E 01
1.55	0.11185788E 01	0.14672507E 01
1.60	0.11234731E 01	0.14797952E 01
1.65	0.11279862E 01	0.14919189E 01
1.70	0.11321477E 01	0.15036336E 01
1.75	0.11359849E 01	0.15149520E 01
1.80	0.11395232E 01	0.15258866E 01
1.85	0.11427864E 01	0.15364501E 01
1.90	0.11457959E 01	0.15466552E 01
1.95	0.11485717E 01	0.15565148E 01
2.00	0.11511327E 01	0.15660413E 01

Table D-37 (Cont'd)

THE ROOTS $t_1[Q\exp(185^\circ)]$

Q	Real t_1	Imag t_1
2.05	0.11534956E 01	0.15752470E 01
2.10	0.11556762E 01	0.15841441E 01
2.15	0.11576888E 01	0.15927444E 01
2.20	0.11595468E 01	0.16010594E 01
2.25	0.11612622E 01	0.16091004E 01
2.30	0.11628464E 01	0.16168781E 01
2.35	0.11643097E 01	0.16244030E 01
2.40	0.11656614E 01	0.16316851E 01
2.45	0.11669100E 01	0.16387344E 01
2.50	0.11680640E 01	0.16455600E 01
2.55	0.11691304E 01	0.16521710E 01
2.60	0.11701158E 01	0.16585760E 01
2.65	0.11710268E 01	0.16647833E 01
2.70	0.11718684E 01	0.16708008E 01
2.75	0.11726464E 01	0.16766361E 01
2.80	0.11733655E 01	0.16822965E 01
2.85	0.11740302E 01	0.16877888E 01
2.90	0.11746443E 01	0.16931199E 01
2.95	0.11752114E 01	0.16982958E 01
3.00	0.11757355E 01	0.17033230E 01
3.05	0.11762194E 01	0.17082068E 01
3.10	0.11766662E 01	0.17129532E 01
3.15	0.11770783E 01	0.17175671E 01
3.20	0.11774585E 01	0.17220537E 01
3.25	0.11778086E 01	0.17264180E 01
3.30	0.11781315E 01	0.17306643E 01
3.35	0.11784288E 01	0.17347971E 01
3.40	0.11787021E 01	0.17388207E 01
3.45	0.11789533E 01	0.17427392E 01
3.50	0.11791839E 01	0.17465561E 01
3.55	0.11793955E 01	0.17502753E 01
3.60	0.11795888E 01	0.17539003E 01
3.65	0.11797659E 01	0.17574345E 01
3.70	0.11799274E 01	0.17608808E 01
3.75	0.11800747E 01	0.17642426E 01
3.80	0.11802084E 01	0.17675228E 01
3.85	0.11803295E 01	0.17707242E 01
3.90	0.11804390E 01	0.17738494E 01
3.95	0.11805380E 01	0.17769009E 01
4.00	0.11806262E 01	0.17798814E 01

Table D-37 (Cont'd)

THE ROOTS $t_1[Q\exp(i85^\circ)]$

Q	Real t_1	Imag t_1
4.05	0.11807057E 01	0.17827930E 01
4.10	0.11807755E 01	0.17856382E 01
4.15	0.11808374E 01	0.17884190E 01
4.20	0.11808920E 01	0.17911376E 01
4.25	0.11809392E 01	0.17937960E 01
4.30	0.11809799E 01	0.17963959E 01
4.35	0.11810139E 01	0.17989394E 01
4.40	0.11810424E 01	0.18014282E 01
4.45	0.11810656E 01	0.18038640E 01
4.50	0.11810828E 01	0.18062485E 01
4.55	0.11810960E 01	0.18085830E 01
4.60	0.11811046E 01	0.18108693E 01
4.65	0.11811090E 01	0.18131086E 01
4.70	0.11811099E 01	0.18153025E 01
4.75	0.11811069E 01	0.18174522E 01
4.80	0.11811005E 01	0.18195591E 01
4.85	0.11810905E 01	0.18216244E 01
4.90	0.11810781E 01	0.18236492E 01
4.95	0.11810635E 01	0.18256346E 01
5.00	0.11810452E 01	0.18275822E 01

Table D-38

THE ROOTS $t_1[Q\exp(i120^{\circ})]$

Q	Real t_1	Imag t_1
0	0.50939649E 00	0.88230059E 00
0.5	0.53335264E 00	0.92379388E 00
0.10	0.55617675E 00	0.96332640E 00
0.15	0.57791819E 00	0.10009836E 01
0.20	0.59862544E 00	0.10368496E 01
0.25	0.61834597E 00	0.10710067E 01
0.30	0.63712609E 00	0.11035348E 01
0.35	0.65501083E 00	0.11345121E 01
0.40	0.67204385E 00	0.11640141E 01
0.45	0.68826737E 00	0.11921141E 01
0.50	0.70372204E 00	0.12188824E 01
0.55	0.71844704E 00	0.12443868E 01
0.60	0.73247990E 00	0.12686925E 01
0.65	0.74585659E 00	0.12918615E 01
0.70	0.75861146E 00	0.13139536E 01
0.75	0.77077727E 00	0.13350254E 01
0.80	0.78238522E 00	0.13551310E 01
0.85	0.79346499E 00	0.13743217E 01
0.90	0.80404471E 00	0.13926463E 01
0.95	0.81415112E 00	0.14101511E 01
1.00	0.82380947E 00	0.14268798E 01
1.05	0.83304375E 00	0.14428741E 01
1.10	0.84187656E 00	0.14581729E 01
1.15	0.85032927E 00	0.14728134E 01
1.20	0.85842206E 00	0.14868306E 01
1.25	0.86617402E 00	0.15002573E 01
1.30	0.87360308E 00	0.15131249E 01
1.35	0.88072622E 00	0.15254624E 01
1.40	0.88755932E 00	0.15372977E 01
1.45	0.89411753E 00	0.15486568E 01
1.50	0.90041494E 00	0.15595642E 01
1.55	0.90646499E 00	0.15700433E 01
1.60	0.91228019E 00	0.15801154E 01
1.65	0.91787246E 00	0.15898015E 01
1.70	0.92325296E 00	0.15991207E 01
1.75	0.92843213E 00	0.16080914E 01
1.80	0.93342005E 00	0.16167306E 01
1.85	0.93822598E 00	0.16250547E 01
1.90	0.94285875E 00	0.16330789E 01
1.95	0.94732666E 00	0.16408174E 01
2.00	0.95163757E 00	0.16482842E 01

Table D-38 (Cont'd)

THE ROOTS t_1[Qexp(i120)]

Q	Real t_1	Imag t_1
2.05	0.95579901E 00	0.16554920E 01
2.10	0.95981773E 00	0.16624525E 01
2.15	0.96370045E 00	0.16691776E 01
2.20	0.96745347E 00	0.16756780E 01
2.25	0.97108258E 00	0.16819637E 01
2.30	0.97459336E 00	0.16880445E 01
2.35	0.97799111E 00	0.16939297E 01
2.40	0.98128080E 00	0.16996276E 01
2.45	0.98446722E 00	0.17051465E 01
2.50	0.98755471E 00	0.17104941E 01
2.55	0.99054762E 00	0.17156778E 01
2.60	0.99344977E 00	0.17207047E 01
2.65	0.99626514E 00	0.17255810E 01
2.70	0.99899732E 00	0.17303131E 01
2.75	0.10016494E 01	0.17349068E 01
2.80	0.10042252E 01	0.17393681E 01
2.85	0.10067274E 01	0.17437021E 01
2.90	0.10091592E 01	0.17479140E 01
2.95	0.10115231E 01	0.17520082E 01
3.00	0.10138218E 01	0.17559898E 01
3.05	0.10160580E 01	0.17598630E 01
3.10	0.10182339E 01	0.17636319E 01
3.15	0.10203519E 01	0.17673001E 01
3.20	0.10224140E 01	0.17708720E 01
3.25	0.10244226E 01	0.17743508E 01
3.30	0.10263794E 01	0.17777400E 01
3.35	0.10282863E 01	0.17810431E 01
3.40	0.10301454E 01	0.17842627E 01
3.45	0.10319579E 01	0.17874021E 01
3.50	0.10337257E 01	0.17904643E 01
3.55	0.10354506E 01	0.17934518E 01
3.60	0.10371338E 01	0.17963672E 01
3.65	0.10387769E 01	0.17992129E 01
3.70	0.10403810E 01	0.18019915E 01
3.75	0.10419478E 01	0.18047052E 01
3.80	0.10434782E 01	0.18073560E 01
3.85	0.10449735E 01	0.18099459E 01
3.90	0.10464349E 01	0.18124771E 01
3.95	0.10478636E 01	0.18149519E 01
4.00	0.10492603E 01	0.18173711E 01

Appendix E

THE HEIGHT GAIN FUNCTIONS FOR $q = 0$ AND $q = \infty$

In this appendix we will discuss a set of FORTRAN programs which can be employed to obtain values for the height gain functions for the two limiting cases of $q = 0$ and $q = \infty$. The case $q = 0$ is important because it occurs in the theory of the propagation of vertically polarized waves around a convex surface which is perfectly conducting. The case $q = \infty$ is encountered in the theory of the propagation of horizontally polarized waves around a perfectly conducting surface. However, these limiting cases are also important in certain other problems in which the convex surface is the interface between a dielectric with the propagation constant

$$k_1^2 = \omega^2 \varepsilon_1 \mu_1 + i\omega\mu_1\sigma_1 \quad , \quad r < a \qquad \text{(E-1)}$$

and "free space" with the propagation constant

$$k^2 = \omega^2 \varepsilon_o \mu_o \quad , \quad r > a \qquad \text{(E-2)}$$

Let q_v and q_h denote the impedance parameters for vertical and horizontal polarization, respectively. For $ka \to \infty$, the impedance parameters are given by

$$q_v = i(ka/2)^{\frac{1}{3}}(k/k_1)\sqrt{1 - (k/k_1)^2} \qquad \text{(E-3)}$$

$$q_h = i(ka/2)^{\frac{1}{3}}(k_1/k)\sqrt{1 - (k/k_1)^2} = (k_1/k)^2 q_v \qquad \text{(E-4)}$$

These are actually very complex relationships since the parameters ε_1, μ_1 and σ_1 are functions of frequency ω. However, if $(k_1/k) \to \infty$, we observe that $q_v \to 0$ and $q_h \to \infty$. Therefore, if the frequency ω is held fixed and the conductivity $\sigma_1 \to \infty$, we have $q_v \to 0$ and $q_h \to \infty$. However, if the conductivity σ_1 is held "fixed" and we

permit the frequency ω to vary, it is possible for ω to become so large that $q_v \to \infty$ even though σ_* may be large numerically. Therefore, the case $q = \infty$ plays a very important role in problems involving the convex interface between a dielectric body and free space. This case is also an interesting one because of the fact that the expansion given in Eq. (2-22) for the roots $t_s(q)$, namely

$$t_s(q) = t_s^\infty + \frac{1}{q} + \frac{t_s^\infty}{3q^3} + \cdots \tag{E-5}$$

can be, for very large values of q, adequately represented by*

$$t_s(q) \approx t_s^\infty + \frac{1}{q} \tag{E-6}$$

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*We must be careful to observe that Eq. (E-6) holds for only a finite number of the roots $t_s(q)$ because eventually the modulus of the root t_s^∞ will become so large that the neglected terms will have to be considered. If one examines the highest powers of t_s^∞ in the coefficients $B_n(t_s^\infty)$ in Eq. (2-22) one will recognize that these are the coefficients in the well known expansion

$$\frac{\tanh^{-1}x}{x} = 1 + \frac{x^2}{3} + \frac{x^4}{5} + \frac{x^6}{7} + \cdots, \qquad x < 1$$

and therefore we will have to replace Eq. (E-6) by the result

$$t_s(q) \approx t_s^\infty + \frac{1}{\sqrt{t_s^\infty}} \tanh^{-1}(\sqrt{t_s^\infty}/q) \quad , |\sqrt{t_s^\infty}/q| < 1$$

in which the modulus of q is very large, and $30^\circ < \arg(q) < 210^\circ$. One will always find that as s is increased that one will reach the situation in which Eq. (2-22) is no longer a convergent series. However, if ξ in the diffraction functions $V_0(\xi,q)$, $V_1(\xi,q)$ and $V_{11}(\xi,q)$ is positive, we can generally observe that the number of terms in the residue series which must be used is finite and hence we can determine how to treat the computation of the root $t_s(q)$ for large values of q.

The approximation in Eq. (E-6) reveals that we can sometimes use the approximation

$$\exp(i\xi t_s) \approx \exp(i\xi t_s^\infty)\exp(i\xi/q) \tag{E-7}$$

which permits us to extend the usefulness of the functions $V_o(\xi,\infty)$ $V_1(\xi,\infty)$ and $V_{11}(\xi,\infty)$ by using the approximations

$$V_o(\xi,q) \approx \exp(i\xi/q)V_o(\xi,\infty) \tag{E-8a}$$

$$V_1(\xi,q) \approx \exp(i\xi/q)V_1(\xi,\infty) \tag{E-8b}$$

$$V_{11}(\xi,q) \approx \exp(i\xi/q)V_{11}(\xi,\infty) \tag{E-8c}$$

In each of these expressions we must have $\xi > 0$ and q must have its argument in the range $30^\circ < \arg(q) < 210^\circ$ and the modulus of q must be "very large". More precise descriptions of the applicability of the approximations in Eq. (E-8) have not yet been established. This is really a very complex problem since Eq. (E-8) can be used only if $\xi > 0$, but it is clear from considering the form

$$\exp[i\xi(t_s - t_s^\infty - \frac{1}{q})] = \exp[i\xi(\frac{t_s^\infty}{3q^3} + \frac{1}{4q^4} + \frac{t_s^{\infty 2}}{5q^5} + \cdots)] \tag{E-9}$$

that one must be certain that the expression in parenthesis must have a positive real part if Eq. (E-6) is to be used when $\xi \to \infty$.

The situation is somewhat different when we consider the case in which $q \to 0$. From Eq. (2-21) we find that in this case we have the approximation

$$t_s(q) \approx t_s^o + \frac{1}{t_s^o}q \tag{E-10}$$

and therefore the "correction term" is dependent upon s and hence we do not find that the functions $V_o(\xi,0)$, $V_1(\xi,0)$ and $V_{11}(\xi,0)$ suffice to describe the behavior when q is very small. We find that we have to define three additional functions $V_o^{(-1)}(\xi,0)$,

$$V_o^{(-1)}(\xi,q) = -i\sqrt{i\pi\xi} \sum_{s=1}^{\infty} \frac{\exp(i\xi t_s)}{t_s(t_s - q^2)} \qquad \text{(E-11a)}$$

$$V_1^{(-1)}(\xi,q) = -i2\sqrt{i\pi} \sum_{s=1}^{\infty} \frac{\exp(i\xi t_s)}{t_s(t_s - q^2)w_1(t_s)} \qquad \text{(E-11b)}$$

$$V_{11}^{(-1)}(\xi,q) = i2\sqrt{-i\pi} \sum_{s=1}^{\infty} \frac{\exp(i\xi t_s)}{t_s(t_s - q^2)[w_1(t_s)]^2} \qquad \text{(E-11c)}$$

We can employ these functions when q is in the vicinity of 0 so as to obtain the approximations

$$V_o(\xi,q) \approx V_o(\xi,0) - \xi q V_o^{(-1)}(\xi,0) \qquad \text{(E-12a)}$$

$$V_1(\xi,q) \approx V_1(\xi,0) - \xi q V_1^{(-1)}(\xi,0) \qquad \text{(E-12b)}$$

$$V_{11}(\xi,q) \approx V_{11}(\xi,0) - \xi q V_{11}^{(-1)}(\xi,0) \qquad \text{(E-12c)}$$

As in the case of the approximations given in Eq. (E-6) and Eq. (E-8), the exact conditions under which the approximations* in Eq. (E-10) and Eq. (E-12) can be used cannot be stated explicitly and we leave undefined what we mean by "small values of q." Since

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*For $s \to \infty$, the approximation in Eq. (E-10) will apparently be valid provided the modulus of q is such that the modulus of the combination $(q/\sqrt{t_s^o})$ is less than unity. This seems to be the case since the dominant terms as $s \to \infty$ are

$$t_s(q) \approx t_s^o + \frac{1}{t_s^o} q \left[1 + \frac{1}{3} z + \frac{1}{5} z^2 + \frac{1}{7} z^3 + \cdots\right], \quad z = \frac{q^2}{t_s^o}$$

and therefore we see that the series converges like $\tanh^{-1} z$ and that $t_s(q) \approx t_s^o + (1/\sqrt{t_s^o})\tanh^{-1}(q/\sqrt{t_s^o})$ and we must require that $(q/\sqrt{t_s^o})$ be less than unity in absolute value.

the form of the coefficients $B_n(t_s^o)$ in Eq. (2-21) reveals that t_s^o occurs only in inverse powers, we can be assured that as s increases that the approximations given in Eq. (E-10) and Eq. (E-12) become increasingly accurate representations provided the modulus of q is "small" and $30^o < \arg(q) < 210^o$.

The above discussion has emphasized the importance of the diffraction functions $V_o(\xi,0)$, $V_1(\xi,0)$ and $V_{11}(\xi,0)$ for $q = 0$ and $V_o(\xi,\infty)$, $V_1(\xi,\infty)$ and $V_{11}(\xi,\infty)$ for $q = \infty$. These functions are dependent only upon the distance parameter ξ and the impedance parameter q. We want now to bring in the dependance of the diffraction phenomenon upon height. In Section 1 we have defined the functions

$$V(x,y,y_o,q) = 2\sqrt{i\pi x}\sum_{s=1}^{\infty}\frac{\exp(ixt_s)w_1(t_s - y)w_1(t_s - y_o)}{t_s[w_1(t_s)]^2 - [w_1'(t_s)]^2} \qquad \text{(E-13)}$$

$$E(x,y,q) = i2\sqrt{\pi}\sum_{s=1}^{\infty}\frac{\exp(ixt_s)w_1(t_s - y)}{t_s[w_1(t_s)]^2 - [w_1'(t_s)]^2} \qquad \text{(E-14)}$$

We can make use of the equation defining t_s to write

$$w_1'(t_s) = qw_1(t_s) \qquad \text{(E-15)}$$

to express the functions $V(x,y,y_o,q)$ and $E(x,y,q)$ in the forms

$$V(x,y,y_o,q) = 2\sqrt{i\pi x}\sum_{s=1}^{\infty}\frac{\exp(ixt_s)}{t_s - q^2}\, f_s(y,q)f_s(y_o,q) \qquad \text{(E-16)}$$

$$E(x,y,q) = i2\sqrt{\pi}\sum_{s=1}^{\infty}\frac{\exp(ixt_s)}{(t_s - q^2)w_1(t_s)}\, f_s(y,q) \qquad \text{(E-17)}$$

where $f_s(y,q)$ is the height gain function

$$f_s(y,q) = \frac{w_1(t_s - y)}{w_1(t_s)} = \frac{Ai[-a_s + y\exp(-i\frac{2}{3}\pi)]}{Ai(-a_s)} \qquad \text{(E-18)}$$

and

$$V(x,y,y_o,q) = -2\sqrt{i\pi x}\; q^{-2} \sum_{s=1}^{\infty} \frac{\exp(ixt_s)}{1 - t_s q^{-2}}\, g_s(y,q) g_s(y_o,q) \qquad \text{((E-19)}$$

$$E(x,y,q) = -i2\sqrt{\pi}\; q^{-2} \sum_{s=1}^{\infty} \frac{\exp(ixt_s)}{(1 - t_s q^{-2}) w_1'(t_s)}\, g_s(y,q) \qquad \text{(E-20)}$$

where $g_s(y,q)$ is the height gain function

$$g_s(y,q) = \frac{w_1(t_s - y)}{w_1'(t_s)} = \frac{Ai[-a_s + y\exp(-i\frac{2}{3}\pi)]}{\exp(i\frac{1}{3}\pi)Ai(-\alpha_s)} \qquad \text{(E-21)}$$

The relationship between the t_s and the a_s is

$$t_s = a_s \exp(i\tfrac{1}{3}\pi)$$

and the form taken by Eq. (E-15) when expressed in terms of the functions $Ai(z)$ and $Ai'(z)$ is

$$Ai'(-a_s) + q\exp(i\tfrac{1}{3}\pi)Ai(-a_s) = 0$$

Since the Airy functions are solution of the differential equation $d^2w(z)/dz^2 = zw(z)$, we observe that the height gain functions $f_s(y,q)$ and $g_s(y,q)$ are solutions of the differential equation

$$\frac{d^2h(y)}{dy^2} = (t_s - y)h(y) \qquad \text{(E-22)}$$

which satisfy the boundary conditions

$$f_s(0,q) = 1 \quad , \quad \frac{df_s(0,q)}{dy} = -q \qquad \text{(E-23)}$$

$$g_s(0,q) = \frac{1}{q} \quad , \quad \frac{dg_s(0,q)}{dy} = -1 \tag{E-24}$$

The functions $V(x,y,y_o,q)$ and $E(x,y,q)$ can be referred to as the diffraction functions for vertical polarization because they often occur in situations in which the electric vector the wave is perpendicular to the surface over which the wave is being propagated. In the description of the propagation of horizontally polarized waves in which the electric vector is parallel to the surface, we usually encounter the diffraction functions

$$W(x,y,y_o,q) = 4\sqrt{-i\pi}\; x^{3/2} \sum_{s=1}^{\infty} \frac{\exp(ixt_s)}{1 - t_s q^{-2}}\, g_s'(y,q) g_s'(y_o,q) \tag{E-25}$$

$$F(x,y,q) = -i2\sqrt{\pi} \sum_{s=1}^{\infty} \frac{\exp(ixt_s)}{(1 - t_s q^{-2}) w_1'(t_s)}\, g_s'(y,q) \tag{E-26}$$

where

$$g_s'(y,q) = \frac{\partial g_s(y,q)}{\partial y}$$

If we compare Eqs. (E-25) and (E-26) with Eqs. (E-19) and (E-20), respectively, we see that

$$W(x,y,y_o,q) = i2xq^2 \frac{\partial^2 V(x,y,y_o,q)}{\partial y\, \partial y_o} \tag{E-27}$$

$$F(x,y,q) = q^2 \frac{\partial E(x,y,q)}{\partial y} \tag{E-28}$$

The functions $V_o(\xi,q)$ and $V_1(\xi,q)$ have been defined so that

$$V(x,0,0,q) = 2V_o(x,q) \tag{E-29}$$

$$E(x,0,q) = V_1(x,q) \tag{E-30}$$

By analogy with these results, we define the functions $U_o(\xi,q)$ and $U_1(\xi,q)$ by means of the relations

$$W(x,0,0,q) = 2U_o(x,q) \tag{E-31}$$

$$F(x,0,q) = U_1(x,q) \tag{E-32}$$

where

$$U_o(\xi,q) = 2\sqrt{-i\pi}\, x^{3/2} \sum_{s=1}^{\infty} \frac{\exp(i\xi t_s)}{1 - t_s q^{-2}} \tag{E-34}$$

$$U_1(\xi,q) = i2\sqrt{\pi} \sum_{s=1}^{\infty} \frac{\exp(i\xi t_s)}{(1 - t_s q^{-2}) w_1'(t_s)} \tag{E-35}$$

We also observe that

$$[-2i\xi q^2 V_o(\xi,q)] \xrightarrow[q \to \infty]{} U_o(\xi,q) \tag{E-36}$$

$$[-qV_1(\xi,q)] \xrightarrow[q \to \infty]{} U_1(\xi,q) \tag{E-37}$$

Let us now return to a consideration of the height gain functions and designate by $f_s(y)$ and $g_s(y)$ the special cases which arise when $q = 0$ and $q = \infty$, respectively. We then have

$$f_s(y) = \frac{w_1(t_s^o - y)}{w_1(t_s^o)} = \frac{Ai[-\beta_s + y\exp(-i\frac{2}{3}\pi)]}{Ai(-\beta_s)} \tag{E-38}$$

where

$$w_1'(t_s^o) = Ai(-\beta_s) = 0$$

and

$$g_s(y) = -\frac{w_1(t_s^\infty - y)}{w_1'(t_s^\infty)} = \frac{Ai[-\alpha_s + y\exp(-i\frac{2}{3}\pi)]}{\exp(-i\frac{2}{3}\pi)Ai'(-\alpha_s)} \tag{E-39}$$

where

$$w_1(t_s^\infty) = Ai(-\alpha_s) = 0$$

We observe that these functions satisfy the differential equations

$$\frac{d^2 f_s(y)}{dy^2} = (t_s^o - y) f_s(y) \tag{E-40}$$

$$\frac{d^2 g_s(y)}{dy^2} = (t_s^\infty - y) g_s(y) \tag{E-41}$$

with the boundary conditions

$$f_s(0) = 1 \quad , \quad \frac{df_s(0)}{dy} = 0 \tag{E-42}$$

$$g_s(0) = 0 \quad , \quad \frac{dg_s(0)}{dy} = 1 \tag{E-43}$$

We have chosen to define $f_s(y,0) = f_s(y)$ and $g_s(y,\infty) = - g_s(y)$ in order that the functions $f_s(y)$ and $g_s(y)$ agree with the definitions in the tables published by Azriliant and Belkina (Ref. 1).

For moderately small values of we can evaluate the height gain functions by means of a Taylor series. Let us develop the theory of the Taylor series by considering solutions of the differential equation

$$\frac{d^2 Y(z)}{dz^2} = zY(z) \tag{E-44}$$

which are expressed in the form

$$Y(t + x) = Y(t)u(x,t) + Y'(t)v(x,t) \tag{E-45a}$$

$$Y'(t + x) = Y(t)u'(x,t) + Y'(t)v'(x,t) \tag{E-45b}$$

The functions $u(x,t)$ and $v(x,t)$ are solutions of the differential equation

$$\frac{d^2y(x)}{dx^2} = (x + t)y(x) \tag{E-46}$$

We will let a prime denote differentiation with respect to x, i.e.,

$$u'(x,t) = \frac{\partial u(x,t)}{\partial x} \quad , \quad v'(x,t) = \frac{\partial v(x,t)}{\partial x}$$

The boundary conditions on $u(x,t)$ and $v(x,t)$ are

$$u(0,t) = 1 \quad , \quad u'(0,t) = 0 \quad , \quad v(0,t) = 0 \quad , \quad v'(0,t) = 1$$

We can express the solution for these functions in the form of Taylor series which involve a set of coefficients $a_n(t)$ and $b_n(t)$ in the following manner:

$$u(x,t) = 1 + \sum_{n=2}^{\infty} a_n(t)\frac{x^n}{n!} \tag{E-47a}$$

$$v(x,t) = x + \sum_{n=2}^{\infty} b_n(t)\frac{x^n}{n!} \tag{E-47b}$$

$$u'(x,t) = \sum_{n=2}^{\infty} na_n(t)\frac{x^{n-1}}{n!} \tag{E-47c}$$

$$v'(x,t) = 1 + \sum_{n=2}^{\infty} nb_n(t)\frac{x^{n-1}}{n!} \tag{E-47d}$$

When the expansions in Eq. (E-47) are inserted in Eq. (E-46), the $a_n(t)$ and $b_n(t)$ are found to satisfy the recursion relations

$$\begin{aligned}
c_2(t) \quad\quad\quad\quad\; - tc_o(t) &= 0 \\
c_3(t) - 1c_o(t) - tc_1(t) &= 0 \\
c_4(t) - 2c_1(t) - tc_2(t) &= 0 \\
c_5(t) - 3c_2(t) - tc_3(t) &= 0
\end{aligned}$$

and, in general,

$$c_n(t) = (n - 2)c_{n-3}(t) + tc_{n-2}(t) \quad , \; n > 3 \qquad \text{(E-48)}$$

where $c_n(t)$ denotes either $a_n(t)$ or $b_n(t)$. The first few terms are

$$\begin{aligned}
a_0(t) &= 1 \quad , & b_0(t) &= 0 \\
a_1(t) &= 0 \quad , & b_1(t) &= 1 \\
a_2(t) &= t \quad , & b_2(t) &= 0 \\
a_3(t) &= 1 \quad , & b_3(t) &= t \\
a_4(t) &= t^2 \quad , & b_4(t) &= 2 \\
a_5(t) &= 4t \quad , & b_5(t) &= t^2
\end{aligned}$$

In Program E-1 we have presented an algorithm for determining the numerical values of the $a_n(x,t)$ and $b_n(x,t)$. The subroutine is identified as SUBRØUTINE STOKES(N,A,B,C,D,U,V). The integer N is the largest value of n for which a coefficient is desired. We have used the FORTRAN variables in the following manner:

$$\begin{aligned}
a_I(x,t) &= A(I) + iC(I) \\
b_I(x,t) &= B(I) + iD(I) \\
t &= U + iV
\end{aligned}$$

In Program E-2 we employ the coefficients obtained from this subroutine to compute Y(t+x). We have restricted this program to real values of t and x, which we denote in FORTRAN by T and X, respectively. The reader will observe that we let Y(t) = WT, Y'(t) = WTP, and Y(t+x) = WTPX. The integer N is the number of terms which we ask SUBRØUTINE TAYSUM(T,X,WT,WPT,WTPX,N) to employ. We wrote a main program in which N = 25 and

```
X      = 1.0D0
T      = 0.0D0
WT     = 0.35502805388781 7D0
WPT    =-0.25881940379280 7D0
```


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Program E-1

```
      SUBRØUTINE STOKES(N,A,B,C,D,U,V)
C------THIS SUBRØUTINE EVALUATES THE CØEFFICIENTS IN THE TAYLOR
C------SERIES OF THE AIRY FUNCTION
      DØUBLE PRECISIØN A(N),B(N),C(N),D(N),U,V,ZERØ,ØNE,TWØ,DBIM2
      DATA ZERØ,ØNE,TWØ/ 0.0D0,1.0D0,2.0D0/
      DBIM2(K) = DBLE(FLØAT(K-2))
      IF(N.LT.5) GØ TØ 20
      A(1) = ZERØ
      C(1) = ZERØ
      B(1) = ØNE
      D(1) = ZERØ
      A(2) = U
      C(2) = ZERØ
      B(2) = ZERØ
      D(2) = ZERØ
      A(3) = ØNE
      C(3) = ZERØ
      B(3) = U
      D(3) = ZERØ
      A(4) = U*U
      C(4) = ZERØ
      B(4) = TWØ
      D(4) = ZERØ
      IF(DABS(V).GT.1.0D-37) GØ TØ  10
C------TRANSFER TØ 10 IF T = CMPLX(U,V) IS NØT REAL
      DØ 2 I = 5,N
      A(I) = DBIM2(I)*A(I-3) + U*A(I-2)
      C(I) = ZERØ
      B(I) = DBIM2(I)*B(I-3) + U*B(I-2)
      D(I) = ZERØ
    2 CØNTINUE
      RETURN
   10 C(2) = V
      D(3) = V
      A(4) = A(4) - V*V
      C(4) = TWØ*U*V
      DØ 1 I = 5,N
      A(I) = DBIM2(I)*A(I-3) + U*A(I-2) - V*C(I-2)
      C(I) = DBIM2(I)*C(I-3) + U*C(I-2) + V*A(I-2)
      B(I) = DBIM2(I)*B(I-3) + U*B(I-2) - V*D(I-2)
      D(I) = DBIM2(I)*D(I-3) + U*D(I-2) + V*B(I-2)
    1 CØNTINUE
      RETURN
   20 WRITE(6,110) N
  110 FØRMAT(43H1SUBRØUTINE STØKES IS BEING ASKED TO USE N =,I5,
     A       34HPRØGRAM EXITS UNDER THIS CØNDITIØN)
      CALL EXIT
      END
```

Program E-2

```
      SUBRØUTINE TAYSUM(T,X,WT,WPT,WTPX,N)
C-----THIS SUBRØUTINE COMPUTES THE AIRY FUNCTION W(T+X) = WT*
C-----U(X,T) + WTP*V(X,T)
      DØUBLE PRECISIØN T,X,WT,WPT,WTPX,WPTPX,SUM,XTNØNF(100)
      DØUBLE PRECISIØN A(100),B(100),C(100),D(100), DFLØAT
      DFLØAT(K) = DBLE(FLØAT(K))
      IF(N.GT.100) GØ TØ 20
      CALL STØKES(N,A,B,C,D,T,0.0D0)
      XTNØNF(1) = X
      DØ 1 I = 2,N
      XTNØNF(I) = X*XTNØNF(I-1)/DFLØAT(I)
    1 CØNTINUE
      SUM = 1.0D0
      DØ 2 I = 1,N
      SUM     = SUM + XTNØNF(I)*A(I)
    2 CØNTINUE
      WTPX    = WT*SUM
      SUM     = X
      DØ 3 I = 1,N
      SUM     = SUM + XTNØNF(I)*B(I)
    3 CØNTINUE
      WTPX    = WTPX + WPT*SUM
      RETURN
   20 WRITE(6,120) N
  120 FØRMAT(43H1SUBRØUTINE TAYSUM IS BEING ASKED TO USE N =,I5,
     A        34HPRØGRAM EXITS UNDER THIS CØNDITIØN)
      CALL EXIT
      END
```

Since this means that we have taken WT = Ai(0) and WTP = Ai′(0), the result obtained, namely

$$WTPX = .1352924163128808+00$$

represents the value of the Airy function

$$Ai(1) = 0.1352924163128814155$$

In Table E-1 we give the values of the coefficients which were generated during the course of this calculation.

The method employed in Program E-2 to compute Y(t+x) does not represent an application of good programming technique if the desired end product is a set of numerical values for the functions

Table E-1

TABLE OF COEFFICIENTS $a_n(0)$ AND $b_n(0)$ FOR $u(x,0)$ AND $v(x,0)$

n	$a_n(0)$	$b_n(0)$
3	.1000000000000000+01	.0000000000000000
4	.0000000000000000	.1999999999999999+01
5	.0000000000000000	.0000000000000000
6	.3999999999999999+01	.0000000000000000
7	.0000000000000000	.1000000000000000+02
8	.0000000000000000	.0000000000000000
9	.2799999999999999+02	.0000000000000000
10	.0000000000000000	.7999999999999999+02
11	.0000000000000000	.0000000000000000
12	.2799999999999999+03	.0000000000000000
13	.0000000000000000	.8799999999999999+03
14	.0000000000000000	.0000000000000000
15	.3639999999999999+04	.0000000000000000
16	.0000000000000000	.1231999999999999+05
17	.0000000000000000	.0000000000000000
18	.5823999999999999+05	.0000000000000000
19	.0000000000000000	.2094399999999999+06
20	.0000000000000000	.0000000000000000
21	.1106559999999999+07	.0000000000000000
22	.0000000000000000	.4188799999999999+07
23	.0000000000000000	.0000000000000000
24	.2434431999999999+08	.0000000000000000
25	.0000000000000000	.9634239999999998+08

$u(x,t)$ and $v(x,t)$. It is more convenient to compute the terms in the series directly rather than to evaluate the $a_n(t)$ and $b_n(t)$ and then sum the polynomial which results when the series is truncated. This is also desirable because the $a_n(t)$ and $b_n(t)$ may "overflow" on the computer (i.e., become so large that they exceed the maximum magnitude of 10^{38} which most FORTRAN compilers permit) and yet the product of the coefficients $a_n(t)$ and $x^n/n!$ may still be significant.*

Let us introduce the notation

$$A_n(x,t) = a_n(t)x^n/n! \qquad \text{(E-49a)}$$

$$B_n(x,t) = b_n(t)x^n/n! \qquad \text{(E-49b)}$$

The $A_n(x,t)$ and $B_n(x,t)$ can be shown to satisfy the recursion formula

$$n(n-1)C_n(x,t) = x^3C_{n-3}(x,t) + tx^2C_{n-2}(x,t) \qquad \text{(E-50)}$$

with the initial values

$$A_o(x,t) = 1 \quad , \quad B_o(x,t) = 0$$

$$A_1(x,t) = 0 \quad , \quad B_1(x,t) = x$$

$$A_2(x,t) = \frac{tx^2}{2} \quad , \quad B_2(x,t) = 0$$

$$A_3(x,t) = \frac{x^3}{6} \quad , \quad B_3(x,t) = \frac{tx^3}{6}$$

$$A_4(x,t) = \frac{t^2x^4}{24} \quad , \quad B_4(x,t) = \frac{x^4}{12}$$

*The factor $x^n/n!$ may also "underflow." For example, since $80! = 0.71569457E+119$, we would have $10^{80}/80! \approx 1.4 \times 10^{-39}$ and this is outside the range 10^{-38} to 10^{38} permitted by most compilers.

The coefficients $A_n(x,t)$ and $B_n(x,t)$ can be evaluated with the SUBRØUTINE TERMS(N,AR,AI,BR,BI,XR,XI,TR,TI) which is given in Program E-3. The reader will observe that we have used the following FORTRAN representations for the values of x, t, A_n and B_n:

$$x = XR + iXI \quad , \quad A_I(x,t) = AR(I) + iAI(I)$$
$$t = TR + iTI \quad , \quad B_I(x,t) = BR(I) + IBI(I)$$

Program E-3

```
      SUBROUTINE TERMS(N,AR,AI,BR,BI,XR,XI,TR,TI)
C-----COMPUTE TERMS IN TAYLOR SERIES EXPANSION OF AIRY FUNCTION
      DOUBLE PRECISION AR(N),AI(N),BR(N),BI(N),XR,XI,TR,TI,DFLOAT,
     A                 ARI,AII,BRI,BII,XSQR,XSQI
      DFLOAT(K)= DBLE(FLOAT(K))
      XSQR = XR*XR - XI*XI
      XSQI = 2.0D0*XR*XI
      AR(1)= 0.0D0
      AI(1)= 0.0D0
      AR(2)= (TR*XSQR - TI*XSQI)/(2.0D0)
      AI(2)= (TR*XSQI + TI*XSQR)/(2.0D0)
      AR(3)= (XR*XSQR - XI*XSQI)/(6.0D0)
      AI(3)= (XR*XSQI + XI*XSQR)/(6.0D0)
      AR(4)= (AR(2)*AR(2) - AI(2)*AI(2))/(6.0D0)
      AI(4)= (AR(2)*AI(2))/(3.0D0)
      BR(1)= XR
      BI(1)= XI
      BR(2)= 0.0D0
      BI(2)= 0.0D0
      BR(3)= TR*AR(3) - TI*AI(3)
      BI(3)= TR*AI(3) + TI*AR(3)
      BR(4)  = (XSQR*XSQR - XSQI*XSQI)/12.0D0
      BI(4)  = (XSQR*XSQI)/6.0D0
      DO 1 I=5,N
C-----THE NEXT STEP GENERATES AN EXPRESSION FRØM WHICH AR(I),ETC.
C-----CAN THEN BE CØNSTRUCTED
      ARI  = XR*AR(I-3) - XI*AI(I-3) + TR*AR(I-2) - TI*AI(I-2)
      AII  = XR*AI(I-3) + XI*AR(I-3) + TR*AI(I-2) + TI*AR(I-2)
      BRI  = XR*BR(I-3) - XI*BI(I-3) + TR*BR(I-2) - TI*BI(I-2)
      BII  = XR*BI(I-3) + XI*BR(I-3) + TR*BI(I-2) + TI*BR(I-2)
      AR(I)= (XSQR*ARI - XSQI*AII)/DFLOAT(I*(I-1))
      AI(I)= (XSQR*AII + XSQI*ARI)/DFLOAT(I*(I-1))
      BR(I)= (XSQR*BRI - XSQI*BII)/DFLOAT(I*(I-1))
      BI(I)= (XSQR*BII + XSQI*BRI)/DFLOAT(I*(I-1))
    1 CONTINUE
      RETURN
      END
```

In Program E-4 we present a main program which we used to sum the series of terms obtained from Program E-3. The series is of the form

$$u(x,t) = 1 + A_2(x,t) + A_3(x,t) + A_4(x,t) + \cdots \tag{E-51a}$$

$$v(x,t) = x + B_2(x,t) + B_3(x,t) + B_4(x,t) + \cdots \tag{E-51b}$$

Program E-4

```
C-----THIS IS A TEST OF SUBRØUTINE TERMS(N,AR,AI,BR,BI,XR,XI,TR,TI)
      DØUBLE PRECISIØN AR(100),AI(100),BR(100),BI(100),XR,XI,TR,TI,
     A         DFLØAT.WTR.WTI.WPTR.WPTI.UR.UI.VR,VI,WTPXR,WTPXI
      WTR     =-0.7258727400
      WTI     = 3.4419277000
      WPTR    =-3.7668668500
      WPTI    =+4.8506065800
      CALL TERMS(40,AR,AI,BR,BI, 1.0D0, 1.0D0, 2.5D0, 2.5D0)
      WRITE(6,110) (I,AR(I),AI(I),BR(I),BI(I),I=1,40)
  110 FORMAT(1H0,I4,4D23.16)
      XR = 1.0D0
      XI = 1.0D0
      UR      = 1.0D0
      UI      = 0.0D0
      DO 10 I= 2,40
      UR      = UR+ AR(I)
      UI      = UI+ AI(I)
   10 CONTINUE
      WRITE(6,500) UR,UI
  500 FORMAT (1H0, 5HUR,UI, 2D25.16)
      VR      = XR
      VI      = XI
      DO 20 I= 2,40
      VR      = VR + BR(I)
      VI      = VI + BI(I)
   20 CONTINUE
      WRITE(6,400) VR,VI
  400 FORMAT (1H0, 5HVR,VI, 2D25.16)
      WTPXR   = WTR*UR-WTI*UI + WPTR*VR-WPTI*VI
      WTPXI   = WTR*UI+WTI*UR + WPTR*VI+WPTI*VR
      WRITE(6,111) WTPXR, WTPXI
  111 FORMAT(1H1,2X,2D25.16)
      CALL EXIT
      END
```

The test that we have conducted in Prog ram E-4 was based upon entries in Harvard Computation Laboratory (Ref. 2) tables of the functions $h_1(z)$ and $h_2(z)$ which satisfy the differential equation

$$\frac{d^2 h_{1,2}(z)}{dz^2} + z h_{1,2}(z) = 0 \qquad \text{(E-52)}$$

If we compare Eq. (E-52) with Eq. (E-44), we see that we can take

$$Y(t+x) = h_1(-t-x)$$
$$Y(t) = h_1(-t)$$
$$Y'(t) = -h_1'(-t)$$

In the test we took

$$t = TR + iTI = 2.5 + i2.5$$
$$x = XR + iXI = 1.0 + i1.0$$

with the initial conditions

$$Y(t) = h_1(-t) = WTR + iWTI = -0.72587274 + i3.44192770$$
$$Y'(t) = -h_1'(-t) = WPTR + iWPTI = 3.76686685 - i4.85060658$$

The computer output

-.2157289088982663+01 -.9312479583220205+01

is in agreement with the entry from the tables, namely

$$Y(t+x) = h_1(-t-x) = WTPXR + iWTPXI = -2.15728909 - i9.31247959$$

The SUBRØUTINE TERMS(N,AR,AI,BR,BI,XR,XI,TR,TI) can be used as a building block for the development of a step-by-step method of integrating the differential equation in Eq. (E-22) to obtain the height gain functions $f_s(y,q)$ and $g_s(y,q)$.

Since we have shown that the limiting cases $q = 0$ and $q = \infty$ play such important roles, we want now to present programs which permit one to evaluate the Taylor series for the height gain functions $f_s(y)$ and $g_s(y)$. These functions are readily expressed in terms of the functions $u(x,t)$ and $v(x,t)$.

Let us express the four height gain functions in the form

$$f_s(y) = 1 + A_1(t_s^o,-y) + A_2(t_s^o,-y) + A_3(t_s^o,-y) + \cdots \tag{E-53}$$

$$f_s'(y) = B_1(t_s^o,-y) + B_2(t_s^o,-y) + B_3(t_s^o,-y) + \cdots \tag{E-54}$$

$$g_s(y) = y + C_1(t_s^\infty,-y) + C_2(t_s^\infty,-y) + C_3(t_s^\infty,-y) + \cdots \tag{E-55}$$

$$g_s'(y) = 1 + D_1(t_s^\infty,-y) + D_2(t_s^\infty,-y) + D_3(t_s^\infty,-y) + \cdots \tag{E-56}$$

where the $A_n(t,-y)$, $B_n(t,-y)$, $C_n(t,-y)$ and $D_n(t,-y)$ satisfy the recursion relation

$$n(n-1)K_n(t,-y) = y^2[-yK_{n-3}(t,-y) + tK_{n-2}(t,-y)] \tag{E-57}$$

with the initial terms

$$A_1 = 0 \quad , \; A_2 = \frac{ty^2}{2} \quad , \; A_3 = -\frac{y^3}{6} \quad , \; A_4 = \frac{t^2y^4}{24}$$

$$B_1 = ty \quad , \; B_2 = -\frac{y^2}{2} \quad , \; B_3 = \frac{t^2y^3}{6} \quad , \; B_4 = -\frac{ty^4}{6}$$

$$C_1 = y \quad , \; C_2 = 0 \quad , \; C_3 = \frac{ty^3}{6} \quad , \; C_4 = -\frac{y^4}{12}$$

$$D_1 = 0 \quad , \; D_2 = \frac{ty^2}{2} \quad , \; D_3 = -\frac{y^3}{3} \quad , \; D_4 = \frac{t^2y^4}{24}$$

Program E-5 contains SUBRØUTINE HGFY(N,Y,FYR,FYI) which will compute

$$f_I(Y) = FYR(I) + iFYI(I) \quad , \; I = 1,2,3,\ldots,N \tag{E-58}$$

The SUBRØUTINE HGFPY(N,Y,FPYR,FPYI) in Program E-6 will generate values for

$$f_I'(Y) = -[FPYR(I) + iFPYI(I)] \; , \; I = 1,2,\ldots,N \tag{E-59}$$

Each of these program require the roots β_s of the Airy function

which are tabulated in Table 2-1. These values are transferred to the subroutine by means of a labeled CØMMØN/DABETA/BETA(50), AIRYF(50). Output from these programs is presented in Tables E-4 and E-5. The results are consistent with the tables prepared by Azriliant and Belkina (Ref. 1). A sample of the tables of Ref. 1 is presented in Tables E-2 and E-3.

Table E-2

SAMPLES OF TABLES OF $f_s(y)$ and $f_s'(y)$ PUBLISHED BY SOVIET AUTHORS

y	Re $f_1(y)$	Im $f_1(y)$	Re $f_2(y)$	Im $f_2(y)$	Re $f_1'(y)$	Im $f_1'(y)$	Re $f_2'(y)$	Im $f_2'(y)$
0,0	1	0	1	0	0	0	0	0
0,1	1,0024	0,0044	1,0079	0.0141	0,0458	0,0894	0,1565	0,2828
0,2	1,0098	0,0177	1,0308	0,0568	0,0811	0,1774	0,2973	0,5740
0,3	1,0182	0,0399	1,0666	0,1294	0,1048	0,2675	0,4157	0,8811
0,4	1,0294	0,0712	1,1129	0,2338	0,1163	0,[illegible]587	0,5041	1,2102
0,5	1,0410	0,1117	1,1661	0,3724	0,1144	0,4555	0,5534	1,5658
0,6	1,0518	0,1613	1,2219	0,5480	0,0982	0,5422	0,5530	1,9506
0,7	1,0572	0,2201	1,2747	0,7636	0,0666	0,6327	0,4911	2,3653
0,8	1,0646	0,2878	1,3177	1,0220	0,0187	0,7203	0,3514	2,8078
0,9	1,0633	0,3640	1,3426	1,3259	−0,0463	0,8031	0,1283	3,2728
1,0	1,0547	0,4482	1,3398	1,6770	−0,1287	0,8787	−0,2024	3,7515
1,1	1,0370	0,5394	1,2981	2,0762	−0.2288	0,9443	−0,6532	4,2308
1,2	1,0084	0,6[illegible]66	1,2017	2,5226	−0,34[illegible]2	0,9969	−1,2394	4,6926
1,3	0,9672	0,7382	1,0453	3,0134	−0,48[illegible]0	1,0331	−1,9746	5,1138
1,4	0,9119	0,8426	0,8044	3,5431	−0,62[illegible]8	1.0496	−2,8703	5,4653
1,5	0,8410	0,9474	0,4656	4,1030	−0,7903	1,0428	−3,9337	5,7124
1,6	0,7535	1,0502	0,0120	4,6807	−0,9615	1,0092	−5,1667	5,8144
1,7	0,6485	1,1482	− 0,5732	5,2595	−1,1385	0,9456	−6,5639	5,[illegible]253
1,8	0,5257	1,2382	− 1,3058	5,8177	−1,3168	0,8494	−8,1105	5,3943
1,9	0,3853	1,3119	− 2,1995	6,3285	−1,4907	0,7182	−9,7603	4,7674
2,0	0,2279	1,3807	− 3,2647	6,7595	−1,6541	0,5508	−11,534	3.789
2,1	0,0651	1,4259	− 4,5073	7,0728	−1,7999	0,3469	−13,317	2,405
2,2	−0,1312	1,4489	− 5,9267	7,2252	−1,9206	0,1075	−15,057	0,564
2,3	−0,3280	1,4463	− 7,5144	7,1190	−2,0086	−0,1651	−16,666	−1,774
2,4	−0,5416	1,4149	− 9,2520	6,8528	−2,0558	−0,4667	−18,036	−4,640
2,5	−0,7375	1,3521	−11,109	6,223	−2,0548	−0,7916	−19,041	−8,043
2,6	−0,9407	1,2560	−13,043	5,227	−1,9986	−1,1323	−19,541	−11,971
2,7	−1,1352	1,1255	−14,996	3,813	−1,8813	−1,4793	−19,383	−16,[illegible]82
2,8	−1,3148	0,9603	−16,892	1,937	−1,6986	−1,8217	−18,405	−21,196
2,9	−1,4727	0,7617	−18,644	− 0,436	−1,4480	−2,1469	−16,446	−26,2[illegible]4
3,0	−1,6021	0,5320	−20,144	− 3,326	−1,1293	−2,4410	−13,353	−31,510
3,1	−1,6963	0,2750	−21,272	− 6,735	−0,7452	−2,6896	−8,990	−36,631
3,2	−1,7491	−0,0039	−21,896	−10,640	−0,3014	−2,8779	−3,256	−41,393
3,3	−1,7549	−0,2981	−21,876	−14,991	0,1931	−2,9415	3,908	−45,485
3,4	−1,7092	−0,5993	−21,067	−19,703	0,7259	−3,0170	12,498	−48,[illegible]51
3,5	−1,6080	−0,8962	−19,331	−24,654	1,2811	−2,9432	22,135	−50,205
3,6	−1,4529	−1,1843	−16,541	−29,[illegible]83	1,8[illegible]99	−2,7614	33,548	−5[illegible],040
3,7	−1,2416	−1,4467	−12,591	−34,568	2,3405	−2,4665	45,561	−47,646
3,8	−0,9782	−1,6738	− 7,411	−39,125	2,8793	−2,0579	58,[illegible]	−42,[illegible]40
3,9	−0,6680	−1,8546	− 0,974	−43,017	3,3112	−1,5399	70,607	−34,689
4,0	−0,3190	−1,9785	6,689	−45,966	3,6512	−0,9224	82,198	−23,542

The subroutines in Programs E-7 and E-8 evaluate the functions

$$g_I(Y) = -\,[GYR(I) + iGYI(I)] \quad , \; I = 1,2,3,\ldots,N \qquad (E\text{-}60)$$

$$g_I'(Y) = GPYR(I) + iGPYI(I) \quad , \; I = 1,2,3,\ldots,N \qquad (E\text{-}61)$$

These programs have been compared with the Soviet tables by means of the sample output given in Tables E-6 and E-7.

Table E-3

SAMPLES OF TABLES OF $g_s(y)$ AND $g_s'(y)$ PUBLISHED BY SOVIET AUTHORS

y	Re $g_1(y)$	Im $g_1(y)$	Re $g_2(y)$	Im $g_2(y)$	Re $g_3(y)$	Re $g_1'(y)$	Im $g_1'(y)$	Re $g_2'(y)$	Im $g_2'(y)$
0,0	0	0	0	0	0	1	0	1	0
0,1	0,10019	0,00034	0,10033	0,00069	0,1004	1,00550	0,01014	1,00985	0,01776
0,2	0,20141	0,00271	0,20257	0,00476	0,2036	1,02251	0,04078	1,03762	0,07171
0,3	0,30452	0,00920	0,30834	0,01620	0,3114	1,04254	0,09247	1,07963	0,16376
0,4	0,41006	0,02193	0,41887	0,03887	0,4259	1,06867	0,16599	1,13193	0,29680
0,5	0,51826	0,04315	0,53486	0,07705	0,5477	1,09552	0,26219	1,18777	0,47443
0,6	0,62906	0,07516	0,65630	0,13538	0,6768	1,11915	0,38169	1,23965	0,70666
0,7	0,74186	0,12033	0,78234	0,21893	0,8114	1,13508	0,52553	1,27788	0,97951
0,8	0,85565	0,18109	0,91103	0,33315	0,9479	1,13822	0,69357	1,29057	1,31448
0,9	0,96889	0,25984	1,03913	0,48378	1,0804	1,12242	0,88523	1,26847	1,70802
1,0	1,07942	0,35888	1,16184	0,67674	1,1999	1,08299	1,09933	1,17974	2,16078
1,1	1,18444	0,48037	1,27254	0,91785	1,2930	1,01181	1,33353	1,02301	2,67080
1,2	1,28051	0,62614	1,36258	1,21262	1,3457	0,90248	1,58420	0,7[illegible]214	3,23257
1,3	1,36344	0,79760	1,42086	1,56576	1,3337	0,74800	1,84025	0,38212	3,83602
1,4	1,42837	0,99555	1,43430	1,98070	1,2310	0,54[illegible]56	2,11290	−0,14343	4,46575
1,5	1,46980	1,22005	1,39651	2,45895	1,0046	0,27688	2,37552	−0,84328	5,09787
1,6	1,48163	1,47017	1,25901	2,99937	0,6160	−0,05140	2,62352	−1,74187	5,70293
1,7	1,45727	1,74365	1,03074	3,59731	0,0242	−0,44724	2,84430	−2,86238	6,24982
1,8	1,38066	2,03764	0,67852	4,24366	−0,8324	−0,91206	3,02334	−4,22329	6,66201
1,9	1,27243	2,34658	0,17766	4,92384	−1,9068	−1,44717	3,14432	−5,83535	6,90664
2,0	1,00823	2,66398	−0,49722	5,61677	−3,3256	−2,04720	3,18946	−7,70298	6,90453
2,1	0,86104	2,98132	−1,3710	6,2939	−5,476	−2,70541	3,13997	−9,8110	6,5759
2,2	0,55559	3,28818	−2,4665	6,9181	−7,896	−3,41024	2,97070	−12,1309	5,8628
2,3	0,17798	3,57224	−3,8026	7,4436	−10,823	−4,14530	2,68092	−14,6115	4,5818
2,4	−0,27377	3,81937	−5,3917	7,8144	−14,277	−4,88910	2,28537	−17,1765	2,7277
2,5	−0,79924	4,01383	−7,2372	7,9658	−18,250	−5,61480	1,62545	−19,7202	0,1783
2,6	−1,39506	4,13861	−9,3306	7,8241	−22,690	−6,29028	0,84052	−22,1052	−3,1481
2,7	−2,05440	4,17591	−11,6473	7,3082	−27,531	−6,87844	−0,12462	−24,1596	−7,3157
2,8	−2,70647	4,10773	−14,1445	6,3317	−32,395	−7,33797	−1,26817	−25,6772	−12,36[illegible]4
2,9	−3,51622	3,91066	−16,7569	4,8069	−37,605	−7,62433	−2,5799	−26,4193	−18,2797
3,0	−4,28403	3,58680	−19,3937	2,6483	−42,436	−7,6913	−4,0395	−26,1188	−25,0206
3,1	−5,04564	3,10481	−21,9363	−0,2209	−46,499	−7,4431	−5,6158	−24,4880	−32,4675
3,2	−5,77296	2,46117	−24,2370	−3,8625	−49,351	−6,9865	−7,2654	−21,2304	−40,4290
3,3	−6,43130	1,65115	−26,1184	−8,3147	−50,381	−6,1338	−8,9325	−16,0558	−48,6361
3,4	−6,96661	0,67624	−27,3754	−13,5637	−48,879	−4,9863	−10,5490	−8,7010	−56,6084
3,5	−7,30967	−0,45433	−27,7787	−19,6324	−44,045	−3,3872	−12,0764	1,0460	−64,1537
3,6	−7,63062	−1,72346	−27,082	−26,378	−36,08	−1,2783	−13,3027	13,3176	−70,4387
3,7	−7,64210	−3,10438	−25,030	−33,660	−20,92	1,1114	−14,2390	28,1343	−74,8[illegible]
3,8	−7,30743	−4,56065	−21,374	−41,267	−0,86	3,8330	−14,7930	45,3716	−76,7361
3,9	−6,86644	−6,04592	−15,685	−46,880	25,86	6,8253	−14,8192	64,7257	−75,2579
4,0	−6,02637	−7,50440	−8,375	−56,179	59,85	9,9473	−14,2432	85,862	−69,546

The roots α_s which are required in Programs E-7 and E-8 have been transmitted to the program by means of a labeled block of COMMON which is obtained by means of the non-executable statement which reads CØMMØN/DALPHA/ALPHA(50),AIRYD(50).

For $q \neq 0$ and $q \neq \infty$, one may find it convenient to use Program E-9 to compute the Airy functions. This program is similar to Program B-1; the difference ebing that this program returns the complex numbers which represent the Airy functions $v(z)$ and $w_1(z)$ which are employed in the papers by Fock and other Soviet writers. The program is called SUBRØUTINE VTWTZ(Z,VT,WT,VTP,WTP) and the relation of the FORTRAN variables to the mathematical quantities are

$$z = Z \; , \; v(z) = VT \; , \; w_1(z) = WT \; , \; v'(z) = VTP \; , \; w_1'(z) = WTP$$

The program should only be used when the modulus of z is less than 6.1.

The author has done extensive work on the case of obtaining the height gain functions when the roots have a large modulus. However, this study is incomplete and no attempt will be made to discuss the methods and the results which have been obtained. This work will be carried to completion and reported in a later publication. The author would like to make one observation regarding this work. In the book by Azriliant and Belkina (Ref. 1) use is made of an approximation which is due to Fock, namely

$$f_s(y,q) \approx \cosh(y\sqrt{t_s}) - \frac{q}{\sqrt{t_s}} \sinh(y\sqrt{t_s}) \; , \quad \left| \frac{y^2}{2\sqrt{t_s}} \right| << 1 \qquad \text{(E-62)}$$

When one attempts to find an asymptotic series which has these terms as the leading terms, the second term (which vanishes when $q = 0$) is apparently cancelled out by the higher order terms. Therefore, until further work has been done and comparisons made with other methods of approximation, the second term in Eq. (E-62) is of questionable value.

Program E-5

```
      SUBROUTINE HGFY(N,Y,FYR,FYI)
      COMMON/DABETA/BETA(50),AIRYF(50)
      DOUBLE PRECISION BETA,AIRYF,TR,TI,A1R,A1I,A2R,A2I,A3R,A3I,AR,AI,
     1        DFLOAT,SR,SI,X,X2,STR,STI,A4R,A4I
      DIMENSION FYR(N),FYI(N)
  199 FORMAT(11H1  FOR Y = ,E16.8,27H  PROGRAM FXIT BECAUSE N = ,I4,
     A        51H  EXCEEDS 50 WHICH IS THE CAPACITY OF HGFY(N,Y,...))
  198 FORMAT(11H1  FOR Y = ,E16.8,62H  PROGRAM EXIT BECAUSE OF LOSS OF A
     ACCURACY DUE TO CANCELLATION//10X,7HKDEX = ,I4,9H  LDEX = ,I4//
     B   10X,54HHGFY(N,Y,...) IS NOT CAPABLE OF HANDLING THIS ARGUMENT)
  197 FORMAT(11H1  FOR Y = ,E16.8,88H  PROGRAM EXIT BECAUSE COMPLETION O
     AF 50 ITERATIONS FAILED TO YIELD CONVERGENCE OF SERIES  //
     B                                10X,7HKDEX = ,I4,9H  LDEX = ,I4//
     C   10X,54HHGFY(N,Y,...) IS NOT CAPABLE OF HANDLING THIS ARGUMENT)
      DFLOAT(J) = DBLE(FLOAT(J))
      IF(N.GT.50) GO TO 99
C-----PROGRAM WILL CAUSE CALL EXIT WHEN N EXCEEDS 50
      X       = DBLE(Y)
      X2      = X*X
      DO 10 K=1,N
      KDEX    = K
      TR      = 0.5D0*BETA(K)
      TI      = 1.7320508075688770D0*TR
      A2R     = TR*(X**2)/2.0D0
      A2I     = 1.7320508075688770D0*A2R
      A3R     =-(X**3)/6.0D0
      A3I     = 0.0D0
      A4R     = (A2R*A2R - A2I*A2I)/6.0D0
      A4I     = (A2R*A2I)/3.0D0
      SR      = 1.0D0 + A2R + A3R + A4R
      SI      =         A2I + A3I + A4I
      STR     = 1.0D0 + DABS(A2R) + DABS(A3R) + DABS(A4R)
      STI     =         DABS(A2I) + DABS(A3I) + DABS(A4I)
C-----STR AND STI ARE TEST QUANTITIES TO TEST FOR EXCESSIVE CANCELLATION
```

```
      DO 20 L=5,50
      LDEX    = L
      AR      = X2*(-X*A2R   + TR*A3R - TI*A3I)/DFLOAT(L*(L-1))
      AI      = X2*(-X*A2I   + TR*A3I + TI*A3R)/DFLOAT(L*(L-1))
      SR      = SR + AR
      SI      = SI + AI
      STR     = STR + DABS(AR)
      STI     = STI + DABS(AI)
      IF((STR/DABS(SR) + STI/DABS(SI)).GT.1.0D12)  GO TO 98
C-----TRANSFER OUT IF CANCELLATION SUGGESTS EXCESSIVE LOSS OF ACCURACY
      IF(DABS(AR + A4R + A3R).LT.1.0D-8) GO TO 89
C-----TRANSFER OUT IF SUM OF LAST THREE TERMS OF REAL PART IS LESS THAN
C-----1.0D-8 WHERE AS INITIAL TERM WAS UNITY.
      A2R     = A3R
      A2I     = A3I
      A3R     = A4R
      A3I     = A4I
      A4R     = AR
      A4I     = AI
   20 CONTINUE
      GO TO 97
C-----WRITE A MESSAGE THAT PROGRAM FAILED TO LEAVE THE LOOP AFTER FIFTY
C-----ITERATIONS AND THEN CALL EXIT
   89 FYR(K) = SNGL(SR)
      FYI(K) = SNGL(SI)
   10 CONTINUE
C-----THIS IS THE NORMAL RETURN WHEN PROGRAM SUCCESSFULLY ESTABLISHES AN
C-----ARRAY FYR(N) AND FYI(N) WHICH CONTAINS THE N HEIGHT GAIN FUNCTIONS
      RETURN
   99 WRITE(6,199) Y,N
      CALL EXIT
   98 WRITE(6,198) Y,KDEX,LDEX
      CALL EXIT
   97 WRITE(6,197) Y,KDEX,LDEX
      CALL EXIT
      END
```

```
      SUBROUTINE HGFPY(N,Y,FPYR,FPYI)
      DIMENSION FPYR(N),FPYI(N)
      COMMON/DABETA/BETA(50),AIRYF(50)
      DOUBLE PRECISION BETA,AIRYF,DFLOAT,TR,TI,X,X2,A1R,A1I,A2R,A2I,A3R,
     A        A3I,SR,SI,STR,STI,TEST,AR,AI
  199 FORMAT(11H1  FOR Y = ,E16.8,27H  PROGRAM EXIT BECAUSE N = ,I4,
     A        52H  EXCEEDS 50 WHICH IS THE CAPACITY OF HGFPY(N,Y,...))
  198 FORMAT(11H1  FOR Y = ,E16.8,62H  PROGRAM EXIT BECAUSE OF LOSS OF A
     ACCURACY DUE TO CANCELLATION//10X,7HKDEX = ,I4,9H  LDEX = ,I4//
     B   10X,55HHGFPY(N,Y,...) IS NOT CAPABLE OF HANDLING THIS ARGUMENT)
  197 FORMAT(11H1  FOR Y = ,E16.8,88H  PROGRAM EXIT BECAUSE COMPLETION O
     AF 50 ITERATIONS FAILED TO YIELD CONVERGENCE OF SERIES//
     B                                   10X,7HKDEX = ,I4,9H  LDEX = ,I4//
     C   10X,55HHGFPY(N,Y,...) IS NOT CAPABLE OF HANDLING THIS ARGUMENT)
      DFLOAT(J) = DBLE(FLOAT(J))
      IF(N.GT.50) GO TO 99
C-----PROGRAM WILL CAUSE EXIT WHEN N EXCEEDS 50
      X       = DBLE(Y)
      X2      = X*X
      DO 10 K=1,N
      KDEX    = K
      TEST    = X*BETA(K)*1.0D-8
      TR      = 0.5D0*BETA(K)
      TI      = 1.732050807568877D0*TR
      A1R     =-TR*X
      A1I     =-TI*X
      A2R     = 0.5D0*(X*X)
      A2I     = 0.0D0
      A3R     = (-X/6.0D0)*(A1R*A1R - A1I*A1I)
      A3I     = (-X/3.0D0)*(A1R*A1I)
      SR      = A1R + A2R + A3R
      SI      = A1I + A2I + A3I
      STR     = DABS(A1R) + DABS(A2R) + DABS(A3R)
      STI     = DABS(A1I) + DABS(A2I) + DABS(A3I)
C-----STR AND STI ARE TEST QUANTITIES TO TEST FOR EXCESSIVE CANCELLATION
```

```
      DO 20 L=4,50
      LDEX    = L
      AR      =-X2*(X*DFLOAT(L-1)*A1R - DFLOAT(L-2)*(TR*A2R - TI*A2I))/
     A          DFLOAT(L*(L-1)*(L-2))
      AI      =-X2*(X*DFLOAT(L-1)*A1I - DFLOAT(L-2)*(TR*A2I + TI*A2R))/
     A          DFLOAT(L*(L-1)*(L-2))
      SR      = SR + AR
      SI      = SI + AI
      STR     = STR + DABS(AR)
      STI     = STI + DABS(AI)
      IF((STR/DABS(SR) + STI/DABS(SI)).GT.1.0D12) GO TO 98
C-----TRANSFER OUT IF CANCELLATION SUGGESTS EXCESSIVE LOSS OF ACCURACY
      IF(DABS(AR + A3R + A2R).LT.TEST) GO TO 89
C-----TRANSFER OUT IF SUM OF LAST THREE TERMS OF REAL PART IS LESS THAN
C-----TEST = 1.0D-3*(MAGNITUDE OF FIRST TERM)
      A1R     = A2R
      A1I     = A2I
      A2R     = A3R
      A2I     = A3I
      A3R     = AR
      A3I     = AI
   20 CONTINUE
      GO TO 97
C-----WRITE A MESSAGE THAT PROGRAM FAILED TO LEAVE THE LOOP AFTER FIFTY
C-----ITERATIONS AND THEN CALL EXIT
   89 FPYR(K)= SNGL(SR)
      FPYI(K)= SNGL(SI)
   10 CONTINUE
      RETURN
C-----THIS IS THE NORMAL RETURN WHEN PROGRAM SUCCESSFULLY ESTABLISHES AN
C-----ARRAY FPYR(N),FPYI(N) WHICH CONTAINS THE N HEIGHT GAIN FUNCTIONS
   99 WRITE(6,199) Y,N
      CALL EXIT
   98 WRITE(6,198) Y,KDEX,LDEX
      CALL EXIT
   97 WRITE(6,197) Y,KDEX,LDEX
      CALL EXIT
      END
```

```
      SUBROUTINE HGGY(N,Y,GYR,GYI)
      DIMENSION GYR(N),GYI(N)
      COMMON/DALPHA/ALPHA(50),AIRYD(50)
      DOUBLE PRECISION ALPHA,AIRYD,DFLOAT,TR,TI,X,X2,A2R,A2I,A3R,A3I,A4R
     A                   ,A4I,SR,SI,STR,STI,TEST,AR,AI
  199 FORMAT(11H1  FOR Y = ,E16.8,27H  PROGRAM EXIT BECAUSE N = ,I4,
     A         51H  EXCEEDS 50 WHICH IS THE CAPACITY OF HGGY(N,Y,...))
  198 FORMAT(11H1  FOR Y = ,E16.8,62H  PROGRAM EXIT BECAUSE OF LOSS OF A
     ACCURACY DUE TO CANCELLATION//10X,7HKDEX = ,I4,9H  LDEX = ,I4//
     B   10X,54HHGGY(N,Y,...) IS NOT CAPABLE OF HANDLING THIS ARGUMENT)
  197 FORMAT(11H1  FOR Y = ,E16.8,88H  PROGRAM EXIT BECAUSE COMPLETION O
     AF 50 ITERATIONS FAILED TO YIELD CONVERGENCE OF SERIES//
     B                                10X,7HKDEX = ,I4,9H  LDEX = ,I4//
     C   10X,54HHGGY(N,Y,...) IS NOT CAPABLE OF HANDLING THIS ARGUMENT)
      DFLOAT(J) = DBLE(FLOAT(J))
      IF(N.GT.50) GO TO 99
C-----PROGRAM WILL CAUSE EXIT WHEN N EXCEEDS 50
      X       = DBLE(Y)
      TEST    = (1.0D-9)*DABS(X)
      DO 10 K=1,N
      KDEX    = K
      TR      = 0.5*ALPHA(K)
      TI      = 1.732050807568877D0*TR
      X2      = X*X
      A2R     = 0.0D0
      A2I     = 0.0D0
      A3R     =-X*X2*TR/6.0D0
      A3I     = 1.732050807568877D0*A3R
      A4R     = X2*X2/12.0D0
      A4I     = 0.0D0
      SR      =-X + A2R + A3R + A4R
      SI      =     A2I + A3I + A4I
      STR     = DABS(X) + DABS(A2R) + DABS(A3R) + DABS(A4R)
      STI     =           DABS(A2I) + DABS(A3I) + DABS(A4I)
C-----STR AND STI ARE TEST QUANTITIES TO TEST FOR EXCESSIVE CANCELLATION
```


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```
      DO 20 L=5,50
      LDEX    = L
      AR      = X2*(-X*A2R + TR*A3R - TI*A3I)/DFLOAT(L*(L-1))
      AI      = X2*(-X*A2I + TR*A3I + TI*A3R)/DFLOAT(L*(L-1))
      SR      = SR + AR
      SI      = SI + AI
      STR     = STR + DABS(AR)
      STI     = STI + DABS(AI)
      IF((STR/DABS(SR) + STI/DABS(SI)).GT.1.0D12) GO TO 98
C-----TRANSFER OUT IF CANCELLATION SUGGESTS EXCESSIVE LOSS OF ACCURACY
      IF(DABS(AR + A4R + A3R).LT.TEST) GO TO 89
C-----TRANSFER OUT IF SUM OF LAST THREE TERMS OF REAL PART IS LESS THAN
C-----TEST = 1.0D-8*(MAGNITUDE OF FIRST TERM)
      A2R     = A3R
      A2I     = A3I
      A3R     = A4R
      A3I     = A4I
      A4R     = AR
      A4I     = AI
   20 CONTINUE
      GO TO 97
C-----WRITE A MESSAGE THAT PROGRAM FAILED TO LEAVE THE LOOP AFTER FIFTY
C-----ITERATIONS AND THEN CALL EXIT
   89 GYR(K) = SNGL(SR)
      GYI(K) = SNGL(SI)
   10 CONTINUE
C-----THIS IS THE NORMAL RETURN WHEN PROGRAM SUCCESSFULLY ESTABLISHES AN
C-----ARRAY GYR(N) AND GYI(N) WHICH CONTAINS THE N HEIGHT GAIN FUNCTIONS
      RETURN
   99 WRITE(6,199) Y,N
      CALL EXIT
   98 WRITE(6,198) Y,KDEX,LDEX
      CALL EXIT
   97 WRITE(6,197) Y,KDEX,LDEX
      CALL EXIT
      END
```

```
      SUBROUTINE HGGPY(N,Y,GPYR,GPYI)
      DIMENSION GPYR(N),GPYI(N)
      COMMON/DALPHA/ALPHA(50),AIRYD(50)
      DOUBLE PRECISION ALPHA,AIRYD,DFLOAT,TR,TI,X,X2,A1R,A1I,A2R,A2I,
     A         A3R,A3I,SR,SI,STR,STI,TEST,AR,AI
  199 FORMAT(11H1   FOR Y = ,E16.8,27H  PROGRAM EXIT BECAUSE N = ,I4,
     A         52H  EXCEEDS 50 WHICH IS THE CAPACITY OF HGGPY(N,Y,...))
  198 FORMAT(11H1   FOR Y = ,E16.8,62H  PROGRAM EXIT BECAUSE OF LOSS OF A
     ACCURACY DUE TO CANCELLATION//10X,7HKDEX = ,I4,9H  LDEX = ,I4//
     B   10X,55HHGGPY(N,Y,...) IS NOT CAPABLE OF HANDLING THIS ARGUMENT)
  197 FORMAT(11H1   FOR Y = ,E16.8,88H  PROGRAM EXIT BECAUSE COMPLETION O
     AF 50 ITERATIONS FAILED TO YIELD CONVERGENCE OF SERIES//
     B                                    10X,7HKDEX = ,I4,9H  LDEX = ,I4//
     C   10X,55HHGGPY(N,Y,...) IS NOT CAPABLE OF HANDLING THIS ARGUMENT)
      DFLOAT(J) = DBLE(FLOAT(J))
      TEST    = 1.0D-8
      IF(N.GT.50) GO TO 99
C-----PROGRAM WILL CAUSE EXIT WHEN N EXCEEDS 50
      X       = DBLE(Y)
      X2      = X*X
      DO 10 K=1,N
      KDEX    = K
      TR      = 0.5D0*ALPHA(K)
      TI      = 1.732050807568877D0*TR
      A1R     = 0.0D0
      A1I     = 0.0D0
      A2R     = 0.5*X2*TR
      A2I     = 1.732050807568877D0*A2R
      A3R     =-X*X2/3.0D0
      A3I     = 0.0D0
      SR      = 1.0D0 + A1R + A2R + A3R
      SI      =         A1I + A2I + A3I
      STR     = 1.0D0 + DABS(A1R) + DABS(A2R) + DABS(A3R)
      STI     =         DABS(A1I) + DABS(A2I) + DABS(A3I)
C-----STR AND STI ARE TEST QUANTITIES TO TEST FOR EXCESSIVE CANCELLATION
```

```
      DO 20 L=4,50
      LDEX    = L
      AR      = X2*(-X*DFLOAT(L-1)*A1R + DFLOAT(L-2)*(TR*A2R - TI*A2I))/
     A          DFLOAT(L*(L-1)*(L-2))
      AI      = X2*(-X*DFLOAT(L-1)*A1I + DFLOAT(L-2)*(TR*A2I + TI*A2R))/
     A          DFLOAT(L*(L-1)*(L-2))
      SR      = SR + AR
      SI      = SI + AI
      STR     = STR + DABS(AR)
      STI     = STI + DABS(AI)
      IF((STR/DABS(SR) + STI/DABS(SI)).GT.1.0D12) GO TO 98
C-----TRANSFER OUT IF CANCELLATION SUGGESTS LOSS OF ACCURACY
      IF(DABS(AR + A3R + A2R).LT.TEST) GO TO 89
C-----TRANSFER OUT IF SUM OF LAST THREE TERMS OF REAL PART IS LESS THAN
C-----TEST = 1.0D-8*(MAGNITUDE OF FIRST TERM)
      A1R     = A2R
      A1I     = A2I
      A2R     = A3R
      A2I     = A3I
      A3R     = AR
      A3I     = AI
   20 CONTINUE
      GO TO 97
C-----WRITE A MESSAGE THAT PROGRAM FAILED TO LEAVE THE LOOP AFTER FIFTY
C-----ITERATIONS AND THEN CALL EXIT
   89 GPYR(K)= SNGL(SR)
      GPYI(K)= SNGL(SI)
   10 CONTINUE
C-----THIS IS THE NORMAL RETURN WHEN PROGRAM SUCCESSFULLY ESTABLISHES AN
C-----ARRAY GPYR(N) AND GPYI(N) WHICH CONTAINS N HEIGHT GAIN FUNCTIONS
      RETURN
   99 WRITE(6,199) Y,N
      CALL EXIT
   98 WRITE(6,198) Y,KDEX,LDEX
      CALL EXIT
   97 WRITE(6,197) Y,KDEX,LDEX
      CALL EXIT
      END
```

Table E-4

THE HEIGHT GAIN FUNCTION $f_s(y)$

$$f_s(y) = \frac{w_1(t_s^o - y)}{w_1(t_s^o)} = \frac{Ai[-\beta_s + y\exp(-i\frac{2}{3}\pi)]}{Ai(-\beta_s)} = FYR(J) + iFYI(J)$$

$$f_s(0) = 1 \;,\; s = J = 1,2,3,\ldots$$

```
TEST OF SUBROUTINE HGFY(N,Y,FYR,FYI) FOR N.EQ.5 AND Y = 0.2
J =  1  FYR(J) =     .10088148+01  FYI(J) =     .17696507-01
J =  2  FYR(J) =     .10307771+01  FYI(J) =     .56839366-01
J =  3  FYR(J) =     .10460582+01  FYI(J) =     .84782937-01
J =  4  FYR(J) =     .10589806+01  FYI(J) =     .10888700+00
J =  5  FYR(J) =     .10705028+01  FYI(J) =     .13075837-00

TEST OF SUBROUTINE HGFY(N,Y,FYR,FYI) FOR N.EQ.5 AND Y = 0.4
J =  1  FYR(J) =     .10293761+01  FYI(J) =     .71239454-01
J =  2  FYR(J) =     .11129232+01  FYI(J) =     .23380095-00
J =  3  FYR(J) =     .11683412+01  FYI(J) =     .35391892-00
J =  4  FYR(J) =     .12132932+01  FYI(J) =     .46016063-00
J =  5  FYR(J) =     .12517816+01  FYI(J) =     .55860890-00

TEST OF SUBROUTINE HGFY(N,Y,FYR,FYI) FOR N.EQ.5 AND Y = 0.6
J =  1  FYR(J) =     .10517905+01  FYI(J) =     .16134623-00
J =  2  FYR(J) =     .12219528+01  FYI(J) =     .54793072-00
J =  3  FYR(J) =     .13223805+01  FYI(J) =     .84805959-00
J =  4  FYR(J) =     .13940288+01  FYI(J) =     .11224667+01
J =  5  FYR(J) =     .14464756+01  FYI(J) =     .13834634+01

TEST OF SUBROUTINE HGFY(N,Y,FYR,FYI) FOR N.EQ.5 AND Y = 0.8
J =  1  FYR(J) =     .10645694+01  FYI(J) =     .28780445-00
J =  2  FYR(J) =     .13177004+01  FYI(J) =     .10219096+01
J =  3  FYR(J) =     .14261585+01  FYI(J) =     .16246052+01
J =  4  FYR(J) =     .14651560+01  FYI(J) =     .21942076+01
J =  5  FYR(J) =     .14528968+01  FYI(J) =     .27487827+01

TEST OF SUBROUTINE HGFY(N,Y,FYR,FYI) FOR N.EQ.5 AND Y = 1.0
J =  1  FYR(J) =     .10547342+01  FYI(J) =     .44819684-00
J =  2  FYR(J) =     .13399089+01  FYI(J) =     .16769136+01
J =  3  FYR(J) =     .13447475+01  FYI(J) =     .27419598+01
J =  4  FYR(J) =     .11899459+01  FYI(J) =     .37751160+01
J =  5  FYR(J) =     .90555649-00  FYI(J) =     .47952715+01
```

Table E-4 (Cont'd)

```
TEST OF SUBROUTINE HGFY(N,Y,FYR,FYI) FOR N.LE.5 AND Y = 1.2
J = 1   FYR(J) =     .10083951+01   FYI(J) =     .63663021-00
J = 2   FYR(J) =     .12048182+01   FYI(J) =     .25224828+01
J = 3   FYR(J) =     .87463003-00   FYI(J) =     .42331848+01
J = 4   FYR(J) =     .19362763-00   FYI(J) =     .59131758+01
J = 5   FYR(J) =    -.79359463-00   FYI(J) =     .75674323+01

TEST OF SUBROUTINE HGFY(N,Y,FYR,FYI) FOR N.LE.5 AND Y = 1.4
J = 1   FYR(J) =     .91188672-00   FYI(J) =     .84261201-00
J = 2   FYR(J) =     .80464242-00   FYI(J) =     .35429361+01
J = 3   FYR(J) =    -.26646827-00   FYI(J) =     .60652344+01
J = 4   FYR(J) =    -.20648704+01   FYI(J) =     .85151403+01
J = 5   FYR(J) =    -.45288388+01   FYI(J) =     .10846764+02

TEST OF SUBROUTINE HGFY(N,Y,FYR,FYI) FOR N.LE.5 AND Y = 1.6
J = 1   FYR(J) =     .75350671-00   FYI(J) =     .10502922+01
J = 2   FYR(J) =     .12381600-01   FYI(J) =     .46806542+01
J = 3   FYR(J) =    -.24350606+01   FYI(J) =     .80816305+01
J = 4   FYR(J) =    -.62891234+01   FYI(J) =     .11211196+02
J = 5   FYR(J) =    -.11461491+02   FYI(J) =     .13887667+02

TEST OF SUBROUTINE HGFY(N,Y,FYR,FYI) FOR N.LE.5 AND Y = 1.8
J = 1   FYR(J) =     .52575394-00   FYI(J) =     .12383454+01
J = 2   FYR(J) =    -.13052221+01   FYI(J) =     .58178070+01
J = 3   FYR(J) =    -.60298929+01   FYI(J) =     .99303788+01
J = 4   FYR(J) =    -.13273552+02   FYI(J) =     .13173055+02
J = 5   FYR(J) =    -.22877772+02   FYI(J) =     .15048999+02

TEST OF SUBROUTINE HGFY(N,Y,FYR,FYI) FOR N.LE.5 AND Y = 2.0
J = 1   FYR(J) =     .22794384-00   FYI(J) =     .13807892+01
J = 2   FYR(J) =    -.32639299+01   FYI(J) =     .67598190+01
J = 3   FYR(J) =    -.11416890+02   FYI(J) =     .10985952+02
J = 4   FYR(J) =    -.23715350+02   FYI(J) =     .12903678+02
J = 5   FYR(J) =    -.39777827+02   FYI(J) =     .11356582+02

TEST OF SUBROUTINE HGFY(N,Y,FYR,FYI) FOR N.LE.5 AND Y = 2.2
J = 1   FYR(J) =    -.13122203-00   FYI(J) =     .14489918+01
J = 2   FYR(J) =    -.59257936+01   FYI(J) =     .72258984+01
J = 3   FYR(J) =    -.18795560+02   FYI(J) =     .10284675+02
J = 4   FYR(J) =    -.37854578+02   FYI(J) =     .80492896+01
J = 5   FYR(J) =    -.62064302+02   FYI(J) =    -.18881157+01
```

Table E-4 (Cont'd)

```
TEST OF SUBROUTINE HGFY(N,Y,FYR,FYI) FOR N.LE.5 AND Y = 2.4
J =  1   FYR(J) =     -.53160496-00   FYI(J) =       .14150044+01
J =  2   FYR(J) =     -.92510689+01   FYI(J) =       .68539393+01
J =  3   FYR(J) =     -.27996995+02   FYI(J) =       .65062997+01
J =  4   FYR(J) =     -.54895929+02   FYI(J) =      -.46683320+01
J =  5   FYR(J) =     -.87200169+02   FYI(J) =      -.31212131+02

TEST OF SUBROUTINE HGFY(N,Y,FYR,FYI) FOR N.LE.5 AND Y = 2.6
J =  1   FYR(J) =     -.94075157-00   FYI(J) =       .12561362+01
J =  2   FYR(J) =     -.13042705+02   FYI(J) =       .52283652+01
J =  3   FYR(J) =     -.38217935+02   FYI(J) =      -.19512682+01
J =  4   FYR(J) =     -.72204016+02   FYI(J) =      -.29243561+02
J =  5   FYR(J) =     -.10829514+03   FYI(J) =      -.84380431+02

TEST OF SUBROUTINE HGFY(N,Y,FYR,FYI) FOR N.LE.5 AND Y = 2.8
J =  1   FYR(J) =     -.13146679+01   FYI(J) =       .96038737-00
J =  2   FYR(J) =     -.16892447+02   FYI(J) =       .19391098+01
J =  3   FYR(J) =     -.47721217+02   FYI(J) =      -.16746924+02
J =  4   FYR(J) =     -.34347471+02   FYI(J) =      -.69692163+02
J =  5   FYR(J) =     -.11160126+03   FYI(J) =      -.16852822+03

TEST OF SUBROUTINE HGFY(N,Y,FYR,FYI) FOR N.LE.5 AND Y = 3.0
J =  1   FYR(J) =     -.16022247+01   FYI(J) =       .53199985-00
J =  2   FYR(J) =     -.20144670+02   FYI(J) =      -.33231721+01
J =  3   FYR(J) =     -.53571033+02   FYI(J) =      -.39164963+02
J =  4   FYR(J) =     -.82208068+02   FYI(J) =      -.12863547+03
J =  5   FYR(J) =     -.75405886+02   FYI(J) =      -.28632849+03

TEST OF SUBROUTINE HGFY(N,Y,FYR,FYI) FOR N.LE.5 AND Y = 3.2
J =  1   FYR(J) =     -.17492878+01   FYI(J) =      -.39692703-02
J =  2   FYR(J) =     -.21898223+02   FYI(J) =      -.10637400+02
J =  3   FYR(J) =     -.51518709+02   FYI(J) =      -.69443727+02
J =  4   FYR(J) =     -.52569112+02   FYI(J) =      -.20497868+03
J =  5   FYR(J) =      .32677336+02   FYI(J) =      -.42961048+03

TEST OF SUBROUTINE HGFY(N,Y,FYR,FYI) FOR N.LE.5 AND Y = 3.4
J =  1   FYR(J) =     -.17093749+01   FYI(J) =      -.59938923-00
J =  2   FYR(J) =     -.21070597+02   FYI(J) =      -.19700594+02
J =  3   FYR(J) =     -.36201143+02   FYI(J) =      -.10588472+03
J =  4   FYR(J) =      .21174397+02   FYI(J) =      -.29067785+03
J =  5   FYR(J) =      .25105467+03   FYI(J) =      -.57035594+03
```

Table E-4 (Cont'd)

```
TEST OF SUBROUTINE HGFY(N,Y,FYR,FYI) FOR N.LE.5 AND Y = 3.6
J =  1  FYR(J) =   -.14529858+01  FYI(J) =    -.11844860+01
J =  2  FYR(J) =   -.16545933+02  FYI(J) =    -.29681987+02
J =  3  FYR(J) =   -.18415579+01  FYI(J) =    -.14385580+03
J =  4  FYR(J) =    .15635155+03  FYI(J) =    -.36695578+03
J =  5  FYR(J) =    .61646407+03  FYI(J) =    -.65015223+03

TEST OF SUBROUTINE HGFY(N,Y,FYR,FYI) FOR N.LE.5 AND Y = 3.8
J =  1  FYR(J) =   -.97820794-00  FYI(J) =    -.16740365+01
J =  2  FYR(J) =   -.74179436+01  FYI(J) =    -.39125926+02
J =  3  FYR(J) =    .56376314+02  FYI(J) =    -.17493213+03
J =  4  FYR(J) =    .36553682+03  FYI(J) =    -.40090809+03
J =  5  FYR(J) =    .11461494+04  FYI(J) =    -.57114778+03

TEST OF SUBROUTINE HGFY(N,Y,FYR,FYI) FOR N.LE.5 AND Y = 4.0
J =  1  FYR(J) =   -.31894834-00  FYI(J) =    -.19787108+01
J =  2  FYR(J) =    .66820224+01  FYI(J) =    -.45959718+02
J =  3  FYR(J) =    .14018605+03  FYI(J) =    -.18656795+03
J =  4  FYR(J) =    .64770570+03  FYI(J) =    -.34419971+03
J =  5  FYR(J) =    .18093090+04  FYI(J) =    -.19430102+03
```

Table E-5

THE HEIGHT GAIN FUNCTION $f_s'(y)$

$$f_s'(y) = -\frac{w_1'(t_s^o - y)}{w_1(t_s^o)} = \frac{Ai'[-\beta_s + y\exp(-i\frac{2}{3}\pi)]}{\exp(i\frac{2}{3}\pi)Ai(-\beta_s)} = -[FPYR(J)+iFPYI(J)]$$

$$f_s'(0) = 0 \;,\; s = J = 1,2,3,\ldots$$

```
TEST OF SUBROUTINE HGFPY(N,Y,FPYR,FPYI) FOR N.LE.5 AND Y = 0.2
J =  1 FPYR(J) =     -.81059795-01 FPYI(J) =     -.17742269-00
J =  2 FPYR(J) =     -.29727631-00 FPYI(J) =     -.57402983-00
J =  3 FPYR(J) =     -.44559926-00 FPYI(J) =     -.86056310-00
J =  4 FPYR(J) =     -.56958488-00 FPYI(J) =     -.11099266+01
J =  5 FPYR(J) =     -.67896298-00 FPYI(J) =     -.13379091+01

TEST OF SUBROUTINE HGFPY(N,Y,FPYR,FPYI) FOR N.LE.5 AND Y = 0.4
J =  1 FPYR(J) =     -.11633140+00 FPYI(J) =     -.35870559-00
J =  2 FPYR(J) =     -.50413105-00 FPYI(J) =     -.12101973+01
J =  3 FPYR(J) =     -.74116873-00 FPYI(J) =     -.18653974+01
J =  4 FPYR(J) =     -.91811065-00 FPYI(J) =     -.24611172+01
J =  5 FPYR(J) =     -.10560531+01 FPYI(J) =     -.30254193+01

TEST OF SUBROUTINE HGFPY(N,Y,FPYR,FPYI) FOR N.LE.5 AND Y = 0.6
J =  1 FPYR(J) =     -.98231439-01 FPYI(J) =     -.54229230-00
J =  2 FPYR(J) =     -.55307698-00 FPYI(J) =     -.19505852+01
J =  3 FPYR(J) =     -.73219433-00 FPYI(J) =     -.31251052+01
J =  4 FPYR(J) =     -.77875100-00 FPYI(J) =     -.42461856+01
J =  5 FPYR(J) =     -.72787657-00 FPYI(J) =     -.53462727+01

TEST OF SUBROUTINE HGFPY(N,Y,FPYR,FPYI) FOR N.LE.5 AND Y = 0.8
J =  1 FPYR(J) =     -.18694966-01 FPYI(J) =     -.72038312-00
J =  2 FPYR(J) =     -.35449129-00 FPYI(J) =     -.28076883+01
J =  3 FPYR(J) =     -.19452000-00 FPYI(J) =     -.46911680+01
J =  4 FPYR(J) =      .26149514-00 FPYI(J) =     -.65586106+01
J =  5 FPYR(J) =      .96223660-00 FPYI(J) =     -.84316372+01

TEST OF SUBROUTINE HGFPY(N,Y,FPYR,FPYI) FOR N.LE.5 AND Y = 1.0
J =  1 FPYR(J) =      .12871639-00 FPYI(J) =     -.87872599-00
J =  2 FPYR(J) =      .20213357-00 FPYI(J) =     -.37514367+01
J =  3 FPYR(J) =      .11772721+01 FPYI(J) =     -.65132960+01
J =  4 FPYR(J) =      .28002930+01 FPYI(J) =     -.92977919+01
J =  5 FPYR(J) =      .50045191+01 FPYI(J) =     -.12083924+02
```

Table E-5 (Cont'd)

```
TEST OF SUBROUTINE HGFPY(N,Y,FPYR,FPYI) FOR N.LE.5 AND Y = 1.2
J =   1 FPYR(J) =     .34620561-00 FPYI(J) =      -.99693047-00
J =   2 FPYR(J) =     .12389611+01 FPYI(J) =      -.46925993+01
J =   3 FPYR(J) =     .37593541+01 FPYI(J) =      -.83762085+01
J =   4 FPYR(J) =     .76136740+01 FPYI(J) =      -.12016350+02
J =   5 FPYR(J) =     .12723719+02 FPYI(J) =      -.15477475+02

TEST OF SUBROUTINE HGFPY(N,Y,FPYR,FPYI) FOR N.LE.5 AND Y = 1.4
J =   1 FPYR(J) =     .62880537-00 FPYI(J) =      -.10496925+01
J =   2 FPYR(J) =     .28696762+01 FPYI(J) =      -.54654790+01
J =   3 FPYR(J) =     .79511998+01 FPYI(J) =      -.98199940+01
J =   4 FPYR(J) =     .15563002+02 FPYI(J) =      -.13710465+02
J =   5 FPYR(J) =     .25607763+02 FPYI(J) =      -.16725555+02

TEST OF SUBROUTINE HGFPY(N,Y,FPYR,FPYI) FOR N.LE.5 AND Y = 1.6
J =   1 FPYR(J) =     .96148112-00 FPYI(J) =      -.10092729+01
J =   2 FPYR(J) =     .51659364+01 FPYI(J) =      -.58149509+01
J =   3 FPYR(J) =     .14073071+02 FPYI(J) =      -.10057363+02
J =   4 FPYR(J) =     .27356528+02 FPYI(J) =      -.12577946+02
J =   5 FPYR(J) =     .44823103+02 FPYI(J) =      -.12353588+02

TEST OF SUBROUTINE HGFPY(N,Y,FPYR,FPYI) FOR N.LE.5 AND Y = 1.8
J =   1 FPYR(J) =     .13168399+01 FPYI(J) =      -.84944202-00
J =   2 FPYR(J) =     .81095923+01 FPYI(J) =      -.53953089+01
J =   3 FPYR(J) =     .22191188+02 FPYI(J) =      -.79146220+01
J =   4 FPYR(J) =     .43098140+02 FPYI(J) =      -.58099227+01
J =   5 FPYR(J) =     .70239089+02 FPYI(J) =       .31833582+01

TEST OF SUBROUTINE HGFPY(N,Y,FPYR,FPYI) FOR N.LE.5 AND Y = 2.0
J =   1 FPYR(J) =     .16541683+01 FPYI(J) =      -.55085786-00
J =   2 FPYR(J) =     .11533250+02 FPYI(J) =      -.37905148+01
J =   3 FPYR(J) =     .31861902+02 FPYI(J) =      -.18401523+01
J =   4 FPYR(J) =     .61567689+02 FPYI(J) =       .10457679+02
J =   5 FPYR(J) =     .98791426+02 FPYI(J) =       .37642753+02

TEST OF SUBROUTINE HGFPY(N,Y,FPYR,FPYI) FOR N.LE.5 AND Y = 2.2
J =   1 FPYR(J) =     .19207697+01 FPYI(J) =      -.10749185+00
J =   2 FPYR(J) =     .15057028+02 FPYI(J) =      -.56630688-00
J =   3 FPYR(J) =     .41806538+02 FPYI(J) =       .99586570+01
J =   4 FPYR(J) =     .79229468+02 FPYI(J) =       .40832852+02
J =   5 FPYR(J) =     .12212040+03 FPYI(J) =       .10028559+03
```

```
TEST OF SUBROUTINE HGFPY(N,Y,FPYR,FPYI) FOR N.LE.5 AND Y =  2.4
J =  1 FPYR(J) =      .20559415+01 FPYI(J) =       .46673623-00
J =  2 FPYR(J) =      .18036326+02 FPYI(J) =       .46370593+01
J =  3 FPYR(J) =      .49564185+02 FPYI(J) =       .29178223+02
J =  4 FPYR(J) =      .89070005+02 FPYI(J) =       .89726979+02
J =  5 FPYR(J) =      .12369509+03 FPYI(J) =       .19957245+03

TEST OF SUBROUTINE HGFPY(N,Y,FPYR,FPYI) FOR N.LE.5 AND Y =  2.6
J =  1 FPYR(J) =      .19987497+01 FPYI(J) =       .11324218+01
J =  2 FPYR(J) =      .19542961+02 FPYI(J) =       .11968350+02
J =  3 FPYR(J) =      .51221023+02 FPYI(J) =       .56815356+02
J =  4 FPYR(J) =      .79563832+02 FPYI(J) =       .15948731+03
J =  5 FPYR(J) =      .76154087+02 FPYI(J) =       .33836818+03

TEST OF SUBROUTINE HGFPY(N,Y,FPYR,FPYI) FOR N.LE.5 AND Y =  2.8
J =  1 FPYR(J) =      .16987926+01 FPYI(J) =       .18219288+01
J =  2 FPYR(J) =      .18408219+02 FPYI(J) =       .21192943+02
J =  3 FPYR(J) =      .41376768+02 FPYI(J) =       .92267553+02
J =  4 FPYR(J) =      .34314187+02 FPYI(J) =       .24737615+03
J =  5 FPYR(J) =     -.59604295+02 FPYI(J) =       .50586680+03

TEST OF SUBROUTINE HGFPY(N,Y,FPYR,FPYI) FOR N.LE.5 AND Y =  3.0
J =  1 FPYR(J) =      .11294327+01 FPYI(J) =       .24412707+01
J =  2 FPYR(J) =      .13357986+02 FPYI(J) =       .31508587+02
J =  3 FPYR(J) =      .13566644+02 FPYI(J) =       .13218696+03
J =  4 FPYR(J) =     -.66771097+02 FPYI(J) =       .34141783+03
J =  5 FPYR(J) =     -.33102658+03 FPYI(J) =       .66608839+03

TEST OF SUBROUTINE HGFPY(N,Y,FPYR,FPYI) FOR N.LE.5 AND Y =  3.2
J =  1 FPYR(J) =      .30141609-00 FPYI(J) =       .28782056+01
J =  2 FPYR(J) =      .32632682+01 FPYI(J) =       .41393879+02
J =  3 FPYR(J) =     -.38615334+02 FPYI(J) =       .16927116+03
J =  4 FPYR(J) =     -.24366706+03 FPYI(J) =       .41563716+03
J =  5 FPYR(J) =     -.78266474+03 FPYI(J) =       .74429638+03

TEST OF SUBROUTINE HGFPY(N,Y,FPYR,FPYI) FOR N.LE.5 AND Y =  3.4
J =  1 FPYR(J) =     -.72602319-00 FPYI(J) =       .30173384+01
J =  2 FPYR(J) =     -.12489749+02 FPYI(J) =       .48554537+02
J =  3 FPYR(J) =     -.11948773+03 FPYI(J) =       .19135496+03
J =  4 FPYR(J) =     -.50865792+03 FPYI(J) =       .42600988+03
J =  5 FPYR(J) =     -.14331648+04 FPYI(J) =       .61530291+03
```

Table E-5 (Cont'd)

```
TEST OF SUBROUTINE HGFPY(N,Y,FPYR,FPYI) FOR N.LE.5 AND Y =  3.6
J =  1 FPYR(J) =     -.18401798+01 FPYI(J) =      .27616575+01
J =  2 FPYR(J) =     -.33539942+02 FPYI(J) =      .50046442+02
J =  3 FPYR(J) =     -.22829963+03 FPYI(J) =      .18136887+03
J =  4 FPYR(J) =     -.85433155+03 FPYI(J) =      .30948775+03
J =  5 FPYR(J) =     -.22376838+04 FPYI(J) =      .10136946+03

TEST OF SUBROUTINE HGFPY(N,Y,FPYR,FPYI) FOR N.LE.5 AND Y =  3.8
J =  1 FPYR(J) =     -.28796604+01 FPYI(J) =      .20580650+01
J =  2 FPYR(J) =     -.58079117+02 FPYI(J) =      .42651922+02
J =  3 FPYR(J) =     -.35555709+03 FPYI(J) =      .11888645+03
J =  4 FPYR(J) =     -.12376604+04 FPYI(J) =     -.10479266+02
J =  5 FPYR(J) =     -.30367860+04 FPYI(J) =     -.10083432+04

TEST OF SUBROUTINE HGFPY(N,Y,FPYR,FPYI) FOR N.LE.5 AND Y =  4.0
J =  1 FPYR(J) =     -.36516627+01 FPYI(J) =      .92241727-00
J =  2 FPYR(J) =     -.82499113+02 FPYI(J) =      .23556542+02
J =  3 FPYR(J) =     -.47938444+03 FPYI(J) =     -.16003852+02
J =  4 FPYR(J) =     -.15630375+04 FPYI(J) =     -.60788676+03
J =  5 FPYR(J) =     -.35006523+04 FPYI(J) =     -.29016662+04
```

Table E-6

THE HEIGHT GAIN FUNCTION $g_s(y)$

$$g_s(y) = -\frac{w_1(t_s^\infty - y)}{w_1'(t_s^\infty)} = \frac{Ai[-\alpha_s + y\exp(-i\frac{2}{3}\pi)]}{\exp(-i\frac{2}{3}\pi)Ai'(-\alpha_s)} = -[GYR(J) + iGYI(J)]$$

$$g_s(0) = 0 \ , \ s = J = 1,2,3,\ldots$$

TEST OF SUBROUTINE HGGY(N,Y,GYR,GYI) FOR N.LE.5 AND Y = 0.2

J =	1	GYR(J) =	-.20141748-00	GYI(J) =	-.27113553-02
J =	2	GYR(J) =	-.20256845-00	GYI(J) =	-.47570512-02
J =	3	GYR(J) =	-.20350454-00	GYI(J) =	-.64424053-02
J =	4	GYR(J) =	-.20432716-00	GYI(J) =	-.79398178-02
J =	5	GYR(J) =	-.20507526-00	GYI(J) =	-.93151245-02

TEST OF SUBROUTINE HGGY(N,Y,GYR,GYI) FOR N.LE.5 AND Y = 0.4

J =	1	GYR(J) =	-.41006308-00	GYI(J) =	-.21932527-01
J =	2	GYR(J) =	-.41886896-00	GYI(J) =	-.38874135-01
J =	3	GYR(J) =	-.42586648-00	GYI(J) =	-.53080253-01
J =	4	GYR(J) =	-.43188708-00	GYI(J) =	-.65887243-01
J =	5	GYR(J) =	-.43725257-00	GYI(J) =	-.77800848-01

TEST OF SUBROUTINE HGGY(N,Y,GYR,GYI) FOR N.LE.5 AND Y = 0.6

J =	1	GYR(J) =	-.62905992-00	GYI(J) =	-.75154160-01
J =	2	GYR(J) =	-.65629295-00	GYI(J) =	-.13537752-00
J =	3	GYR(J) =	-.67681631-00	GYI(J) =	-.18721087-00
J =	4	GYR(J) =	-.69353877-00	GYI(J) =	-.23490903-00
J =	5	GYR(J) =	-.70759956-00	GYI(J) =	-.28005432-00

TEST OF SUBROUTINE HGGY(N,Y,GYR,GYI) FOR N.LE.5 AND Y = 0.8

J =	1	GYR(J) =	-.85565010-00	GYI(J) =	-.18108702-00
J =	2	GYR(J) =	-.91101921-00	GYI(J) =	-.33314802-00
J =	3	GYR(J) =	-.94792631-00	GYI(J) =	-.46808085-00
J =	4	GYR(J) =	-.97355965-00	GYI(J) =	-.59506267-00
J =	5	GYR(J) =	-.99077038-00	GYI(J) =	-.71739352-00

TEST OF SUBROUTINE HGGY(N,Y,GYR,GYI) FOR N.LE.5 AND Y = 1.0

J =	1	GYR(J) =	-.10794139+01	GYI(J) =	-.35888197-00
J =	2	GYR(J) =	-.11618280+01	GYI(J) =	-.67672353-00
J =	3	GYR(J) =	-.11999053+01	GYI(J) =	-.96758191-00
J =	4	GYR(J) =	-.12082572+01	GYI(J) =	-.12468703+01
J =	5	GYR(J) =	-.11927380+01	GYI(J) =	-.15197225+01

Table E-6 (Cont'd)

```
TEST OF SUBROUTINE HGGY(N,Y,GYR,GYI) FOR N.LE.5 AND Y = 1.2
J =  1   GYR(J) =    -.12805049+01   GYI(J) =    -.62613682-00
J =  2   GYR(J) =    -.13625718+01   GYI(J) =    -.12126042+01
J =  3   GYR(J) =    -.13457168+01   GYI(J) =    -.17639314+01
J =  4   GYR(J) =    -.12559362+01   GYI(J) =    -.23006450+01
J =  5   GYR(J) =    -.11037300+01   GYI(J) =    -.28281220+01

TEST OF SUBROUTINE HGGY(N,Y,GYR,GYI) FOR N.LE.5 AND Y = 1.4
J =  1   GYR(J) =    -.14283726+01   GYI(J) =    -.99554102-00
J =  2   GYR(J) =    -.14342927+01   GYI(J) =    -.19606679+01
J =  3   GYR(J) =    -.12309604+01   GYI(J) =    -.29242437+01
J =  4   GYR(J) =    -.86024064-00   GYI(J) =    -.38448675+01
J =  5   GYR(J) =    -.33914699-00   GYI(J) =    -.47421474+01

TEST OF SUBROUTINE HGGY(N,Y,GYR,GYI) FOR N.LE.5 AND Y = 1.6
J =  1   GYR(J) =    -.14816288+01   GYI(J) =    -.14701586+01
J =  2   GYR(J) =    -.12590161+01   GYI(J) =    -.29993241+01
J =  3   GYR(J) =    -.61592690-00   GYI(J) =    -.44711455+01
J =  4   GYR(J) =     .38519678-00   GYI(J) =    -.58824038+01
J =  5   GYR(J) =     .17178123+01   GYI(J) =    -.72100006+01

TEST OF SUBROUTINE HGGY(N,Y,GYR,GYI) FOR N.LE.5 AND Y = 1.8
J =  1   GYR(J) =    -.13898627+01   GYI(J) =    -.20376276+01
J =  2   GYR(J) =    -.67854464-00   GYI(J) =    -.42435969+01
J =  3   GYR(J) =     .83254799-00   GYI(J) =    -.63290778+01
J =  4   GYR(J) =     .30534214+01   GYI(J) =    -.82225474+01
J =  5   GYR(J) =     .59392369+01   GYI(J) =    -.98378531+01

TEST OF SUBROUTINE HGGY(N,Y,GYR,GYI) FOR N.LE.5 AND Y = 2.0
J =  1   GYR(J) =    -.10982454+01   GYI(J) =    -.26639637+01
J =  2   GYR(J) =     .49716704-00   GYI(J) =    -.56166993+01
J =  3   GYR(J) =     .35258340+01   GYI(J) =    -.82487614+01
J =  4   GYR(J) =     .78516312+01   GYI(J) =    -.10322998+02
J =  5   GYR(J) =     .13381091+02   GYI(J) =    -.11603099+02

TEST OF SUBROUTINE HGGY(N,Y,GYR,GYI) FOR N.LE.5 AND Y = 2.2
J =  1   GYR(J) =    -.55561009-00   GYI(J) =    -.32881657+01
J =  2   GYR(J) =     .24664162+01   GYI(J) =    -.69180658+01
J =  3   GYR(J) =     .78964868+01   GYI(J) =    -.97147733+01
J =  4   GYR(J) =     .15494031+02   GYI(J) =    -.11088800+02
J =  5   GYR(J) =     .25022208+02   GYI(J) =    -.10484983+02
```

Table E-6 (Cont'd)

```
TEST OF SUBROUTINE HGGY(N,Y,GYR,GYI) FOR N.LE.5 AND Y = 2.4
 J =  1   GYR(J) =      .27374032-00   GYI(J) =      -.38193570+01
 J =  2   GYR(J) =      .53915074+01   GYI(J) =      -.78143335+01
 J =  3   GYR(J) =      .14277759+02   GYI(J) =      -.98524704+01
 J =  4   GYR(J) =      .26422244+02   GYI(J) =      -.86644955+01
 J =  5   GYR(J) =      .41188855+02   GYI(J) =      -.30870224+01

TEST OF SUBROUTINE HGGY(N,Y,GYR,GYI) FOR N.LE.5 AND Y = 2.6
 J =  1   GYR(J) =      .13950253+01   GYI(J) =      -.41386146+01
 J =  2   GYR(J) =      .93302394+01   GYI(J) =      -.78241112+01
 J =  3   GYR(J) =      .22700072+02   GYI(J) =      -.73669500+01
 J =  4   GYR(J) =      .40315197+02   GYI(J) =      -.30035844-00
 J =  5   GYR(J) =      .60531960+02   GYI(J) =       .15570195+02

TEST OF SUBROUTINE HGGY(N,Y,GYR,GYI) FOR N.LE.5 AND Y = 2.8
 J =  1   GYR(J) =      .27664256+01   GYI(J) =      -.41077460+01
 J =  2   GYR(J) =      .14144237+02   GYI(J) =      -.63317790+01
 J =  3   GYR(J) =      .32596669+02   GYI(J) =      -.56693160-00
 J =  4   GYR(J) =      .55356056+02   GYI(J) =       .17565259+02
 J =  5   GYR(J) =      .78479660+02   GYI(J) =       .51758324+02

TEST OF SUBROUTINE HGGY(N,Y,GYR,GYI) FOR N.LE.5 AND Y = 3.0
 J =  1   GYR(J) =      .42839799+01   GYI(J) =      -.35868418+01
 J =  2   GYR(J) =      .19393325+02   GYI(J) =      -.26485033+01
 J =  3   GYR(J) =      .42437174+02   GYI(J) =       .12454100+02
 J =  4   GYR(J) =      .67293973+02   GYI(J) =       .48798756+02
 J =  5   GYR(J) =      .85253765+02   GYI(J) =       .11175233+03

TEST OF SUBROUTINE HGGY(N,Y,GYR,GYI) FOR N.LE.5 AND Y = 3.2
 J =  1   GYR(J) =      .57723219+01   GYI(J) =      -.24612336+01
 J =  2   GYR(J) =      .24236689+02   GYI(J) =       .38620754+01
 J =  3   GYR(J) =      .49352720+02   GYI(J) =       .33397172+02
 J =  4   GYR(J) =      .68471164+02   GYI(J) =       .96383370+02
 J =  5   GYR(J) =      .63874614+02   GYI(J) =       .19890821+03

TEST OF SUBROUTINE HGGY(N,Y,GYR,GYI) FOR N.LE.5 AND Y = 3.4
 J =  1   GYR(J) =      .69865948+01   GYI(J) =      -.67642115-00
 J =  2   GYR(J) =      .27375179+02   GYI(J) =       .13583117+02
 J =  3   GYR(J) =      .46879109+02   GYI(J) =       .63027858+02
 J =  4   GYR(J) =      .47189535+02   GYI(J) =       .16031080+03
 J =  5   GYR(J) =     -.10878451+02   GYI(J) =       .30849614+03
```

Table E-6 (Cont'd)

```
TEST OF SUBROUTINE HGGY(N,Y,GYR,GYI) FOR N.LE.5 AND Y = 3.6
J =  1  GYR(J) =     .76309413+01  GYI(J) =     .17233598+01
J =  2  GYR(J) =     .27081870+02  GYI(J) =     .26374540+02
J =  3  GYR(J) =     .35017028+02  GYI(J) =     .10011961+03
J =  4  GYR(J) =    -.11951250+02  GYI(J) =     .23445623+03
J =  5  GYR(J) =    -.17060078+03  GYI(J) =     .42005070+03

TEST OF SUBROUTINE HGGY(N,Y,GYR,GYI) FOR N.LE.5 AND Y = 3.8
J =  1  GYR(J) =     .73975016+01  GYI(J) =     .45605559+01
J =  2  GYR(J) =     .21374433+02  GYI(J) =     .41265697+02
J =  3  GYR(J) =     .86936059-00  GYI(J) =     .14016021+03
J =  4  GYR(J) =    -.12586035+03  GYI(J) =     .30268367+03
J =  5  GYR(J) =    -.44767795+03  GYI(J) =     .48795804+03

TEST OF SUBROUTINE HGGY(N,Y,GYR,GYI) FOR N.LE.5 AND Y = 4.0
J =  1  GYR(J) =     .60264868+01  GYI(J) =     .75043356+01
J =  2  GYR(J) =     .83760788+01  GYI(J) =     .56178005+02
J =  3  GYR(J) =    -.59869993+02  GYI(J) =     .17408120+03
J =  4  GYR(J) =    -.30790649+03  GYI(J) =     .33502712+03
J =  5  GYR(J) =    -.85910077+03  GYI(J) =     .43275864+03
```

Table E-7

THE HEIGHT GAIN FUNCTION $g_s'(y)$

$$g_s'(y) = \frac{w_1'(t_s^\infty - y)}{w_1'(t_s^\infty)} = \frac{Ai'[-\alpha_s + y\exp(-i\frac{2}{3}\pi)]}{Ai'(-\alpha_s)} = GPYR(J) + iGPYI(J)$$

$$g_s'(0) = 1 \; , \; s = J = 1,2,3,\ldots$$

TEST OF SUBROUTINE HGGPY(N,Y,GPYR,GPYI) FOR N = 5 AND Y = 0.2

J = 1	GPYR(J) = .10205133+01	GPYI(J) =	.40780293-01
J = 2	GPYR(J) = .10376182+01	GPYI(J) =	.71713101-01
J = 3	GPYR(J) = .10514853+01	GPYI(J) =	.97301131-01
J = 4	GPYR(J) = .10635846+01	GPYI(J) =	.12011312+00
J = 5	GPYR(J) = .10745647+01	GPYI(J) =	.14112810-00

TEST OF SUBROUTINE HGGPY(N,Y,GPYR,GPYI) FOR N = 5 AND Y = 0.4

J = 1	GPYR(J) = .10686725+01	GPYI(J) =	.16598491-00
J = 2	GPYR(J) = .11319266+01	GPYI(J) =	.29678874-00
J = 3	GPYR(J) = .11809760+01	GPYI(J) =	.40806884-00
J = 4	GPYR(J) = .12221912+01	GPYI(J) =	.50955490-00
J = 5	GPYR(J) = .12580464+01	GPYI(J) =	.60489720-00

TEST OF SUBROUTINE HGGPY(N,Y,GPYR,GPYI) FOR N = 5 AND Y = 0.6

J = 1	GPYR(J) = .11191494+01	GPYI(J) =	.38188630-00
J = 2	GPYR(J) = .12396467+01	GPYI(J) =	.70064884-00
J = 3	GPYR(J) = .13225516+01	GPYI(J) =	.98251755-00
J = 4	GPYR(J) = .13829684+01	GPYI(J) =	.12471880+01
J = 5	GPYR(J) = .14269534+01	GPYI(J) =	.15017994+01

TEST OF SUBROUTINE HGGPY(N,Y,GPYR,GPYI) FOR N = 5 AND Y = 0.8

J = 1	GPYR(J) = .11382220+01	GPYI(J) =	.69356551-00
J = 2	GPYR(J) = .12905714+01	GPYI(J) =	.13144618+01
J = 3	GPYR(J) = .13572284+01	GPYI(J) =	.18865691+01
J = 4	GPYR(J) = .13666646+01	GPYI(J) =	.24389009+01
J = 5	GPYR(J) = .13303431+01	GPYI(J) =	.29811059+01

TEST OF SUBROUTINE HGGPY(N,Y,GPYR,GPYI) FOR N = 5 AND Y = 1.0

J = 1	GPYR(J) = .10829915+01	GPYI(J) =	.10993282+01
J = 2	GPYR(J) = .11797408+01	GPYI(J) =	.21607415+01
J = 3	GPYR(J) = .10917066+01	GPYI(J) =	.31750151+01
J = 4	GPYR(J) = .86557374-00	GPYI(J) =	.41735974+01
J = 5	GPYR(J) = .51959375-00	GPYI(J) =	.51638765+01

Table E-7 (Cont'd)

TEST OF SUBROUTINE HGGPY(N,Y,GPYR,GPYI) FOR N=5 AND Y = 1.2

J =	1 GPYR(J) =	.90248087-00	GPYI(J) =	.15841909+01
J =	2 GPYR(J) =	.76245495-00	GPYI(J) =	.32325230+01
J =	3 GPYR(J) =	.23929378-00	GPYI(J) =	.48471341+01
J =	4 GPYR(J) =	-.59906833-00	GPYI(J) =	.64430758+01
J =	5 GPYR(J) =	-.17250233+01	GPYI(J) =	.80119323+01

TEST OF SUBROUTINE HGGPY(N,Y,GPYR,GPYI) FOR N=5 AND Y = 1.4

J =	1 GPYR(J) =	.54156506-00	GPYI(J) =	.21128957+01
J =	2 GPYR(J) =	-.14339948-00	GPYI(J) =	.44652860+01
J =	3 GPYR(J) =	-.15857376+01	GPYI(J) =	.67792939+01
J =	4 GPYR(J) =	-.36947826+01	GPYI(J) =	.90105704+01
J =	5 GPYR(J) =	-.64321790+01	GPYI(J) =	.11100826+02

TEST OF SUBROUTINE HGGPY(N,Y,GPYR,GPYI) FOR N=5 AND Y = 1.6

J =	1 GPYR(J) =	-.51387051-01	GPYI(J) =	.26235091+01
J =	2 GPYR(J) =	-.17416135+01	GPYI(J) =	.57028639+01
J =	3 GPYR(J) =	-.48432740+01	GPYI(J) =	.86331017+01
J =	4 GPYR(J) =	-.92393076+01	GPYI(J) =	.11218970+02
J =	5 GPYR(J) =	-.14869065+02	GPYI(J) =	.13273506+02

TEST OF SUBROUTINE HGGPY(N,Y,GPYR,GPYI) FOR N=5 AND Y = 1.8

J =	1 GPYR(J) =	-.91263425-00	GPYI(J) =	.30233315+01
J =	2 GPYR(J) =	-.42231832+01	GPYI(J) =	.66619482+01
J =	3 GPYR(J) =	-.99887680+01	GPYI(J) =	.97463735+01
J =	4 GPYR(J) =	-.18042143+02	GPYI(J) =	.11747671+02
J =	5 GPYR(J) =	-.28245572+02	GPYI(J) =	.12184427+02

TEST OF SUBROUTINE HGGPY(N,Y,GPYR,GPYI) FOR N=5 AND Y = 2.0

J =	1 GPYR(J) =	-.20471672+01	GPYI(J) =	.31894595+01
J =	2 GPYR(J) =	-.77028226+01	GPYI(J) =	.69044852+01
J =	3 GPYR(J) =	-.17314343+02	GPYI(J) =	.90290804+01
J =	4 GPYR(J) =	-.30557677+02	GPYI(J) =	.83626230+01
J =	5 GPYR(J) =	-.47034778+02	GPYI(J) =	.38175676+01

TEST OF SUBROUTINE HGGPY(N,Y,GPYR,GPYI) FOR N=5 AND Y = 2.2

J =	1 GPYR(J) =	-.34102030+01	GPYI(J) =	.29767052+01
J =	2 GPYR(J) =	-.12130638+02	GPYI(J) =	.58328312+01
J =	3 GPYR(J) =	-.26686734+02	GPYI(J) =	.49078859+01
J =	4 GPYR(J) =	-.46274705+02	GPYI(J) =	-.22380004+01
J =	5 GPYR(J) =	-.69732343+02	GPYI(J) =	-.17808027+02

Table E-7 (Cont'd)

```
TEST OF SUBROUTINE HGGPY(N,Y,GPYR,GPYI) FOR N=5 AND Y = 2.4
J =  1 GPYR(J) =     -.46890671+01 GPYI(J) =      .22353970+01
J =  2 GPYR(J) =     -.17176183+02 GPYI(J) =      .27277597+01
J =  3 GPYR(J) =     -.37176061+02 GPYI(J) =     -.46111128+01
J =  4 GPYR(J) =     -.62801281+02 GPYI(J) =     -.24300828+02
J =  5 GPYR(J) =     -.90939591+02 GPYI(J) =     -.60339312+02

TEST OF SUBROUTINE HGGPY(N,Y,GPYR,GPYI) FOR N=5 AND Y = 2.6
J =  1 GPYR(J) =     -.62962455+01 GPYI(J) =      .84056528-00
J =  2 GPYR(J) =     -.22104870+02 GPYI(J) =     -.31478631+01
J =  3 GPYR(J) =     -.46606030+02 GPYI(J) =     -.21667563+02
J =  4 GPYR(J) =     -.74694360+02 GPYI(J) =     -.62347681+02
J =  5 GPYR(J) =     -.98851490+02 GPYI(J) =     -.13155881+03

TEST OF SUBROUTINE HGGPY(N,Y,GPYR,GPYI) FOR N=5 AND Y = 2.8
J =  1 GPYR(J) =     -.75379594+01 GPYI(J) =     -.12681035+01
J =  2 GPYR(J) =     -.25676934+02 GPYI(J) =     -.12359819+02
J =  3 GPYR(J) =     -.51117337+02 GPYI(J) =     -.47955532+02
J =  4 GPYR(J) =     -.72257309+02 GPYI(J) =     -.11962638+03
J =  5 GPYR(J) =     -.72654373+02 GPYI(J) =     -.23578718+03

TEST OF SUBROUTINE HGGPY(N,Y,GPYR,GPYI) FOR N=5 AND Y = 3.0
J =  1 GPYR(J) =     -.76913535+01 GPYI(J) =     -.40394570+01
J =  2 GPYR(J) =     -.26118650+02 GPYI(J) =     -.25020012+02
J =  3 GPYR(J) =     -.44920766+02 GPYI(J) =     -.83732820+02
J =  4 GPYR(J) =     -.40800705+02 GPYI(J) =     -.19536877+03
J =  5 GPYR(J) =      .13956369+02 GPYI(J) =     -.36737613+03

TEST OF SUBROUTINE HGGPY(N,Y,GPYR,GPYI) FOR N=5 AND Y = 3.2
J =  1 GPYR(J) =     -.69865280+01 GPYI(J) =     -.72653465+01
J =  2 GPYR(J) =     -.21230481+02 GPYI(J) =     -.40427937+02
J =  3 GPYR(J) =     -.20512666+02 GPYI(J) =     -.12642486+03
J =  4 GPYR(J) =      .38794494+02 GPYI(J) =     -.28073326+03
J =  5 GPYR(J) =      .21595664+03 GPYI(J) =     -.50088553+03

TEST OF SUBROUTINE HGGPY(N,Y,GPYR,GPYI) FOR N=5 AND Y = 3.4
J =  1 GPYR(J) =     -.49054146+01 GPYI(J) =     -.10548927+02
J =  2 GPYR(J) =     -.87015509+01 GPYI(J) =     -.56692151+02
J =  3 GPYR(J) =      .30297857+02 GPYI(J) =     -.16897067+03
J =  4 GPYR(J) =      .18702579+03 GPYI(J) =     -.35377117+03
J =  5 GPYR(J) =      .55864176+03 GPYI(J) =     -.57875136+03
```

Table E-7 (Cont'd)

```
TEST OF SUBROUTINE HGGPY(N,Y,GPYR,GPYI) FOR  N=5 AND Y = 3.6
J =  1 GPYR(J) =    -.12754405+01 GPYI(J) =     -.13302737+02
J =  2 GPYR(J) =     .13316375+02 GPYI(J) =     -.70437594+02
J =  3 GPYR(J) =     .11419202+03 GPYI(J) =     -.19829828+03
J =  4 GPYR(J) =     .41888331+03 GPYI(J) =     -.37457464+03
J =  5 GPYR(J) =     .10688715+04 GPYI(J) =     -.49956673+03

TEST OF SUBROUTINE HGGPY(N,Y,GPYR,GPYI) FOR  N=5 AND Y = 3.8
J =  1 GPYR(J) =     .38328255+01 GPYI(J) =     -.14793096+02
J =  2 GPYR(J) =     .45369728+02 GPYI(J) =     -.76755306+02
J =  3 GPYR(J) =     .23275065+03 GPYI(J) =     -.19462313+03
J =  4 GPYR(J) =     .73207103+03 GPYI(J) =     -.28293075+03
J =  5 GPYR(J) =     .17176650+04 GPYI(J) =     -.11363028+03

TEST OF SUBROUTINE HGGPY(N,Y,GPYR,GPYI) FOR N=5 AND Y = 4.0
J =  1 GPYR(J) =     .99971268+01 GPYI(J) =     -.14243368+02
J =  2 GPYR(J) =     .85679182+02 GPYI(J) =     -.69585072+02
J =  3 GPYR(J) =     .37759458+03 GPYI(J) =     -.13256712+03
J =  4 GPYR(J) =     .10903817+04 GPYI(J) =     -.21427482+01
J =  5 GPYR(J) =     .23790268+04 GPYI(J) =      .76325448+03
```

```
      SUBROUTINE VTWTZ(Z,VT,WT,VTP,WTP)
      COMPLEX          Z,VT,WT,VTP,WTP
      DOUBLE PRECISION AZERO,A(22),BZERO,B(22),CZERO,C(22),DZERO,D(22)
      DOUBLE PRECISION DRZ,DIZ,D2RZ,D2IZ,D3RZ,D3IZ,FR,FI,FPR,FPI,GR,GI,
     A       GPR,GPI,AREAL(23),AIMAG(23),ZERO,TWO,CA,CB,
     B       DVTR,DVTI,DVTPR,DVTPI,DWTR,DWTI,DWTPR,DWTPI
      DATA AZERO,BZERO,(A(M),B(M),M=1,11) /
     A           0.93043671693   D  0 ,      0.67829872514   D  0 ,
     B          31.01455723097   D  0 ,     11.30497875240   D  0 ,
     C         206.76371487316   D  0 ,     53.83323215431   D  0 ,
     D         574.34365242545   D  0 ,    119.62940478735   D  0 ,
     E         870.21765519008   D  0 ,    153.37103177865   D  0 ,
     F         828.77871922864   D  0 ,    127.80919314888   D  0 ,
     G         541.68543740434   D  0 ,     74.74221821572   D  0 ,
     H         257.94544638302   D  0 ,     32.35593862152   D  0 ,
     I          93.45849506631   D  0 ,     10.78531287384   D  0 ,
     J          26.62635187074   D  0 ,      2.85325737403   D  0 ,
     K           6.121000430056  D  0 ,      0.613603736351  D  0 ,
     L           1.159280384480  D  0 ,      0.109376780098  D  0 /
      DATA (A(M),B(M),M=12,22)                              /
     A           0.184012759441  D  0 ,      0.16422939955   D- 1 ,
     B           0.24833030964   D- 1 ,      0.2105505122    D- 2 ,
     C           0.2884208010    D- 2 ,      0.233167788     D- 3 ,
     D           0.291334142     D- 3 ,      0.22528289      D- 4 ,
     E           0.25827495      D- 4 ,      0.1915671       D- 5 ,
     F           0.2025686       D- 5 ,      0.144470        D- 6 ,
     G           0.141557        D- 6 ,      0.9729          D- 8 ,
     H           0.8870          D- 8 ,      0.589           D- 9 ,
     I           0.501           D- 9 ,      0.32            D-10 ,
     J           0.26            D-10 ,      0.2             D-11 ,
     K           0.1             D-11 ,      0.0             D  0 /
      DATA CZERO,DZERO,(C(M),D(M),M=1,11) /
     A           0.46521835846   D  0 ,      0.67829872514   D  0 ,
     B           6.20291144619   D  0 ,     45.21991500962   D  0 ,
     C          25.84546435915   D  0 ,    376.83262508015   D  0 ,
     D          52.21305931140   D  0 ,   1196.29404787350   D  0 ,
     E          62.15840394215   D  0 ,   1993.82341312250   D  0 ,
     F          48.75168936639   D  0 ,   2044.94709038206   D  0 ,
     G          27.08427187022   D  0 ,   1420.10214609865   D  0 ,
```

```
     H          11.21501940796    D  0 ,       711.83064967351    D  0 ,
     I           3.59455750255    D  0 ,       269.63282184603    D  0 ,
     J           0.91815006451    D  0 ,        79.89120647290    D  0 ,
     K           0.191281263439   D  0 ,        19.021715826880   D  0 ,
     L           0.33122296699    D- 1 ,         3.718810523339   D  0 /
      DATA  (C(M),D(M),M=12,22)                /
     A           0.4842441038     D- 2 ,         0.607648778323   D  0 ,
     B           0.605683682      D- 3 ,         0.84220204896    D- 1 ,
     C           0.65550182       D- 4 ,         0.10026214869    D- 1 ,
     D           0.6198599        D- 5 ,         0.1036301278     D- 2 ,
     E           0.516550         D- 6 ,         0.93867869       D- 4 ,
     F           0.38220          D- 7 ,         0.7512435        D- 5 ,
     G           0.2528           D- 8 ,         0.535074         D- 6 ,
     H           0.150            D- 9 ,         0.34135          D- 7 ,
     I           0.8              D-11 ,         0.1962           D- 8 ,
     J           0.0              D  0 ,         0.102            D- 9 ,
     K           0.0              D  0 ,         0.5              D-11 /
      DATA ZERO,TWO,CA,CB/0.0D0,2.0D0,1.171416657372D0,0.676317722467D0/
      IF(CABS(Z).GT.(6.1)) GO TO 99
      M=3+INT(20.*CABS(Z)/6.1)
      DRZ=DBLE(REAL(-Z))
      DIZ=DBLE(REAL((0.,1.)*Z))
      D2RZ=DRZ*DRZ-DIZ*DIZ
      D2IZ=TWO*DRZ*DIZ
      D3RZ=DRZ*D2RZ-DIZ*D2IZ
      D3IZ=DRZ*D2IZ+DIZ*D2RZ
      D3RZ = D3RZ/(-200.0D0)
      D3IZ = D3IZ/(-200.0D0)
      DO 39  I=1,23
      AREAL(I)= ZERO
      AIMAG(I)= ZERO
   39 CONTINUE
      IF(M.GT.22)  M=22
      DO 1 N=M,1,-1
      AREAL(N)=D3RZ*AREAL(N+1) - D3IZ*AIMAG(N+1) + A(N)
      AIMAG(N)=D3RZ*AIMAG(N+1) + D3IZ*AREAL(N+1)
    1 CONTINUE
      FR = D3RZ*AREAL(1)-D3IZ*AIMAG(1)+AZERO
      FI = D3RZ*AIMAG(1)+D3IZ*AREAL(1)
```

```
      DO 2 N=M,1,-1
      AREAL(N)=D3RZ*AREAL(N+1) - D3IZ*AIMAG(N+1) + B(N)
      AIMAG(N)=D3RZ*AIMAG(N+1) + D3IZ*AREAL(N+1)
    2 CONTINUE
      GPR= D3RZ*AREAL(1)-D3IZ*AIMAG(1)+BZERO
      GPI= D3RZ*AIMAG(1)+D3IZ*AREAL(1)
      GR = DRZ*GPR-DIZ*GPI
      GI = DRZ*GPI+DIZ*GPR
      DO 3 N=M,1,-1
      AREAL(N)=D3RZ*AREAL(N+1)-D3IZ*AIMAG(N+1)+C(N)
      AIMAG(N)=D3RZ*AIMAG(N+1)+D3IZ*AREAL(N+1)
    3 CONTINUE
      GPR = D3RZ*AREAL(1)-D3IZ*AIMAG(1)+CZERO
      GPI = D3RZ*AIMAG(1)+D3IZ*AREAL(1)
      FPR =-D2RZ*GPR + D2IZ*GPI
      FPI =-D2RZ*GPI - D2IZ*GPR
      DO 4 N=M,1,-1
      AREAL(N)=D3RZ*AREAL(N+1)-D3IZ*AIMAG(N+1)+D(N)
      AIMAG(N)=D3RZ*AIMAG(N+1)+D3IZ*AREAL(N+1)
    4 CONTINUE
      GPR = D3RZ*AREAL(1)-D3IZ*AIMAG(1)+DZERO
      GPI = D3RZ*AIMAG(1)+D3IZ*AREAL(1)
C-----NOW CONSTRUCT FOCK AIRY FUNCTIONS
      DVTR    = CB*(FR + GR)
      DVTI    = CB*(FI + GI)
      DVTPR   =-CB*(FPR + GPR)
      DVTPI   =-CB*(FPI + GPI)
      DWTR    = CA*(FR - GR) - DVTI
      DWTI    = CA*(FI - GI) + DVTR
      DWTPR   =-CA*(FPR - GPR) - DVTPI
      DWTPI   =-CA*(FPI - GPI) + DVTPR
      VT      = CMPLX(SNGL(DVTR),SNGL(DVTI))
      VTP     = CMPLX(SNGL(DVTPR),SNGL(DVTPI))
      WT      = CMPLX(SNGL(DWTR),SNGL(DWTI))
      WTP     = CMPLX(SNGL(DWTPR),SNGL(DWTPI))
      RETURN
   99 WRITE(6,110) Z
  110 FØRMAT(25H1VTWTZ CANNØT HANDLE Z = ,(E15.8,E15.8))
      CALL EXIT
      END
```

REFERENCES FOR APPENDIX E

1. P. A. Azriliant and M. G. Belkina, Numerical Results of the Theory of Diffraction of Radio Waves Around the Earth's Surface, Moscow, Soviet Radio Press, 1957. (The author is endebted to his colleague Mr. M. D. Friedman for preparing an English translation of this important publication. The would like to see this report made available to people interested in diffraction theory but the cost of printing the text, tables, and graphs is prohibitive and the demand for this material so limited that no attempts have been made to seek a publisher.)

2. Harvard University Computation Laboratory, Tables of the Modified Hankel Functions of Order One-Third and of Their Derivatives, Annals of the Computation Laboratory of Harvard University, Vol. II, Cambridge, Mass., Harvard University Press, 1945

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DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Lockheed Aircraft Corp., Lockheed Missiles & Space Co., 1111 Lockheed Way, Sunnyvale, Calif.	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

Numerical Investigation of Electromagnetic Scattering and Diffraction by Convex Objects

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)* Scientific Report. Final. Period covered: 16 Nov 64 through 15 Nov 65. Accepted: 14 Mar 66.

5. AUTHOR(S) *(Last name, first name, initial)*

Logan, Nelson A.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
Dec 65	707	240

8a. CONTRACT OR GRANT NO. AF19(628)-4393	9a. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT AND TASK NO. 460004	M-76-66-1
c. DOD ELEMENT 62405304	9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*
d. DOD SUBELEMENT 674600	AFCRL-66-153

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11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Hq AFCRL, OAR (CRD) United States Air Force L.G. Hanscom Field, Bedford, Mass.

13. ABSTRACT

The solutions $D(x,y,y_o,q)$ of the partial differential equation D_{yy} + $iD_x + yD = 0$ are employed as a mathematical model for the fields reflected from and diffracted around an opaque convex surface. This solution involves the Airy function $w_1(t)$ which is a solution of $w''_1(t)$ $tw_1(t) = 0$. FORTRAN computer programs and tables are given for the appropriate Airy functions. Representations for the roots of the equation $w'_1(t_s) - qw_1(t_s) = 0$ which play a fundamental role in the theory are discussed. As a function of q, these roots satisfy the Riccati equation $(t - q^2)(dt/dq) = 1$. The Riccati equation is used to develop several representations for these roots. FORTRAN programs are presented for the evaluation of these constants. Tables of the roots are given for certain cases which have been shown to be of practical significance. Asymptotic expansions are developed for the height gain function $F(y)$ which is a solution of $F''(y) + (k^2 f(y) - \lambda_s)F(y) = 0$. The theory of representing functions by a series of Chebyshev polynomials is reviewed and a discussion is given of the significance of these expansions for future work in diffraction theory.

DD FORM 1 JAN 64 1473

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Diffraction Functions	8	3				
Airy Functions	8	3				
Electromagnetic Theory	8	2				
Height Gain Function	8	3				
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